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Exact likelihood inference for two exponential populations under joint Type-II hybrid censoring scheme

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Abstract: When Type-II hybrid censoring is used on two samples in a combined manner, the exact inference for two exponential populations is developed in this paper. The two unknown exponential mean parameters' conditional maximum likelihood and Bayesian estimators are determined. The maximum likelihood estimators' conditional moment generating functions and conditional exact distributions are then calculated. For the unknown parameters, the exact, approximate, and Bayes credible confidence intervals are also constructed. In addition, a Monte Carlo simulation study is carried out to evaluate the performance of the two estimation methods and also the three confidence intervals. Finally, using a real data set, some numerical results are presented.

Keywords: Exponential distribution, Type-II Hybrid censoring, Joint censoring, Maximum likelihood estimation, Bayesian estimation, Confidence interval.

1 Introduction

Due to a variety of factors, the experimenter may choose to end the experiment before failing all units on the test in reliability analysis. Censored data refers to the results of such experiments. There are numerous different types of censoring schemes, with Type-I and Type-II being the most frequent. The experimenter terminates the life testing experiment at a pre-determined time T in the Type-I censoring scheme, whereas the experimenter terminates the life testing experiment at the time of the r^{th} failure in the Type-II censoring method. Surveys of censorship schemes can be found in papers [1,2,3,4,5].

Epstein [6] proposed the Type-I hybrid censoring scheme (Type-I HCS), in which the life testing experiment is ended after a pre-determined number r out of n items fails or a pre-determined time T on test is reached. MIL-STD-781 C [7] has employed the Type-I HCS as a reliability acceptance test. However, the Type-I HCS may result in the data having too few observations. As a result, Childs et al. [8] presented the Type-II hybrid censoring scheme (Type-II HCS), in which the life-testing experiment ends when one of the aforementioned two termination rules is achieved. It is better to employ Type-II HCS since it guarantees that the number of observations in the data is at least r, resulting in more efficient inferential processes than Type-I HCS. The literature on hybrid censoring and associated inferential approaches is vast; see, for example, [9,10,11]. The new discussion paper [12] provides an in-depth review of different developments in hybrid censoring approach and its applications.

We can utilise the joint Type-II censoring scheme to perform comparative life-tests of items from different lines of manufacturing. Assume two independent samples of sizes m and n are chosen from two product lines and placed on a life-testing experiment at the same time. Under the joint Type-II censoring scheme, the experiment is ended after a pre-specified number of failures are recorded. Balakrishnan and Rasouli [13] studied the exact inference using a joint Type-II censored sample from two exponential populations. They established exact inferential methods based on maximum likelihood (ML) estimators and compared their performance to that of other approaches such as Bayesian and bootstrap; for a generalization of their results to progressive Type-II censoring, see paper [14]. In this paper, we extend these findings to the scenario where the two samples are censored using a joint Type-II hybrid censoring scheme.

The following is a description of this model. Assume that $X_1,...,X_m$ are the lifetimes of m specimens of product A and they are independent and identically distributed

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(iid) random variables derived from the distribution function $F(x)$ and density function $f(x)$. Assume that $Y_1,...,Y_n$ are the lifetimes of n specimens of product B and they are iid random variables derived from the distribution function $G(x)$ and the density function $g(x)$. Assume that $W_1 \le ... \le W_N$ denote the order statistics of the combined sample of $N = m+n$ random variables $\{X_1,...,X_m;Y_1,...,Y_n\}$, and the experiment ends at time $T^* = \max\{W_r,T\}$, where $1 \le r \le N$ and $T \in (0, \infty)$ are pre-determined.

Let D represent the total number of failures up to T. Then D is a discrete random variable with the following probability mass function

$$P(D=d) = \sum_{k=max(0,d-n)}^{min(m,d)} \binom{m}{k}\binom{n}{d-k} p_1{}^k q_1{}^{m-k} p_2{}^{d-k} q_2{}^{n-d+k}, \quad d = 0,1,...,N, \tag{1}$$

where $p_1 = F(T)$, $q_1 = 1-F(T)$, $p_2 = G(T)$ and $q_2 = 1-G(T)$.

Therefore, under the joint Type-II hybrid censoring scheme described above, the observable data consist of $(\mathbf{Z},\mathbf{W})$ where $\mathbf{Z} = (Z_1,...,Z_{r^*})$ and $\mathbf{W} = (W_1,...,W_{r^*})$ with

$$r^* = \begin{cases} r, & \text{if } T^* = W_r,\ D = 0,1,...,r-1, \\ D, & \text{if } T^* = T,\quad D = r,r+1,...,N, \end{cases}$$

and $Z_i = 1$ or 0 according as W_i is from an X- or Y-failure.

The likelihood function of $(\mathbf{Z},\mathbf{W})$ is given by

$$L(\theta_1,\theta_2,\mathbf{z},\mathbf{w}) = \frac{m!n!}{(m-m_{r^*})!(n-n_{r^*})!}\prod_{i=1}^{r^*} f(w_i)^{z_i} g(w_i)^{(1-z_i)} \{\bar{F}(T^*)\}^{m-m_{r^*}} \{\bar{G}(T^*)\}^{n-n_{r^*}} \tag{2}$$

where $\bar{F} = 1-F$, $\bar{G} = 1-G$, $M_{r^*} = \sum_{i=1}^{r^*} Z_i$ is the number of X-failures in $\mathbf{W}$ and $N_{r^*} = \sum_{i=1}^{r^*}(1-Z_i)$ is the number of Y-failures in $\mathbf{W}$.

The content of this work is arranged in the following manner. In Section 2, we consider the case of two exponential distributions based on joint Type-II hybrid censored data and compute the ML estimators of the two scale parameters, after which we generate the exact conditional moment generating function of the ML estimators and then use them to obtain the means, variances, and mean squared errors of these estimators. The exact, approximate, and Bayesian techniques of forming confidence intervals (CIs) for unknown parameters are discussed in Section 3. Finally, in Section 4, Monte Carlo simulation and numerical results are provided to illustrate all of the inferential approaches presented here.

2 Methods

The conditional ML estimators of the unknown parameters are calculated in this section, followed by the conditional moment generating functions and conditional exact distributions of the ML estimators. Assume the distributions of the two populations are exponential with the following survival functions

$$\bar{F}(x) = e^{-x/\theta_1} \text{ and } \bar{G}(x) = e^{-x/\theta_2},\ x > 0,\ \theta_1 > 0,\ \theta_2 > 0.$$

In this case, the likelihood function of $(\mathbf{Z},\mathbf{W})$ in (2) simplifies to

$$L(\theta_1,\theta_2,\mathbf{z},\mathbf{w}) = \frac{m!n!}{(m-m_{r^*})!(n-n_{r^*})!\theta_1^{m_{r^*}}\theta_2^{n_{r^*}}} e^{\left[-\frac{u_1}{\theta_1}-\frac{u_2}{\theta_2}\right]}, \tag{3}$$

where

$$u_1 = \sum_{i=1}^{r^*} z_i w_i + (m-m_{r^*})T^*$$

and

$$u_2 = \sum_{i=1}^{r^*} (1-z_i) w_i + (n-n_{r^*})T^*.$$

From this likelihood function, we readily obtain the MLEs of θ_1 and θ_2 as

$$\hat{\theta}_1 = \frac{u_1}{m_{r^*}} = \begin{cases} \frac{1}{m_r}\left(\sum_{i=1}^{r} z_i w_i + (m-m_r)w_r\right), & D = 0,1,...,r-1, \\ \frac{1}{m_D}\left(\sum_{i=1}^{D} z_i w_i + (m-m_D)T\right), & D = r,r+1,...,N, \end{cases} \tag{4}$$

and

$$\hat{\theta}_2 = \frac{u_2}{n_{r^*}} = \begin{cases} \frac{1}{n_r}\left(\sum_{i=1}^{r} (1-z_i) w_i + (n-n_r)w_r\right), & D = 0,1,..,r-1, \\ \frac{1}{n_D}\left(\sum_{i=1}^{D} (1-z_i) w_i + (n-n_D)T\right), & D = r,r+1,..,N. \end{cases} \tag{5}$$

Remark 1. From the ML estimators in (4) and (5), it can be seen immediately that if $T < W_r$ and $M_r = 0$ (or r), then $\hat{\theta}_1$ (or $\hat{\theta}_2$) does not exist. Also, if $W_r < T$ and $M_D = 0$ (or D), then $\hat{\theta}_1$ (or $\hat{\theta}_2$) does not exist. Hence, the ML estimators in (4) and (5) are only conditional ML estimators, conditioned on

$$max\{1, r-n\} \le M_r \le min\{r-1, m\}$$

or

$$max\{1, D-n\} \le M_D \le min\{D-1, m\},$$

corresponding to $T < W_r$ or $W_r < T$, respectively. Therefore, we need to discuss the sampling distributions and other properties of the ML estimators only conditional on the event $A = A_1 \cup A_2$ where

$$A_1 = (max\{1, r-n\} \leq M_r \leq min\{r-1, m\})$$

and

$$A_2 = (max\{1, D-n\} \leq M_D \leq min\{D-1, m\}).$$

The primary findings described in the subsequent theorems will be developed using the following Lemma.

Lemma 2. Let $a_j > 0$, for $j = 1, 2, ..., s$. Then, we have

$$\int_0^T \int_0^{w_s} ... \int_0^{w_2} e^{-\sum_{j=1}^s a_j w_j} dw_1 ... dw_{s-1} dw_s = \sum_{i=0}^{s} c_{i,s}(\mathbf{a}_s) e^{-b_{i,s}(\mathbf{a}_s)T}, \quad (6)$$

where $\mathbf{a}_s = (a_1, a_2, ..., a_s)$,

$$c_{i,s}(\mathbf{a}_s) = \frac{(-1)^i}{\left(\prod_{j=1}^{i} \sum_{k=s-i+1}^{s-i+j} a_k\right)\left(\prod_{j=1}^{s-i} \sum_{k=j}^{s-i} a_k\right)} \quad (7)$$

and

$$b_{i,s}(\mathbf{a}_s) = \sum_{j=s-i+1}^{s} a_j, \quad (8)$$

in which we adopt the usual conventions that $\prod_{k=1}^{0} d_j \equiv 1$ and $\sum_{k=i}^{i-1} d_j \equiv 0$.

For a proof of this result and some generalizations of it, one may refer to paper [15].

Theorem 3.

1.Conditional on $D = d$, $d = 0, 1, ..., r-1$, the joint probability mass function of $\mathbf{Z}_r = (Z_1, ..., Z_r)$ is

$$P(\ \mathbf{Z}_r = \mathbf{z}_r | D = d) = \frac{C_r}{\theta_1{}^{m_r} \theta_2{}^{n_r} P(D = d)} \sum_{i=0}^{d} \frac{c_{i,d}(\mathbf{a}_d)\ e^{-\{b_{i,d}(\mathbf{a}_d) + \frac{m-m_r}{\theta_1} + \frac{n-n_r}{\theta_2} + \sum_{j=d+1}^{r} a_j\}T}}{\prod_{j=1}^{r-d} (\frac{m-m_r}{\theta_1} + \frac{n-n_r}{\theta_2} + \sum_{k=1}^{j} a_{r-k+1})}, \quad (9)$$

for $Q_1 = \{\mathbf{z}_r = (z_1, ..., z_r) : z_j = 0\ or\ 1\}$, where $C_r = \frac{m!n!}{(m-m_r)!(n-n_r)!}$, and $c_{i,d}(\mathbf{a}_d)$ and $b_{i,d}(\mathbf{a}_d)$ as in (7) and (8), respectively, with $s = d$ and $a_j = \frac{z_j}{\theta_1} + \frac{1-z_j}{\theta_2}$, for $j = 1, ..., r$;

2.Conditional on $D = d$, $d = r, r+1, ..., N$, the joint probability mass function of $\mathbf{Z}_d = (Z_1, ..., Z_d)$ is

$$P(\ \mathbf{Z}_d = \mathbf{z}_d | D = d) = \frac{C_d}{\theta_1^{m_d} \theta_2^{n_d} P(D = d)} \sum_{i=0}^{d} c_{i,d}(\mathbf{a}_d) e^{-\{b_{i,d}(\mathbf{a}_d) + \frac{m-md}{\theta_1} + \frac{n-nd}{\theta_2}\}T}, \quad (10)$$

for $Q_2 = \{\mathbf{z}_d = (z_1, ..., z_d) : z_j = 0\ or\ 1\}$, where $C_d = \frac{m!n!}{(m-m_d)!(n-n_d)!}$, and $c_{i,d}(\mathbf{a}_d)$ and $b_{i,d}(\mathbf{a}_d)$ as in (7) and (8), respectively, with $s = d$ and $a_j = \frac{z_j}{\theta_1} + \frac{1-z_j}{\theta_2}$, for $j = 1, ..., d$;

3.Thence, conditional on $D = d$, $d = 0, 1, ..., r-1$, the probability mass function of $M_r = \sum_{j=1}^{r} Z_j$, for $\ell = 0, 1, ..., r$, is

$$P(M_r = \ell | D = d) = \frac{C_{\ell,r}}{\theta_1^{\ell} \theta_2^{r-\ell} P(D = d)} \sum_{\mathbf{z}_r \in Q_1^*} ... \sum \sum_{i=0}^{d} \frac{c_{i,d}(\mathbf{a}_d)}{\omega_{\ell,d}} e^{-\delta_{i,\ell,d}T}, \quad (11)$$

for $Q_{1\ell}^* = \{\mathbf{z}_r = (z_1, ..., z_r) : z_j = 0\ or\ 1,\ \sum_{j=1}^{r} z_j = \ell\}$, where $C_{\ell,r} = \frac{m!n!}{(m-\ell)!(n-r+\ell)!}$, $\omega_{\ell,d} = \prod_{j=1}^{r-d} (\frac{m-\ell}{\theta_1} + \frac{n-r+\ell}{\theta_2} + \sum_{k=1}^{j} a_{r-k+1})$, and $\delta_{i,\ell,d} = b_{i,d}(\mathbf{a}_d) + \frac{m-\ell}{\theta_1} + \frac{n-r+\ell}{\theta_2} + \sum_{k=d+1}^{r} a_k$;

4.Thence, conditional on $D = d$, $d = r, r+1, ..., N$, the probability mass function of $M_d = \sum_{j=1}^{d} Z_j$, for $\ell = 0, 1, ..., d$, is

$$P(M_d = \ell | D = d) = \frac{C_{\ell,d}^*}{\theta_1^{\ell} \theta_2^{d-\ell} P(D = d)} \sum_{\mathbf{z}_d \in Q_{2\ell}^*} ... \sum \sum_{i=0}^{d} c_{i,d}(\mathbf{a}_d) e^{-\delta_{i,\ell,d}^* T}, \quad (12)$$

for $Q_{2\ell}^* = \{\mathbf{z}_d = (z_1, ..., z_d) : z_j = 0\ or\ 1,\ \sum_{j=1}^{d} z_j = \ell\}$, where $C_{\ell,d}^* = \frac{m!n!}{(m-\ell)!(n-d+\ell)!}$ and $\delta_{i,\ell,d}^* = b_{i,d}(\mathbf{a}_d) + \frac{m-\ell}{\theta_1} + \frac{n-d+\ell}{\theta_2}$.

Proof. Since, for $d = 0, 1, .., r-1$, the conditional joint density function of $(W_1, ..., W_r; \mathbf{Z}_r)$, given $D = d$, is given by

$$f(w_1, .., w_r, \mathbf{z}_r | D = d) = \frac{C_r}{P(D = d)} \prod_{i=1}^{r} f(w_i)^{z_i} g(w_i)^{(1-z_i)} \{\bar{F}(w_r)\}^{m-m_r} \{\bar{G}(w_r)\}^{n-n_r}$$

$$= \frac{C_r}{\theta_1{}^{m_r} \theta_2{}^{n_r} P(D = d)} e^{-\{\sum_{j=1}^{r} a_j w_j + (\frac{m-m_r}{\theta_1} + \frac{n-n_r}{\theta_2}) w_r\}},$$

$$0 < w_1 < ... < w_d < T < w_{d+1} < ... < w_r < \infty.$$

Then, we obtain the joint probability mass function of $\mathbf{Z}_r = (Z_1, ..., Z_r)$ as

$$P(\mathbf{Z}_r = \mathbf{z}_r | D = d) = \int_T^{\infty} .. \int_{w_{r-1}}^{\infty} \int_0^T .. \int_0^{w_2} f(w_1, .., w_r, \mathbf{z}_r | D = d) dw_1..dw_d dw_r..dw_{d+1}$$
$$= \frac{C_r}{\theta_1^{m_r} \theta_2^{n_r} P(D=d)} \sum_{i=0}^{d} c_{i,d}(\mathbf{a}_d) \times \frac{e^{-\left\{b_{i,d}(\mathbf{a}_d) + \frac{m-m_r}{\theta_1} + \frac{n-n_r}{\theta_2} + \sum_{j=d+1}^{r} a_j\right\}T}}{\prod_{j=1}^{r-d} \left(\frac{m-m_r}{\theta_1} + \frac{n-n_r}{\theta_2} + \sum_{k=1}^{j} a_{r-k+1}\right)}, \quad (13)$$

as presented in (9).

2. Since, for $d = r, r+1, .., N$, the conditional joint density function of $(W_1, ..., W_d; \mathbf{Z}_d)$, given $D = d$, is given by

$$f(w_1, .., w_d, \mathbf{z}_d | D = d) = \frac{C_d}{P(D=d)} \times \prod_{i=1}^{d} f(w_i)^{z_i} g(w_i)^{(1-z_i)} \{\bar{F}(T)\}^{m-m_d} \{\bar{G}(T)\}^{n-n_d}$$
$$= \frac{C_d}{\theta_1^{m_d} \theta_2^{n_d} P(D=d)} e^{-\{\sum_{j=1}^{d} a_j w_j + (\frac{m-md}{\theta_1} + \frac{n-nd}{\theta_2})T\}},$$
$$0 < w_1 < ... < w_d < T.$$

Then, we obtain the joint probability mass function of $\mathbf{Z}_d = (Z_1, ..., Z_d)$ as

$$P(\mathbf{Z}_d = \mathbf{z}_d | D = d) = \int_0^T \int_0^{w_d} .. \int_0^{w_2} f(w_1, .., w_d, \mathbf{z}_d | D = d) dw_1..dw_{d-1} dw_d$$
$$= \frac{C_d}{\theta_1^{m_d} \theta_2^{n_d} P(D=d)} \sum_{i=0}^{d} c_{i,d}(\mathbf{a}_d) e^{-\{b_{i,d}(\mathbf{a}_d) + \frac{m-md}{\theta_1} + \frac{n-nd}{\theta_2}\}T}, \quad (14)$$

as presented in (10).

3. From (9), the formula $P(M_r = \ell | D = d)$ in (11) follows easily.

4. From (10), the formula $P(M_d = \ell | D = d)$ in (12) follows easily.

3 The exact conditional distribution of $\widehat{\theta}_1$

Theorem 4. Conditional on the event A, the moment generating function (mgf) of $\hat{\theta}_1$ is given by

$$M_{\hat{\theta}_1}(t) = \sum_{d=0}^{r-1} \frac{1}{P(\ell_{1,r} \le M_r \le \ell_{2,r} | D = d)} \left\{ \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{\mathbf{z}_r \in Q_{1\ell}^*} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,r} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell} \theta_2^{r-\ell} \omega_{\ell,d}} \prod_{j=1}^{i} (1 - \phi_{j,i,\ell,d} t)^{-1} \prod_{j=1}^{d-i} (1 - \psi_{j,i,\ell,d} t)^{-1} \prod_{j=1}^{r-d} (1 - \chi_{j,i,\ell,d} t)^{-1} e^{-(1-\gamma_{i,\ell,d} t)\delta_{i,\ell,d} T} \right\}$$
$$+ \sum_{d=r}^{N} \frac{1}{P(\ell_{1,d} \le M_d \le \ell_{2,d} | D = d)} \left\{ \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{\mathbf{z}_d \in Q_{2\ell}^*} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,d}^* c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell} \theta_2^{d-\ell}} \prod_{j=1}^{i} (1 - \phi_{j,i,\ell,d} t)^{-1} \prod_{j=1}^{d-i} (1 - \psi_{j,i,\ell,d} t)^{-1} e^{-(1-\gamma_{i,\ell,d}^* t)\delta_{i,\ell,d}^* T} \right\}, \quad (15)$$

where $\ell_{1,r} = max\{1, r-n\}$, $\ell_{2,r} = min\{r-1, m\}$, $\ell_{1,d} = max\{1, d-n\}$, $\ell_{2,d} = min\{d-1, m\}$,

$$\phi_{j,i,\ell,d} = \frac{\sum_{k=d-i+1}^{d-i+j} z_k}{\ell \sum_{k=d-i+1}^{d-i+j} a_k}, \quad \psi_{j,i,\ell,d} = \frac{\sum_{k=j}^{d-i} z_k}{\ell \sum_{k=j}^{d-i} a_k},$$

$$\chi_{j,i,\ell,d} = \frac{m - \ell + \sum_{k=1}^{j} z_{r-k+1}}{\ell\left(\frac{m-\ell}{\theta_1} + \frac{n-r+\ell}{\theta_2} + \sum_{k=1}^{j} a_{r-k+1}\right)},$$

$$\gamma_{i,\ell,d} = \frac{m - \ell + \sum_{k=d-i+1}^{r} z_k}{\ell \delta_{i,\ell,d}}, \quad \gamma_{i,\ell,d}^* = \frac{m - \ell + \sum_{k=d-i+1}^{d} z_k}{\ell \delta_{i,\ell,d}^*}.$$

Proof. Conditioning on the event A, we have

$$M_{\hat{\theta}_1}(t) = E(e^{t\hat{\theta}_1} | A)$$
$$= \sum_{d=0}^{r-1} E(e^{t\hat{\theta}_1} | D = d, \ell_{1,r} \le M_r \le \ell_{2,r}) P(D = d)$$
$$+ \sum_{d=r}^{N} E(e^{t\hat{\theta}_1} | D = d, \ell_{1,d} \le M_d \le \ell_{2,d}) P(D = d). \quad (16)$$

First, for $d = 0, 1, .., r-1$, we have

$$E(e^{t\hat{\theta}_1}|D=d, \ell_{1,r} \le M_r \le \ell_{2,r})$$
$$= \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} E(e^{t\hat{\theta}_1}|D=d, M_r=\ell)$$
$$\times P(M_r=\ell|D=d, \ell_{1,r} \le M_r \le \ell_{2,r})$$
$$= \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{z_r=0}^{1} \cdots \sum_{z_1=0}^{1} E(e^{t\hat{\theta}_1}|D=d, M_r=\ell, \mathbf{Z}_r=\mathbf{z}_r)$$
$$\times P(\mathbf{Z}_r=\mathbf{z}_r|D=d, M_r=\ell)$$
$$\times P(M_r=\ell|D=d, \ell_{1,r} \le M_r \le \ell_{2,r})$$
$$= \frac{1}{P(\ell_{1,r} \le M_r \le \ell_{2,r}|D=d)P(D=d)}$$
$$\times \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{\mathbf{z}_r \in Q_{1\ell}^*} \cdots \sum \frac{C_{\ell,r}}{\theta_1^{\ell}\theta_2^{r-\ell}} \int_T^{\infty} \cdots \int_{w_{r-1}}^{\infty} \int_0^T \cdots \int_0^{w_2}$$
$$e^{-(\sum_{j=1}^{r} A_{j,\ell}(t)w_j + B_{\ell,r}(t)w_r)} dw_1 \ldots dw_d dw_r \ldots dw_{d+1},$$

where $A_{j,\ell}(t) = a_j - \frac{z_j}{\ell}t$, for $j = 1, \ldots, r$, and $B_{\ell,r}(t) = \frac{m-\ell}{\theta_1} + \frac{n-r+\ell}{\theta_2} - \frac{m-\ell}{\ell}t$.

After completing the necessary integration and applying Lemma 2, we now have

$$E(e^{t\hat{\theta}_1}|D=d, \ell_{1,r} \le M_r \le \ell_{2,r})$$
$$= \frac{1}{P(\ell_{1,r} \le M_r \le \ell_{2,r}|D=d)P(D=d)}$$
$$\times \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{\mathbf{z}_r \in Q_{1\ell}^*} \cdots \sum \sum_{i=0}^{r-1} \frac{C_{\ell,r} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}} \prod_{j=1}^{i} (1-\phi_{j,i,\ell,d}t)^{-1}$$
$$\prod_{j=1}^{d-i}(1-\psi_{j,i,\ell,d}t)^{-1} \prod_{j=1}^{r-d}(1-\chi_{j,i,\ell,d}t)^{-1} e^{-(1-\gamma_{i,\ell,d}t)\delta_{i,\ell,d}T}. \tag{17}$$

Next, for $d = r, r+1, .., N$, we have

$$E(e^{t\hat{\theta}_1}|D=d, \ell_{1,d} \le M_d \le \ell_{2,d})$$
$$= \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} E(e^{t\hat{\theta}_1}|D=d, M_d=\ell)$$
$$\times P(M_d=\ell|D=d, \ell_{1,d} \le M_d \le \ell_{2,d})$$
$$= \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{z_d=0}^{1} \cdots \sum_{z_1=0}^{1} E(e^{t\hat{\theta}_1}|D=d, M_d=\ell, \mathbf{Z}_d=\mathbf{z}_d)$$
$$\times P(\mathbf{Z}_d=\mathbf{z}_d|D=d, M_d=\ell)$$
$$\times P(M_d=\ell|D=d, \ell_{1,d} \le M_d \le \ell_{2,d})$$
$$= \frac{1}{P(\ell_{1,d} \le M_d \le \ell_{2,d}|D=d)P(D=d)}$$
$$\times \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{\mathbf{z}_d \in Q_{2\ell}^*} \cdots \sum \frac{C_{\ell,d}^*}{\theta_1^{\ell}\theta_2^{d-\ell}} \int_0^T \int_0^{w_d} \cdots \int_0^{w_2}$$
$$e^{-(\sum_{j=1}^{d} A_{j,\ell}^*(t)w_j + B_{\ell,d}^*(t)T)} dw_1 \ldots dw_{d-1} dw_d,$$

where $A_{j,\ell}^*(t) = a_j - \frac{z_j}{\ell}t$, for $j = 1, \ldots, d$, and $B_{\ell,d}^*(t) = \frac{m-\ell}{\theta_1} + \frac{n-d+\ell}{\theta_2} - \frac{m-\ell}{\ell}t$.

After completing the necessary integration and applying Lemma 2, we now have

$$E(e^{t\hat{\theta}_1}|D=d, \ell_{1,d} \le M_d \le \ell_{2,d})$$
$$= \frac{1}{P(\ell_{1,d} \le M_d \le \ell_{2,d}|D=d)P(D=d)}$$
$$\times \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{\mathbf{z}_d \in Q_{2\ell}^*} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,d}^* c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}} \prod_{j=1}^{i} (1-\phi_{j,i,\ell,d}t)^{-1}$$
$$\times \prod_{j=1}^{d-i}(1-\psi_{j,i,\ell,d}t)^{-1} e^{-(1-\gamma_{i,\ell,d}^*t)\delta_{i,\ell,d}^*T}. \tag{18}$$

We can get the formula in (15) by substituting (17) and (18) into (16).

Remark 5.

1. $(1-ct)^{-1}$ is the mgf of the exponential distribution with scale parameter c;
2. e^{ct} is the mgf of the degenerate distribution localized at a point c.

Theorem 6. Conditional on the event A, the density of the MLE $\hat{\theta}_1$ is given by

$$f_{\hat{\theta}_1}(x) = \sum_{d=0}^{r-1} \frac{1}{P(\ell_{1,r} \le M_r \le \ell_{2,r}|D=d)}$$
$$\left\{ \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{\mathbf{z}_r \in Q_{1\ell}^*} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,r} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}} e^{-\delta_{i,\ell,d}T} g_{X_{i,\ell,d}}(x) \right\}$$
$$+ \sum_{d=r}^{N} \frac{1}{P(\ell_{1,d} \le M_d \le \ell_{2,d}|D=d)}$$
$$\left\{ \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{\mathbf{z}_d \in Q_{2\ell}^*} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,d}^* c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}} e^{-\delta_{i,\ell,d}^*T} h_{X_{i,\ell,d}^*}(x) \right\}. \tag{19}$$

where, $X_{i,\ell,d} \overset{d}{=} X_{1_{i,\ell,d}} + X_{2_{i,\ell,d}} + X_{3_{i,\ell,d}} + X_{4_{i,\ell,d}}$, with $X_{1_{i,\ell,d}} = \sum_{j_1=1}^{i} X_{1j_1}$, $X_{2_{i,\ell,d}} = \sum_{j_2=1}^{d-i} X_{2j_2}$ and $X_{3_{i,\ell,d}} = \sum_{j_3=1}^{r-d} X_{3j_3}$, with X_{1j_1} $(j_1 = 1, \ldots, i)$, X_{2j_2} $(j_2 = 1, \ldots, d-i)$ and X_{3j_3} $(j_3 = 1, \ldots, r-d)$ being independent random variables having exponential distributions with scale parameters $\phi_{j_1,i,\ell,d}$, $\psi_{j_2,i,\ell,d}$ and $\chi_{j_3,i,\ell,d}$, respectively, $X_{4_{i,\ell,d}}$ being a random variable distributed as degenerate localized at a point $\gamma_{i,\ell,d}\delta_{i,\ell,d}T$, and $X_{i,\ell,d}^* \overset{d}{=} X_{1_{i,\ell,d}} + X_{2_{i,\ell,d}} + X_{3_{i,\ell,d}}^*$, with $X_{3_{i,\ell,d}}^*$ being a random variable distributed as degenerate localized at a point $\gamma_{i,\ell,d}^*\delta_{i,\ell,d}^*T$.

Proof. The conditional mgf of θ_1 in (15) and Remark 5 instantly lead to this result.

Corollary 7. From (19), we can obtain

$$E(\hat{\theta}_1)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\left\{\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}e^{-\delta_{i,\ell,d}T}\times\sum_{j=1}^{i}\phi_{j,i,\ell,d}+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}+\sum_{j=1}^{r-d}\chi_{j,i,\ell,d}+\gamma_{i,\ell,d}\delta_{i,\ell,d}T\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\left\{\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}e^{-\delta^*_{i,\ell,d}T}\times\sum_{j=1}^{i}\phi_{j,i,\ell,d}+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}+\gamma^*_{i,\ell,d}\delta^*_{i,\ell,d}T\right\}, \quad (20)$$

$$E(\hat{\theta}_1^2)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}e^{-\delta_{i,\ell,d}T}$$
$$\times\left\{\sum_{j=1}^{i}\phi^2_{j,i,\ell,d}+\sum_{j=1}^{d-i}\psi^2_{j,i,\ell,d}+\sum_{j=1}^{r-d}\chi^2_{j,i,\ell,d}+\gamma^2_{i,\ell,d}\delta^2_{i,\ell,d}T^2+\sum_{j=1}^{i}\sum_{k=1}^{i}\phi_{j,i,\ell,d}\phi_{k,i,\ell,d}+\sum_{j=1}^{d-i}\sum_{k=1}^{d-i}\psi_{j,i,\ell,d}\psi_{k,i,\ell,d}\right.$$
$$+\sum_{j=1}^{r-d}\sum_{k=1}^{r-d}\chi_{j,i,\ell,d}\chi_{k,i,\ell,d}+2\sum_{j=1}^{i}\sum_{k=1}^{d-i}\phi_{j,i,\ell,d}\psi_{k,i,\ell,d}+2\sum_{j=1}^{i}\sum_{k=1}^{r-d}\phi_{j,i,\ell,d}\chi_{k,i,\ell,d}+2\sum_{j=1}^{d-i}\sum_{k=1}^{r-d}\psi_{j,i,\ell,d}\chi_{k,i,\ell,d}$$
$$\left.+2\gamma_{i,\ell,d}\delta_{i,\ell,d}T\left(\sum_{j=1}^{i}\phi_{j,i,\ell,d}+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}+\sum_{j=1}^{r-d}\chi_{j,i,\ell,d}\right)\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}e^{-\delta^*_{i,\ell,d}T}\left\{\sum_{j=1}^{i}\phi^2_{j,i,\ell,d}\right.$$
$$+\sum_{j=1}^{d-i}\psi^2_{j,i,\ell,d}+\gamma^{*2}_{i,\ell,d}\delta^{*2}_{i,\ell,d}T^2+\sum_{j=1}^{i}\sum_{k=1}^{i}\phi_{j,i,\ell,d}\phi_{k,i,\ell,d}+\sum_{j=1}^{d-i}\sum_{k=1}^{d-i}\psi_{j,i,\ell,d}\psi_{k,i,\ell,d}+2\sum_{j=1}^{i}\sum_{k=1}^{d-i}\phi_{j,i,\ell,d}\psi_{k,i,\ell,d}$$
$$\left.+2\gamma^*_{i,\ell,d}\delta^*_{i,\ell,d}T\left(\sum_{j=1}^{i}\phi_{j,i,\ell,d}+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}\right)\right\}. \quad (21)$$

Then, using these two expressions, $Var(\hat{\theta}_1)$ and $MSE(\hat{\theta}_1)$ may be easily derived.

The mgf of $\hat{\theta}_1$ in (15) may be rewritten as

$$M_{\hat{\theta}_1}(t)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\left\{\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}e^{-\delta_{i,\ell,d}T}\times\prod_{j=1}^{k_1}(1-\lambda_{j,i,\ell,d}t)^{-r_j}e^{\alpha_{i,\ell,d}t}\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\left\{\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}e^{-\delta^*_{i,\ell,d}T}\times\prod_{j=1}^{k_2}(1-\lambda^*_{j,i,\ell,d}t)^{-s_j}e^{\alpha^*_{i,\ell,d}t}\right\},$$

where $\lambda_{1,i,\ell,d},\ldots,\lambda_{k_1,i,\ell,d}$ are the distinct values of $\{\phi_{1,i,\ell,d},..,\phi_{i,i,\ell,d},\psi_{1,i,\ell,d},..,\psi_{d-i,i,\ell,d},\chi_{1,i,\ell,d},..,\chi_{r-d,i,\ell,d}\}$ with frequencies $r_1,..,r_{k_1}$, respectively, such that $r_1+...+r_{k_1}=r$, and $\lambda^*_{1,i,\ell,d},\ldots,\lambda^*_{k_2,i,\ell,d}$ are the distinct values of $\{\phi_{1,i,\ell,d}\cdots\phi_{i,i,\ell,d},\psi_{1,i,\ell,d}\cdots\psi_{d-i,i,\ell,d}\}$ with frequencies $s_1,\ldots,s_{k_2}$, respectively, such that $s_1+\cdots+s_{k_2}=d$, $\alpha_{i,\ell,d}=\gamma_{i,\ell,d}\delta_{i,\ell,d}T$ and $\alpha^*_{i,\ell,d}=\gamma^*_{i,\ell,d}\delta^*_{i,\ell,d}T$.

Then, by using partial fractions method, we can express the mgf of $\hat{\theta}_1$ as

$$M_{\hat{\theta}_1}(t)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\left\{\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}e^{-\delta_{i,\ell,d}T}\times\sum_{j=1}^{k_1}\sum_{q=1}^{r_j}A_{q,j,i,\ell,d}(1-\lambda_{j,i,\ell,d}t)^{-q}e^{\alpha_{i,\ell,d}t}\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\left\{\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}e^{-\delta^*_{i,\ell,d}T}\times\sum_{j=1}^{k_2}\sum_{q=1}^{s_j}B_{q,j,i,\ell,d}(1-\lambda^*_{j,i,\ell,d}t)^{-q}e^{\alpha^*_{i,\ell,d}t}\right\}, \quad (22)$$

where $A_{q,j,i,\ell,d}$'s are the coefficients obtained by writing the product $\prod_{j=1}^{k_1}(1-\lambda_{j,i,\ell,d}t)^{-r_j}$ in the partial fraction form $\sum_{j=1}^{k_1}\sum_{q=1}^{r_j}A_{q,j,i,\ell,d}(1-\lambda_{j,i,\ell,d}t)^{-q}$, and $B_{q,j,i,\ell,d}$'s are the coefficients obtained by writing the product

$\prod_{j=1}^{k_2}(1-\lambda^*_{j,i,\ell,d}t)^{-s_j}$ in the partial fraction form $\sum_{j=1}^{k_2}\sum_{q=1}^{s_j}B_{q,j,i,\ell,d}(1-\lambda^*_{j,i,\ell,d}t)^{-q}$.

Since $(1-ct)^{-d}e^{At}$ is the mgf of the random variable $X+A$, where X has the gamma distribution with shape parameter d and scale parameter $1/c$, we readily obtain from the above expression the tail probability of the MLE $\hat{\theta}_1$ as

$$
\begin{aligned}
P\,(\,\hat{\theta}_1>b)=&\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le\ell_{2,r}|D=d)}\\
&\left\{\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q^*_{1\ell}}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}\,e^{-\delta_{i,\ell,d}T}\right.\\
&\left.\times\sum_{j=1}^{k_1}\sum_{q=1}^{r_j}\frac{A_{q,j,i,\ell,d}}{(q-1)!}\,\Gamma\left(q,\frac{1}{\lambda_{j,i,\ell,d}}\langle b-\alpha_{i,\ell,d}\rangle\right)\right\}\\
&+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le\ell_{2,d}|D=d)}\\
&\left\{\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q^*_{2\ell}}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}\,e^{-\delta^*_{i,\ell,d}T}\right.\\
&\left.\times\sum_{j=1}^{k_2}\sum_{q=1}^{s_j}\frac{B_{q,j,i,\ell,d}}{(q-1)!}\,\Gamma\left(q,\frac{1}{\lambda^*_{j,i,\ell,d}}\langle b-\alpha^*_{i,\ell,d}\rangle\right)\right\},\qquad(23)
\end{aligned}
$$

where $\langle x\rangle=\max\{x,0\}$ and $\Gamma(a,z)=\int_z^{\infty}t^{a-1}e^{-t}dt$ is the incomplete gamma function.

3.1 The exact conditional distribution of $\hat{\theta}_2$

Because of the symmetry between $\hat{\theta}_1$ and $\hat{\theta}_2$, we may write down the following results for $\hat{\theta}_2$ without proof from the preceding results for $\hat{\theta}_1$.

Theorem 8. Conditional on the event A, the mgf of the MLE $\hat{\theta}_2$ is given by

$$
\begin{aligned}
M_{\hat{\theta}_2}(t)=&\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le\ell_{2,r}|D=d)}\\
&\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q^*_{1\ell}}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}\prod_{j=1}^{i}(1-\phi^*_{j,i,\ell,d}t)^{-1}\\
&\times\prod_{j=1}^{d-i}(1-\psi^*_{j,i,\ell,d}t)^{-1}\prod_{j=1}^{r-d}(1-\chi^*_{j,i,\ell,d}t)^{-1}e^{-(1-\xi_{i,\ell,d}t)\delta_{i,\ell,d}T}\\
&+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le\ell_{2,d}|D=d)}\\
&\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q^*_{2\ell}}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}\prod_{j=1}^{i}(1-\Phi_{j,i,\ell,d}t)^{-1}\\
&\times\prod_{j=1}^{d-i}(1-\Psi_{j,i,\ell,d}t)^{-1}e^{-(1-\xi^*_{i,\ell,d}t)\delta^*_{i,\ell,d}T},\qquad(24)
\end{aligned}
$$

where

$$
\begin{aligned}
&\phi^*_{j,i,\ell,d}=\frac{\sum_{k=d-i+1}^{d-i+j}(1-z_k)}{(r-\ell)\sum_{k=d-i+1}^{d-i+j}a_k},\quad\psi^*_{j,i,\ell,d}=\frac{\sum_{k=j}^{d-i}(1-z_k)}{(r-\ell)\sum_{k=j}^{d-i}a_k},\\
&\Phi_{j,i,\ell,d}=\frac{\sum_{k=d-i+1}^{d-i+j}(1-z_k)}{(d-\ell)\sum_{k=d-i+1}^{d-i+j}a_k},\quad\Psi_{j,i,\ell,d}=\frac{\sum_{k=j}^{d-i}(1-z_k)}{(d-\ell)\sum_{k=j}^{d-i}a_k},\\
&\xi^*_{i,\ell,d}=\frac{n-d+\ell+\sum_{k=d-i+1}^{d}(1-z_k)}{(d-\ell)\delta^*_{i,\ell,d}},\\
&\xi_{i,\ell,d}=\frac{n-r+\ell+\sum_{k=d-i+1}^{r}(1-z_k)}{(r-\ell)\delta_{i,\ell,d}},\\
&\chi^*_{j,i,\ell,d}=\frac{n-r+\ell+\sum_{k=1}^{j}(1-z_{r-k+1})}{(r-\ell)(\frac{m-\ell}{\theta_1}+\frac{n-r+\ell}{\theta_2}+\sum_{k=1}^{j}a_{r-k+1})}.
\end{aligned}
$$

Theorem 9. Conditional on the event A, the density of the MLE $\hat{\theta}_2$ is given by

$$
\begin{aligned}
f_{\hat{\theta}_2}(x)=&\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le\ell_{2,r}|D=d)}\\
&\left\{\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q^*_{1\ell}}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}\,e^{-\delta_{i,\ell,d}T}g_{Y_{i,\ell,d}}(x)\right\}\\
&+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le\ell_{2,d}|D=d)}\\
&\left\{\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q^*_{2\ell}}\cdots\sum\sum_{i=0}^{d}\frac{C^*_{\ell,d}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}\,e^{-\delta^*_{i,\ell,d}T}h_{Y^*_{i,\ell,d}}(x)\right\}.\qquad(25)
\end{aligned}
$$

where $Y_{i,\ell,d}\overset{d}{=}Y_{1_{i,\ell,d}}+Y_{2_{i,\ell,d}}+Y_{3_{i,\ell,d}}+Y_{4_{i,\ell,d}}$, with $Y_{1_{i,\ell,d}}=\sum_{j_1=1}^{i}Y_{1j_1}$, $Y_{2_{i,\ell,d}}=\sum_{j_2=1}^{d-i}Y_{2j_2}$ and $Y_{3_{i,\ell,d}}=\sum_{j_3=1}^{r-d}Y_{3j_3}$, with Y_{1j_1} $(j_1=1,\ldots,i)$, Y_{2j_2} $(j_2=1,\ldots,d-i)$ and Y_{3j_3} $(j_3=1,\ldots,r-d)$ being independent random variables having exponential distributions with scale parameters $\phi^*_{j_1,i,\ell,d}$, $\psi^*_{j_2,i,\ell,d}$ and $\chi^*_{j_3,i,\ell,d}$, respectively, and $Y_{4_{i,\ell,d}}$ being a random variable distributed as degenerate localized at a point $\xi_{i,\ell,d}\delta_{i,\ell,d}T$, and $Y^*_{i,\ell,d}\overset{d}{=}Y^*_{1_{i,\ell,d}}+Y^*_{2_{i,\ell,d}}+Y^*_{3_{i,\ell,d}}$, with $Y^*_{1_{i,\ell,d}}=\sum_{j_1=1}^{i}Y^*_{1j_1}$, $Y^*_{2_{i,\ell,d}}=\sum_{j_2=1}^{d-i}Y^*_{2j_2}$, with $Y^*_{1j_1}$ $(j_1=1,\ldots,i)$, and $Y^*_{2j_2}$ $(j_2=1,\ldots,d-i)$ being independent random variables having exponential distributions with scale parameters $\Phi_{j_1,i,\ell,d}$ and $\Psi_{j_2,i,\ell,d}$, respectively, and $Y^*_{3_{i,\ell,d}}$ being a random variable distributed as degenerate localized at a point $\xi^*_{i,\ell,d}\delta^*_{i,\ell,d}T$.

Corollary 10. From (25), we immediately obtain

$$E(\hat{\theta}_2)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}\,e^{-\delta_{i,\ell,d}T}\left\{\sum_{j=1}^{i}\phi_{j,i,\ell,d}^*+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}^*+\sum_{j=1}^{r-d}\chi_{j,i,\ell,d}^*+\xi_{i,\ell,d}\delta_{i,\ell,d}T\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,d}^*c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}\,e^{-\delta_{i,\ell,d}^*T}\left\{\sum_{j=1}^{i}\Phi_{j,i,\ell,d}+\sum_{j=1}^{d-i}\Psi_{j,i,\ell,d}+\xi_{i,\ell,d}^*\delta_{i,\ell,d}^*T\right\} \tag{26}$$

and

$$E(\hat{\theta}_2^2)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}\,e^{-\delta_{i,\ell,d}T}$$
$$\times\left\{\sum_{j=1}^{i}\phi_{j,i,\ell,d}^{*2}+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}^{*2}+\sum_{j=1}^{r-d}\chi_{j,i,\ell,d}^{*2}+\xi_{i,\ell,d}^{2}\delta_{i,\ell,d}^{2}T^2+\sum_{j=1}^{i}\sum_{k=1}^{i}\phi_{j,i,\ell,d}^*\phi_{k,i,\ell,d}^*+\sum_{j=1}^{d-i}\sum_{k=1}^{d-i}\psi_{j,i,\ell,d}^*\psi_{k,i,\ell,d}^*\right.$$
$$+\sum_{j=1}^{r-d}\sum_{k=1}^{r-d}\chi_{j,i,\ell,d}^*\chi_{k,i,\ell,d}^*+2\sum_{j=1}^{i}\sum_{k=1}^{d-i}\phi_{j,i,\ell,d}^*\psi_{k,i,\ell,d}^*+2\sum_{j=1}^{i}\sum_{k=1}^{r-d}\phi_{j,i,\ell,d}^*\chi_{k,i,\ell,d}^*+2\sum_{j=1}^{d-i}\sum_{k=1}^{r-d}\psi_{j,i,\ell,d}^*\chi_{k,i,\ell,d}^*$$
$$\left.+2\xi_{i,\ell,d}\delta_{i,\ell,d}T\Big(\sum_{j=1}^{i}\phi_{j,i,\ell,d}^*+\sum_{j=1}^{d-i}\psi_{j,i,\ell,d}^*+\sum_{j=1}^{r-d}\chi_{j,i,\ell,d}^*\Big)\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,d}^*c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}\,e^{-\delta_{i,\ell,d}^*T}$$
$$\times\left\{\sum_{j=1}^{i}\Phi_{j,i,\ell,d}^2+\sum_{j=1}^{d-i}\Psi_{j,i,\ell,d}^2+\sum_{j=1}^{i}\sum_{k=1}^{i}\Phi_{j,i,\ell,d}\Phi_{k,i,\ell,d}+\sum_{j=1}^{d-i}\sum_{k=1}^{d-i}\Psi_{j,i,\ell,d}\Psi_{k,i,\ell,d}+2\sum_{j=1}^{i}\sum_{k=1}^{d-i}\Phi_{j,i,\ell,d}\Psi_{k,i,\ell,d}\right.$$
$$\left.+\xi_{i,\ell,d}^{*2}\delta_{i,\ell,d}^{*2}T^2+2\xi_{i,\ell,d}^*\delta_{i,\ell,d}^*T\Big(\sum_{j=1}^{i}\Phi_{j,i,\ell,d}+\sum_{j=1}^{d-i}\Psi_{j,i,\ell,d}\Big)\right\}. \tag{27}$$

Then, using these two expressions, $Var(\hat{\theta}_2)$ and $MSE(\hat{\theta}_2)$ may be easily derived.

Corollary 11. The tail probability of the MLE $\hat{\theta}_2$ is calculated as follows:

$$P(\hat{\theta}_2>b)=\sum_{d=0}^{r-1}\frac{1}{P(\ell_{1,r}\le M_r\le \ell_{2,r}|D=d)}\left\{\sum_{\ell=\ell_{1,r}}^{\ell_{2,r}}\sum_{\mathbf{z}_r\in Q_{1\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,r}c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{r-\ell}\omega_{\ell,d}}\,e^{-\delta_{i,\ell,d}T}\times\sum_{j=1}^{k_1}\sum_{q=1}^{r_j}\frac{A_{q,j,i,\ell,d}^*}{(q-1)!}\,\Gamma\Big(q,\frac{1}{\rho_{j,i,\ell,d}}\langle b-\beta_{i,\ell,d}\rangle\Big)\right\}$$
$$+\sum_{d=r}^{N}\frac{1}{P(\ell_{1,d}\le M_d\le \ell_{2,d}|D=d)}\left\{\sum_{\ell=\ell_{1,d}}^{\ell_{2,d}}\sum_{\mathbf{z}_d\in Q_{2\ell}^*}\cdots\sum\sum_{i=0}^{d}\frac{C_{\ell,d}^*c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell}\theta_2^{d-\ell}}\,e^{-\delta_{i,\ell,d}^*T}\times\sum_{j=1}^{k_2}\sum_{q=1}^{s_j}\frac{B_{q,j,i,\ell,d}^*}{(q-1)!}\,\Gamma\Big(q,\frac{1}{\rho_{j,i,\ell,d}^*}\langle b-\beta_{i,\ell,d}^*\rangle\Big)\right\}, \tag{28}$$

where $\rho_{1,i,\ell,d},\ldots,\rho_{k_1,i,\ell,d}$ are the distinct values of $\{\phi_{1,i,\ell,d}^*,\ldots,\phi_{i,i,\ell,d}^*,\psi_{1,i,\ell,d}^*,\ldots,\psi_{d-i,i,\ell,d}^*,\chi_{1,i,\ell,d}^*,\ldots,\chi_{r-d,i,\ell,d}^*\}$ with frequencies $r_1,\ldots,r_{k_1}$, respectively, such that $r_1+\cdots+r_{k_1}=r$, and $\rho_{1,i,\ell,d}^*,\ldots,\rho_{k_2,i,\ell,d}^*$ are the distinct values of $\{\Phi_{1,i,\ell,d}\ldots\Phi_{i,i,\ell,d},\Psi_{1,i,\ell,d}\ldots\Psi_{d-i,i,\ell,d}\}$ with frequencies $s_1,\ldots,s_{k_2}$, respectively, such that $s_1+\cdots+s_{k_2}=d$, $\beta_{i,\ell,d}=\xi_{i,\ell,d}\delta_{i,\ell,d}T$ and $\beta_{i,\ell,d}^*=\xi_{i,\ell,d}^*\delta_{i,\ell,d}^*T$, and $A_{q,j,i,\ell,d}^*$'s are the coefficients obtained by writing the product $\prod_{j=1}^{k_1}(1-\rho_{j,i,\ell,d}t)^{-r_j}$ in the partial fraction form $\sum_{j=1}^{k_1}\sum_{q=1}^{r_j}A_{q,j,i,\ell,d}^*(1-\rho_{j,i,\ell,d}t)^{-q}$, and $B_{q,j,i,\ell,d}^*$'s are the coefficients obtained by writing the product $\prod_{j=1}^{k_2}(1-\rho_{j,i,\ell,d}^*t)^{-s_j}$ in the partial fraction form $\sum_{j=1}^{k_2}\sum_{q=1}^{s_j}B_{q,j,i,\ell,d}^*(1-\rho_{j,i,\ell,d}^*t)^{-q}$.

3.2 The exact conditional joint distribution of $(\hat{\theta}_1,\ \hat{\theta}_2)$

We can obtain the conditional joint *mgf* of $(\hat{\theta}_1,\hat{\theta}_2)$ by using the same steps as we did for conditional marginal distributions.

Theorem 12. Conditional on the event A, the joint mgf of $(\hat{\theta}_1, \hat{\theta}_2)$ is given by

$$M_{\hat{\theta}_1,\hat{\theta}_2}(t_1,t_2) = \sum_{d=0}^{r-1} \frac{1}{P(\ell_{1,r} \le M_r \le \ell_{2,r}|D=d)} \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{\mathbf{z}_r \in Q^*_{1\ell}} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,r} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell} \theta_2^{r-\ell} \omega_{\ell,d}} e^{-(1-\{\gamma_{i,\ell,d}t_1 + \xi_{i,\ell,d}t_2\})\delta_{i,\ell,d}T}$$
$$\times \prod_{j=1}^{i} (1-\{\phi_{j,i,\ell,d}t_1 + \phi^*_{j,i,\ell,d}t_2\})^{-1} \times \prod_{j=1}^{d-i} (1-\{\psi_{j,i,\ell,d}t_1 + \psi^*_{j,i,\ell,d}t_2\})^{-1} \times \prod_{j=1}^{r-d} (1-\{\chi_{j,i,\ell,d}t_1 + \chi^*_{j,i,\ell,d}t_2\})^{-1}$$
$$+ \sum_{d=r}^{N} \frac{1}{P(\ell_{1,d} \le M_d \le \ell_{2,d}|D=d)} \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{\mathbf{z}_d \in Q^*_{2\ell}} \cdots \sum \sum_{i=0}^{d} \frac{C^*_{\ell,d} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell} \theta_2^{d-\ell}} e^{-(1-\{\gamma^*_{i,\ell,d}t_1 + \xi^*_{i,\ell,d}t_2\})\delta^*_{i,\ell,d}T}$$
$$\times \prod_{j=1}^{i} (1-\{\phi_{j,i,\ell,d}t_1 + \Phi_{j,i,\ell,d}t_2\})^{-1} \times \prod_{j=1}^{d-i} (1-\{\psi_{j,i,\ell,d}t_1 + \Psi_{j,i,\ell,d}t_2\})^{-1}. \quad (29)$$

Corollary 13. From (29), we can readily obtain

$$E(\hat{\theta}_1\hat{\theta}_2) = \sum_{d=0}^{r-1} \frac{1}{P(\ell_{1,r} \le M_r \le \ell_{2,r}|D=d)} \sum_{\ell=\ell_{1,r}}^{\ell_{2,r}} \sum_{\mathbf{z}_r \in Q^*_{1\ell}} \cdots \sum \sum_{i=0}^{d} \frac{C_{\ell,r} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell} \theta_2^{r-\ell} \omega_{\ell,d}} e^{-\delta_{i,\ell,d}T}$$
$$\times \Bigg\{ \sum_{j=1}^{i} \phi_{j,i,\ell,d}\phi^*_{j,i,\ell,d} + \sum_{j=1}^{d-i} \psi_{j,i,\ell,d}\psi^*_{j,i,\ell,d} + \sum_{j=1}^{r-d} \chi_{j,i,\ell,d}\chi^*_{j,i,\ell,d} + \sum_{j=1}^{i} \phi_{j,i,\ell,d} \Bigg(\sum_{k=1}^{i} \phi^*_{k,i,\ell,d} + \sum_{k=1}^{d-i} \psi^*_{k,i,\ell,d} + \sum_{k=1}^{r-d} \chi^*_{k,i,\ell,d} + \xi_{i,\ell,d}\delta_{i,\ell,d}T \Bigg)$$
$$+ \sum_{j=1}^{d-i} \psi_{j,i,\ell,d} (\sum_{k=1}^{i} \phi^*_{k,i,\ell,d} + \sum_{k=1}^{d-i} \psi^*_{k,i,\ell,d} + \sum_{k=1}^{r-d} \chi^*_{k,i,\ell,d} + \xi_{i,\ell,d}\delta_{i,\ell,d}T) + \sum_{j=1}^{r-d} \chi_{j,i,\ell,d} (\sum_{k=1}^{i} \phi^*_{k,i,\ell,d} + \sum_{k=1}^{d-i} \psi^*_{k,i,\ell,d}$$
$$+ \sum_{k=1}^{r-d} \chi^*_{k,i,\ell,d} + \xi_{i,\ell,d}\delta_{i,\ell,d}T) + \gamma_{i,\ell,d}\delta_{i,\ell,d}T (\sum_{k=1}^{i} \phi^*_{k,i,\ell,d} + \sum_{k=1}^{d-i} \psi^*_{k,i,\ell,d} + \sum_{k=1}^{r-d} \chi^*_{k,i,\ell,d} + \xi_{i,\ell,d}\delta_{i,\ell,d}T) \Bigg\}$$
$$+ \sum_{d=r}^{N} \frac{1}{P(\ell_{1,d} \le M_d \le \ell_{2,d}|D=d)} \sum_{\ell=\ell_{1,d}}^{\ell_{2,d}} \sum_{\mathbf{z}_d \in Q^*_{2\ell}} \cdots \sum \sum_{i=0}^{d} \frac{C^*_{\ell,d} c_{i,d}(\mathbf{a}_d)}{\theta_1^{\ell} \theta_2^{d-\ell}} e^{-\delta^*_{i,\ell,d}T}$$
$$\times \Bigg\{ \sum_{j=1}^{i} \phi_{j,i,\ell,d}\Phi_{j,i,\ell,d} + \sum_{j=1}^{d-i} \psi_{j,i,\ell,d}\Psi_{j,i,\ell,d} + \sum_{j=1}^{i} \phi_{j,i,\ell,d} (\sum_{k=1}^{i} \Phi_{k,i,\ell,d} + \sum_{k=1}^{d-i} \Psi_{k,i,\ell,d} + \xi^*_{i,\ell,d}\delta^*_{i,\ell,d}T)$$
$$+ \sum_{j=1}^{d-i} \psi_{j,i,\ell,d} (\sum_{k=1}^{i} \Phi_{k,i,\ell,d} + \sum_{k=1}^{d-i} \Psi_{k,i,\ell,d} + \xi^*_{i,\ell,d}\delta^*_{i,\ell,d}T) + \gamma^*_{i,\ell,d}\delta^*_{i,\ell,d}T (\sum_{k=1}^{i} \Phi_{k,i,\ell,d} + \sum_{k=1}^{d-i} \Psi_{k,i,\ell,d} + \xi^*_{i,\ell,d}\delta^*_{i,\ell,d}T) \Bigg\}, \quad (30)$$

from which the covariance and correlation coefficient between the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ can also be readily obtained.

3.3 Confidence intervals

We discuss various approaches for forming CIs for the unknown parameters θ_1 and θ_2 in this subsection. We derive the exact CIs for θ_1 and θ_2 using (23) and (28), respectively. We also provide the approximate CIs for θ_1 and θ_2 for larger sample sizes. Finally, we construct credible CIs for θ_1 and θ_2 using the Bayesian technique.

3.3.1 Exact confidence intervals

To guarantee the invertibility for the parameters θ_1 and θ_2, we assume that the tail probabilities of $\hat{\theta}_1$ and $\hat{\theta}_2$ presented in (23) and (28) are increasing functions of θ_1 and θ_2, respectively. This approach has been utilised in other works, including [8] and [15], to create the exact CI in different contexts. We then have a $100(1-\alpha)\%$ confidence interval for θ_1 is $(\theta_{1L}, \theta_{1U})$, where θ_{1L} and θ_{1U} are such that $P_{\theta_{1L}}(\hat{\theta}_1 > \hat{\theta}_{1obs}) = \frac{\alpha}{2}$ and $P_{\theta_{1U}}(\hat{\theta}_1 > \hat{\theta}_{1obs}) = 1 - \frac{\alpha}{2}$ with $\hat{\theta}_{1obs}$ being the observed values of $\hat{\theta}_1$. Also, we have a $100(1-\alpha)\%$ confidence interval for θ_2 is $(\theta_{2L}, \theta_{2U})$, where θ_{2L} and θ_{2U} are such that $P_{\theta_{2L}}(\hat{\theta}_2 > \hat{\theta}_{2obs}) = \frac{\alpha}{2}$ and $P_{\theta_{2U}}(\hat{\theta}_1 > \hat{\theta}_{2obs}) = 1 - \frac{\alpha}{2}$ with $\hat{\theta}_{2obs}$ being the observed values of $\hat{\theta}_2$.

3.3.2 Approximate confidence intervals

For large m and n, the Fisher information matrix of θ_1 and θ_2 is

$$I(\theta_1,\theta_2) = \begin{bmatrix} I_{11}(\theta_1,\theta_2) & \hat{I}_{12}(\theta_1,\theta_2) \\ I_{21}(\theta_1,\theta_2) & \hat{I}_{22}(\theta_1,\theta_2) \end{bmatrix}, \quad (31)$$

where

$$I_{ij}(\theta_1,\theta_2) = -E\left\{\frac{\partial^2 lnL(\theta_1,\theta_2,\mathbf{z},\mathbf{w})}{\partial\theta_i\partial\theta_j}\right\},\ i,j=1,2.$$

From the likelihood function $L(\theta_1,\theta_2,\mathbf{z},\mathbf{w})$ in (3), it is simply to get $I_{12}(\theta_1,\theta_2) = I_{21}(\theta_1,\theta_2) = 0$, and the observed Fisher information matrix of θ_1 and θ_2 is then given by

$$I(\theta_1,\theta_2) = \begin{bmatrix} \frac{-\partial^2 lnL(\theta_1,\theta_2,\mathbf{z},\mathbf{w})}{\partial\theta_1^2} & 0 \\ 0 & \frac{-\partial^2 lnL(\theta_1,\theta_2,\mathbf{z},\mathbf{w})}{\partial\theta_2^2}) \end{bmatrix}_{\theta_1=\hat{\theta}_1,\theta_2=\hat{\theta}_2} \quad (32)$$

where

$$\frac{\partial^2 lnL(\theta_1,\theta_2,\mathbf{z},\mathbf{w})}{\partial\theta_1^2} = \frac{m_{r^*}}{\theta_1^2} - \frac{2u_1}{\theta_1^3}$$

and

$$\frac{\partial^2 lnL(\theta_1,\theta_2,\mathbf{z},\mathbf{w})}{\partial\theta_2^2} = \frac{n_{r^*}}{\theta_2^2} - \frac{2u_2}{\theta_2^3}.$$

Then, by using the asymptotic normality of the MLEs, we can express the two-sided $100(1-\alpha)\%$ approximate CI for θ_1 and θ_2 as

$$\hat{\theta}_1 \pm Z_{\alpha/2}\frac{\sum_{i=1}^{r^*} z_i w_i + (m-m_{r^*})T^*}{\sqrt{\{\sum_{i=1}^{r^*} z_i\}^3}}$$

and

$$\hat{\theta}_2 \pm Z_{\alpha/2}\frac{\sum_{i=1}^{r^*} (1-z_i) w_i + (n-n_{r^*})T^*}{\sqrt{\{\sum_{i=1}^{r^*} (1-z_i)\}^3}},$$

where $Z_{\alpha/2}$ is the upper $\alpha/2$ percentile of the standard normal distribution.

3.3.3 Bayes credible confidence intervals

From a Bayesian perspective, the prior distributions of θ_1 and θ_2 can be viewed as independent inverse gamma prior distributions, namely $IG(a_1,b_1)$ and $IG(a_2,b_2)$, respectively. The joint prior function of θ_1 and θ_2 is then

$$\pi(\theta_1,\theta_2) \propto \frac{1}{\theta_1^{a_1+1}\theta_2^{a_2+1}}\, e^{-(b_1/\theta_1+b_2/\theta_2)}. \quad (33)$$

From the likelihood function in (3) and the joint prior function in (33), we have posterior joint density function as

$$\pi(\theta_1,\theta_2|\mathbf{x}) = \frac{(u_1+b_1)^{m_{r^*}+a_1}(u_2+b_2)^{n_{r^*}+a_2}}{\Gamma(m_{r^*}+a_1)\Gamma(n_{r^*}+a_2)} \times \frac{1}{\theta_1^{m_{r^*}+a_1+1}}\, e^{-(u_1+b_1)/\theta_1}\frac{1}{\theta_2^{n_{r^*}+a_2+1}}\, e^{-(u_2+b_2)/\theta_2}. \quad (34)$$

We can see from (34) that the joint posterior density function of $(\theta_1,\ \theta_2)$ is a product of two independent density functions, and so the marginal posterior density functions of θ_1 and θ_2, given the data, are $IG(m_{r^*}+a_1,u_1+b_1)$ and $IG(n_{r^*}+a_2,u_2+b_2)$, respectively. As a result, the Bayes estimators for θ_1 and θ_2 under the squared error loss function are

$$\hat{\theta}_1 = \frac{u_1+b_1}{m_{r^*}+a_1-1} \quad \text{and} \quad \hat{\theta}_2 = \frac{u_2+b_2}{n_{r^*}+a_2-1}. \quad (35)$$

Let $V_1 = \frac{2(u_1+b_1)}{\theta_1}$ and $V_2 = \frac{2(u_2+b_2)}{\theta_2}$. Evidently, the pivots V_1 and V_2 follow $\chi^2_{2(m_{r^*}+a_1)}$ and $\chi^2_{2(n_{r^*}+a_2)}$ distributions, respectively, provided $2(m_{r^*}+a_1)$ and $2(n_{r^*}+a_2)$ are positive integers. In this case, the $100(1-\alpha)\%$ Bayes credible intervals for θ_1 and θ_2 are

$$\left(\frac{2(u_1+b_1)}{\chi^2_{2(m_{r^*}+a_1),1-\alpha/2}}, \frac{2(u_1+b_1)}{\chi^2_{2(m_{r^*}+a_1),\alpha/2}}\right)$$
and
$$\left(\frac{2(u_2+b_2)}{\chi^2_{2(n_{r^*}+a_2),1-\alpha/2}}, \frac{2(u_2+b_2)}{\chi^2_{2(n_{r^*}+a_2),\alpha/2}}\right). \quad (36)$$

4 Results and discussion

In this section, we analyse the performance of the two estimation methods as well as the three confidence intervals using Monte Carlo simulation. There are also some numerical results that are based on real data.

4.1 Monte Carlo simulation

To evaluate the performance of the conditional ML and Bayesian estimates, as well as the three confidence intervals stated in the prior sections, a simulation study was done. We evaluated using five different sample sizes $(m,\ n)$ and several choices for r and T. We also chose (2, 5) and (1, 3) as the exponential scale parameters $(\theta_1,\ \theta_2)$. We then calculated conditional ML and Bayesian estimates of θ_1 and θ_2 for each of these cases. In addition, for θ_1 and θ_2, we calculated the 95% exact, approximate and Bayes credible confidence intervals.

We calculated the means $\hat{\theta}_1$ and $\hat{\theta}_2$ of the conditional ML and Bayesian estimates, as well as their mean square errors (MSE) and the average widths (AW) of 95% confidence intervals and the associated coverage probabilities (CP), by repeating the process 1000 times. Table 1 shows the means and mean square errors of conditional ML and Bayesian estimates for $\theta_1 = 2$ and $\theta_2 = 5$. When $\theta_1 = 1$ and $\theta_2 = 3$, the means and mean square errors of the conditional ML and Bayesian estimates are provided in Table 2. Table 3 shows the average widths and coverage probability of 95% confidence intervals for $\theta_1 = 2$ and $\theta_2 = 5$. When $\theta_1 = 1$

Table 1: The average values and the mean square errors of the conditional ML and Bayesian estimates when $\theta_1 = 2$ and $\theta_2 = 5$, for different choices of m, n, r and T.

			θ_1				θ_2			
			ML		*Bayesian*		*ML*		*Bayesian*	
(m,n)	r	T	$\hat{\theta}_1$	MSE_1	$\hat{\theta}_1$	MSE_1	$\hat{\theta}_2$	MSE_2	$\hat{\theta}_2$	MSE_2
(6,6)	6	4	2.235	2.017	2.137	0.839	6.572	26.256	5.613	7.291
		6	2.117	1.081	2.098	0.715	6.018	19.336	5.497	6.819
		8	2.077	0.937	2.072	0.628	5.651	11.042	5.377	5.163
		10	2.054	0.832	2.054	0.572	5.414	7.215	5.292	4.742
		20	2.042	0.795	2.044	0.539	5.156	4.840	5.079	3.154
(10,8)	9	4	2.125	0.977	2.094	0.829	6.073	19.185	5.621	7.310
		6	2.078	0.547	2.052	0.417	5.594	8.998	5.430	5.668
		8	2.058	0.494	2.038	0.383	5.404	7.133	5.294	4.575
		10	2.052	0.477	2.035	0.378	5.316	5.971	5.260	4.348
		20	2.042	0.444	2.026	0.356	5.132	3.632	5.058	2.936
(12,12)	12	4	2.075	0.520	2.072	0.393	5.727	8.284	5.491	4.733
		6	2.051	0.426	2.048	0.345	5.423	4.971	5.296	3.399
		8	2.033	0.366	2.030	0.306	5.267	3.487	5.188	2.675
		10	2.030	0.357	2.027	0.301	5.225	3.115	5.157	2.491
		20	2.028	0.345	2.023	0.295	5.094	2.398	5.044	1.944
(15,12)	15	4	2.089	0.371	2.085	0.326	5.645	10.614	5.448	5.569
		6	2.064	0.320	2.059	0.286	5.316	4.546	5.224	3.670
		8	2.048	0.296	2.044	0.266	5.189	3.337	5.118	2.777
		10	2.041	0.285	2.038	0.259	5.149	3.123	5.096	2.643
		20	2.035	0.279	2.033	0.251	5.008	2.268	5.004	2.034
(15,15)	18	4	2.057	0.362	2.037	0.300	5.602	6.494	5.502	4.429
		6	2.035	0.316	2.018	0.269	5.363	3.856	5.329	3.062
		8	2.020	0.286	2.006	0.249	5.260	2.874	5.254	2.429
		10	2.014	0.275	2.002	0.238	5.225	2.513	5.223	2.210
		20	2.011	0.274	2.001	0.235	5.190	1.842	5.113	1.662

Table 2: The average values and the mean square errors of the conditional ML and Bayesian estimates when $\theta_1 = 1$ and $\theta_2 = 3$, for different choices of m, n, r and T.

			θ_1				θ_2			
			ML		*Bayesian*		*ML*		*Bayesian*	
(m,n)	r	T	$\hat{\theta}_1$	MSE_1	$\hat{\theta}_1$	MSE_1	$\hat{\theta}_2$	MSE_2	$\hat{\theta}_2$	MSE_2
(6,6)	6	2	1.119	0.507	1.069	0.210	4.037	9.232	3.380	2.605
		3	1.059	0.271	1.050	0.179	3.715	7.373	3.323	2.493
		4	1.037	0.234	1.036	0.157	3.503	5.576	3.265	2.262
		5	1.028	0.208	1.027	0.143	3.376	3.958	3.223	1.878
		10	1.021	0.199	1.021	0.135	3.115	1.823	3.064	1.216
(10,8)	9	2	1.063	0.170	1.047	0.133	3.813	8.333	3.419	2.890
		3	1.039	0.137	1.026	0.104	3.505	5.569	3.323	2.526
		4	1.029	0.123	1.019	0.096	3.311	3.044	3.228	1.907
		5	1.026	0.119	1.017	0.095	3.232	2.536	3.177	1.689
		10	1.021	0.111	1.013	0.089	3.142	1.392	3.054	1.120
(12,12)	12	2	1.038	0.130	1.035	0.098	3.543	3.521	3.364	2.101
		3	1.021	0.107	1.024	0.086	3.324	2.295	3.219	1.377
		4	1.009	0.091	1.014	0.076	3.215	1.515	3.156	1.156
		5	1.007	0.089	1.013	0.075	3.159	1.263	3.113	0.959
		10	1.004	0.086	1.011	0.074	3.065	0.887	3.034	0.719
(15,12)	15	2	1.042	0.093	1.042	0.081	3.499	4.662	3.329	2.456
		3	1.029	0.080	1.029	0.071	3.265	2.165	3.194	1.565
		4	1.022	0.074	1.022	0.066	3.159	1.490	3.111	1.192
		5	1.019	0.071	1.019	0.065	3.110	1.198	3.068	0.998
		10	1.017	0.070	1.016	0.063	3.021	0.877	3.008	0.769
(15,15)	18	2	1.028	0.090	1.018	0.070	3.471	3.154	3.363	1.945
		3	1.018	0.079	1.009	0.067	3.279	1.925	3.245	1.363
		4	1.010	0.072	1.003	0.062	3.199	1.231	3.179	0.992
		5	1.007	0.069	1.002	0.059	3.154	1.013	3.152	0.859
		10	1.005	0.068	1.001	0.058	3.069	0.700	3.083	0.645

and $\theta_2 = 3$, the average widths and coverage probabilities of 95% confidence intervals are reported in Table 4.

We can see from the results in Tables 1 and 2 that the estimate of θ_1 is highly consistent even for smaller T, whereas the estimate of θ_2 only becomes stable for greater T. This is to be predicted because when θ_1 is smaller than θ_2, when T is small, the exponential population with parameter θ_1 would have caused the majority of the failures seen, whereas the exponential population with parameter θ_2 would have caused very few failures. As one would predict, when T is increased, the biases and mean square errors of the Bayesian estimates are also fewer than those of the ML estimates for all various choices of m, n, r, and T. Furthermore, even for small m and n, all estimates' biases and mean square errors diminish as T increases.

Tables 3 and 4 show that the exact conditional method always has a roughly 95% coverage probability, whereas the approximate method is not at all adequate (as low as 88% in some cases). We also notice that the Bayesian technique has relatively consistent coverage probabilities (near to the nominal level of 95%); nevertheless, when m and n are both small, all of these methods have reduced coverage probability. As a result, even for small m and n, the average widths of all confidence intervals diminish as T is increased.

Table 3: The average widths and the coverage probabilities of 95% confidence intervals when $\theta_1 = 2$ and $\theta_2 = 5$, for different choices of m, n and T.

			θ_1						θ_2					
			Exact		*Approx.*		*Bayesian*		*Exact*		*Approx.*		*Bayesian*	
(m,n)	r	T	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*
(6,6)	6	4	94.8	4.53	87.7	4.17	95.3	3.67	93.9	20.67	88.7	17.75	91.7	10.77
		6	94.8	4.12	87.6	3.57	94.7	3.39	94.3	15.81	89.9	13.39	91.9	9.54
		8	95.1	3.85	87.6	3.39	94.9	3.26	94.5	13.34	88.2	11.00	91.5	8.68
		10	94.9	3.73	87.6	3.30	95.0	3.19	94.1	11.92	88.1	9.75	91.0	8.17
		20	94.8	3.51	87.6	3.25	95.2	3.15	94.4	9.82	88.0	8.38	91.3	7.20
(10,8)	9	4	94.2	3.65	90.5	2.93	93.0	2.80	95.3	16.90	89.2	13.61	93.1	9.83
		6	94.7	3.25	90.8	2.68	93.6	2.58	94.7	12.51	88.3	10.17	92.4	8.41
		8	94.6	3.15	90.7	2.59	93.6	2.51	94.5	10.63	87.9	8.93	91.5	7.60
		10	94.5	3.01	90.7	2.56	93.8	2.49	94.4	9.82	87.1	8.30	91.3	7.25
		20	94.8	2.93	90.7	2.53	93.8	2.46	94.5	8.54	87.3	7.08	91.5	6.39
(12,12)	12	4	95.3	3.43	90.6	2.59	94.8	2.53	95.2	11.93	91.5	9.71	93.3	8.02
		6	94.6	3.09	90.3	2.40	94.3	2.36	94.6	9.69	92.0	7.78	93.0	6.79
		8	94.7	3.00	90.3	2.31	94.6	2.29	94.5	8.75	92.3	6.90	92.3	6.18
		10	95.2	2.92	90.2	2.29	94.9	2.27	94.3	7.87	91.7	6.52	92.1	5.90
		20	95.2	2.85	90.3	2.27	94.8	2.25	94.1	7.26	91.5	5.85	92.0	5.36
(15,12)	15	4	94.7	3.24	93.5	2.31	95.9	2.28	94.0	10.84	90.7	9.67	92.2	7.99
		6	94.8	2.97	93.1	2.16	95.9	2.13	94.3	9.06	91.4	7.58	92.3	6.69
		8	95.3	2.92	93.1	2.10	95.5	2.07	93.8	8.16	91.2	6.78	90.9	6.08
		10	94.8	2.88	93.1	2.07	95.5	2.05	94.1	7.40	90.2	6.42	91.1	5.82
		20	94.9	2.81	93.1	2.06	95.1	2.03	93.9	7.00	90.6	5.74	91.0	5.30
(15,15)	18	4	95.2	3.01	93.1	2.28	95.9	2.22	95.2	9.81	92.9	8.34	94.3	7.32
		6	94.8	2.91	92.7	2.13	94.8	2.08	94.6	8.06	92.8	6.81	93.8	6.21
		8	94.6	2.89	92.6	2.07	94.8	2.03	95.3	7.53	92.9	6.14	94.1	5.69
		10	94.7	2.82	92.6	2.05	95.4	2.10	94.7	7.01	93.7	5.80	93.8	5.42
		20	94.4	2.75	92.5	2.04	95.2	2.00	94.5	6.55	93.4	5.22	93.5	4.92

4.2 Numerical example

We will use the data in [1] (Table 4.1, p. 462) to illustrate all of the inferential results established for the exponential distribution. The original data was 60 times to breakdown of an insulating fluid subjected to high-voltage stress. The data set is divided into 6 groups, each containing 10 insulating fluids. The two groups 1 and 4 are considered

Table 4: The average widths and the coverage probabilities of 95% confidence intervals when $\theta_1 = 1$ and $\theta_2 = 3$, for different choices of m, n and T.

			θ_1						θ_2					
			Exact		*Approx.*		*Bayesian*		*Exact*		*Approx.*		*Bayesian*	
(m,n)	r	T	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*	*CP*	*AW*
(6,6)	6	2	94.8	2.34	87.8	2.09	95.2	1.84	93.5	12.84	90.7	11.78	91.5	6.76
		3	95.3	2.22	87.5	1.79	94.7	1.69	93.8	9.81	89.0	8.92	92.6	6.02
		4	94.8	2.17	87.6	1.70	94.9	1.63	93.7	8.87	89.1	7.39	91.8	5.51
		5	95.1	2.10	87.6	1.65	95.0	1.59	93.4	7.12	88.2	6.51	91.1	5.16
		10	94.9	1.98	87.6	1.63	95.2	1.57	93.5	6.74	88.0	5.15	91.2	4.40
(10,8)	9	2	94.4	1.50	90.3	1.46	93.0	1.40	95.2	10.21	89.9	9.41	94.4	6.33
		3	94.7	1.41	90.4	1.34	93.6	1.29	94.3	8.90	90.0	7.04	92.9	5.47
		4	94.7	1.34	90.4	1.29	93.6	1.26	93.7	6.87	87.9	5.78	91.9	4.86
		5	94.8	1.31	90.4	1.28	93.8	1.24	93.5	6.51	87.7	5.27	91.1	4.53
		10	94.8	1.29	90.4	1.26	93.8	1.23	93.3	5.22	87.3	4.33	91.2	3.91
(12,12)	12	2	95.2	1.33	90.6	1.29	94.8	1.26	94.6	7.05	91.2	6.53	93.4	5.26
		3	94.8	1.28	90.2	1.20	94.3	1.18	94.7	6.14	91.8	5.10	93.9	4.35
		4	94.1	1.27	90.2	1.15	94.6	1.14	93.8	5.26	91.4	4.43	92.7	3.94
		5	95.0	1.20	90.2	1.14	94.9	1.13	94.5	5.01	91.8	4.09	92.2	3.68
		10	94.9	1.19	90.2	1.13	94.8	1.13	95.5	4.35	91.5	3.55	92.3	3.25
(15,12)	15	2	94.8	1.18	93.5	1.16	95.9	1.14	95.3	6.49	91.3	6.06	93.5	5.23
		3	95.2	1.11	93.1	1.08	95.9	1.06	94.5	5.45	91.1	5.00	92.4	4.33
		4	94.6	1.09	93.1	1.05	95.5	1.03	94.2	5.07	90.7	4.35	91.6	3.87
		5	94.6	1.07	93.1	1.04	95.5	1.02	94.1	4.81	90.9	4.03	91.2	3.62
		10	94.5	1.05	93.1	1.03	95.3	1.02	94.5	4.05	90.7	3.51	91.3	3.23
(15,15)	18	2	94.8	1.16	93.1	1.14	95.2	1.11	95.3	6.01	92.4	5.66	94.5	4.80
		3	95.3	1.09	92.7	1.07	94.8	1.04	94.5	5.10	93.1	4.46	93.9	4.00
		4	95.2	1.07	92.6	1.04	94.8	1.01	94.4	4.72	93.3	3.93	93.6	3.60
		5	94.9	1.04	92.6	1.02	95.4	1.00	94.8	4.12	92.9	3.65	94.1	3.38
		10	94.9	1.03	92.5	1.02	95.2	1.00	94.7	3.86	93.4	3.18	93.3	3.00

here, and the associated failure times data are shown in Table 5.

Table 5: Groups 1 and 4 of the times to breakdown of insulating fluids from Nelson (1982).

Group 1	1.89	4.03	1.54	0.31	0.66	1.70	2.17	1.82	9.99	2.24
Group 4	1.17	3.87	2.80	0.70	3.82	0.02	0.50	3.72	0.06	3.57

We assume these data come from two exponential populations, each having a mean of 2.6 and 2. Assume that, on groups 1 and 4, joint Type-II hybrid censoring with $r = 5$ and T as 1, 2, 3, 4, and 7 occurred. The conditional ML estimates of θ_1 and θ_2, as well as the estimates of their standard deviations and mean square errors, were then computed for all T choices. In addition, we computed Bayesian estimates of θ_1 and θ_2 using an informative prior with $(a_1, b_1, a_2, b_2) = (2, 2, 2, 3)$, and the results are shown in Table 6. For all choices of T, the 95% exact, approximate, and Bayes credible confidence intervals for θ_1 and θ_2 are calculated and reported in Table 7.

Table 6 shows that the biases and mean square errors of the Bayesian estimates are fewer than those of the ML estimates for all different choices of T. We also notice that when T increases, the biases and mean square errors of all estimations reduce.

We can see from Tables 7 that the approximate confidence intervals are not as efficient as the exact conditional confidence intervals obtained from Section 2 results. We also see that Bayesian approaches produce findings that are quite close to exact confidence intervals. As a result, we see that the widths of all confidence intervals shrink as T increases.

Table 6: The Bayesian and ML estimates of θ_1 and θ_2 and the corresponding standard deviations, mean square errors, and correlation coefficient based on groups 1 and 4.

T	$\hat{\theta}_{1ML}$	$\hat{\theta}_{1B}$	$SD_{\hat{\theta}_1}$	$MSE_{\hat{\theta}_1}$	$\hat{\theta}_{2ML}$	$\hat{\theta}_{2B}$	$SD_{\hat{\theta}_2}$	$MSE_{\hat{\theta}_2}$	$\rho(\hat{\theta}_1, \hat{\theta}_2)$
1	4.49	3.66	1.74	3.71	1.82	1.86	4.53	22.14	-0.10
2	2.65	2.56	1.53	3.35	2.49	2.41	4.26	20.38	-0.05
3	2.29	2.26	1.43	2.89	2.88	2.75	3.82	19.16	0.06
4	2.54	2.48	1.36	2.18	2.02	2.03	3.20	18.55	0.08
7	2.60	2.54	1.21	1.91	2.03	2.02	2.99	17.84	0.11

Table 7: The 95% exact, approximate and Bayes credible confidence intervals for θ_1 and θ_2 for different choices of T based on groups 1 and 4.

	θ_1			θ_2		
T	*Exact*	*Approx.*	*Bayesian*	*Exact*	*Approx.*	*Bayesian*
1	(0.00, 8.43)	(0.000, 10.67)	(1.25, 10.07)	(0.52, 2.94)	(0.05, 3.60)	(0.80, 3.76)
2	(0.65, 3.01)	(0.541, 4.76)	(1.24, 5.19)	(0.62, 3.97)	(0.32, 4.66)	(1.11, 4.78)
3	(0.87, 2.93)	(0.711, 3.87)	(1.19, 4.24)	(0.81, 4.70)	(0.59, 5.16)	(1.34, 5.28)
4	(0.95, 3.54)	(0.789, 4.29)	(1.31, 4.66)	(1.02, 2.82)	(0.78, 3.27)	(1.12, 3.43)
7	(1.00, 3.73)	(0.908, 4.28)	(1.38, 4.62)	(1.12, 2.81)	(0.78, 3.27)	(1.13, 3.42)

5 Conclusions

The issue of deriving the exact distributions of maximum likelihood estimators when Type-II hybrid censoring is used on two samples from two exponential populations in a combined manner was discussed in this paper. The conditional maximum likelihood and Bayesian estimators of the two unknown exponential mean parameters were first calculated. The conditional moment generating functions and conditional exact distributions of the maximum likelihood estimators were then calculated. We also calculated the exact, approximation, and Bayes credible confidence intervals for the two unknown parameters. Finally, using real data, we provided a Monte Carlo simulation study as well as some numerical results.

Availability of data and materials

All data generated or analysed during this study are included in this published article.

Competing Interests

The author declares that they have no competing interests.

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References

[1] W. Nelson, *Applied Life Data Analysis*, John Wiley & Sons, New York, NY (1982).
[2] N. Balakrishnan, A.C. Cohen, *Order Statistics and Inference: Estimation Methods*, Academic Press, Boston (1991).
[3] A.C. Cohen, *Truncated and Censored Samples: Theory and Applications*, Marcel Dekker, New York (1991).
[4] N. Balakrishnan, R. Aggarwala, *Progressive Censoring: Theory, Methods and Applications*, Birkhäuser, Boston (2000).
[5] A.R. Shafay, N. Balakrishnan, Y. Abdel-Aty, Bayesian inference based on a jointly type-II censored sample from two exponential populations, *Journal of Statistical Computation and Simulation*, **84**, 2427-2440 (2014).
[6] B. Epstein, Truncated life tests in the exponential case, *Annals of Mathematical Statistics*, **25**, 555-564 (1954).
[7] MIL-STD-781 C, *Reliability Design Qualification and Production Acceptance Test, Exponential Distribution*, U.S. Government Printing Office, Washington, D.C. (1977).
[8] A. Childs, B. Chandrasekar, N. Balakrishnan, D. Kundu, Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution, *Annals of the Institute of Statistical Mathematics*, **55**, 319-330 (2003).
[9] B. Chandrasekar, A. Childs, N. Balakrishnan, Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring, *Naval Research Logistics*, **51**, 994-1004 (2004).
[10] S. Park, N. Balakrishnan, G. Zheng, Fisher information in hybrid censored data, *Statistical Probability Letters*, **78**, 2781-2786 (2008).
[11] S. Park, N. Balakrishnan, On simple calculation of the Fisher information in hybrid censoring schemes, *Statistical Probability Letters*, **79**, 1311-1319 (2009).
[12] N. Balakrishnan, D. Kundu, Hybrid censoring; Models, inferential results and applications, *Computational Statistics & Data Analysis*, **57**, 166-209 (2013).
[13] N. Balakrishnana, A. Rasouli, Exact likelihood inference for two exponential populations under joint Type-II censoring, *Computational Statistics & Data Analysis*, **52**, 2725-2738 (2008).
[14] A. Rasouli, N. Balakrishnan, Exact likelihood inference for two exponential populations under joint progressive Type-II censoring, *Communications in Statistics - Theory and Methods*, **39**, 2172-2191 (2010).
[15] N. Balakrishnan, A. Childs, B. Chandrasekar, An efficient computational method for moments of order statistics under progressive censoring, *Statistics & Probability Letters*, **60**, 359-365 (2002).

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