Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160201

On Some Characterizations to NBRULC Class with Hypotheses Testing Application

Walid B. H. Etman, Rashad M. EL-Sagheer, Shaban E. Abu-Youssef and Amr Sadek*

Mathematics Department, Faculty of Science, AL-Azhar University, Nasr City 11884, Cairo, Egypt

Received: 2 Sep. 2021, Revised: 22 Nov. 2021, Accepted: 23 Dec. 2021
Published online: 1 Mar. 2022

Abstract: In this paper, we investigate the probabilistic characteristics for new better than renewal used in Laplace transform in increasing convex order class, the closure properties under various reliability operations such as convolution, mixture, homogeneous Poisson shock model are studied. A new hypothesis test is constructed to test exponentiality against, based on U-statistic. Pitman's asymptptic efficiency of the test are studied and compared with other tests. The percentiles of this test statistic are tabulated and the power of the new test statistics are calculated in case of censored and non-censored data. Finally, the right-censored data problem is given in some medical and engineering science applications to evaluate the performances of our test.

Keywords: Convolution; Mixture; Poisson shock model; U-statistic and increasing convex order.

1 Introduction

The increasing convex ordering of random variables, which is defined through the comparison of the expectation of increasing and convex functions, has found wide applications in the fields of reliability, economics, queuing, epidemics, scheduling, and so on.

Many applications of the increasing convex order have been found in queueing theory, reliability, operations research, economics and so on. For example, Stoyan [1] used this ordering to find the optimal sample size in experimental design. Ross [2] presented some applications of this ordering in comparison of queues and stochastic processes.

Characterization and new definitions of aging classes can be produced by using some ordering of distribution. Aging means phenomenon whereby an older system has a shorter remaining lifetime in some statical sense than a younger one.

In reliability theory and its applications, the lifetime distribution classes (e.g. Increasing failure rate (IFR), Increasing failure rate average $(IFRA)$, new better than used (NBU) have been extensively studied in order to obtain useful bounds for the reliability of components or units. For the same reason, it is important to study the preservation properties of life distribution classes under a variety of reliability operations. These operations are formed by the corresponding systems consisting of similar or dissimilar components. Two fundamental reliability operations are formed by the serial and parallel systems, i.e. the minimum and maximum of the component lifetimes, respectively.

Another popular operation is the convolution which corresponds to the cold standby systems. A lifetime distribution class is called closed under a reliability operation if the lifetime distributions of the components belonging to the class imply that the lifetime distribution of the system also belongs to the same class.

In theory of reliability, it has always been found very useful to classify life distributions according to their aging behaviors. Barlow and Broshan [3] have studied the characteristics of aging for many different classes of life distributions, the most famous of which are Increasing failure rate (IFR), Increasing failure rate average $(IFRA)$, new better than used (NBU) and their duals decreasing failure rate (DFR), decreasing failure rate average $(DFRA)$, new worse than used (NWU).

Cao and Wang [4] have proposed new better than used convex ordering $(NBUC)$ aging properties and their dual new worse than used convex ordering $(NWUC)$ properties to be an extension of NBU. Pellerey [5] studied most of the characteristics related to $NBUC$. Abu-Youssef et al. [6,7] for new better than used convex ordering moment generation function $(NBUC_{mgf})$ class. Mahmoud

* Corresponding author e-mail: Rashadmath27@gmail.com

et al. [8] for new better than used in increasing convex in Laplace transform order ($NBUCL$) class.

Mahmoud et al. [9] for exponential better than equilibrium life in convex ($EBELC$) class. Al-Gashgari et al. [10] for new better than used in the increasing convex average order ($NBUCA$) class based on Laplace transform. Abouammoh et al. [11] studied properties for new better than renewal used ($NBRU$) class. Testing exponentiality against some classes of life distributions has been introduced by many researchers. EL-Sagheer et al. [12] for new better than renewal used in Laplace transform ($NBRUL$) class based on Laplace transform order.

EL-Sagheer et al. [13] for new better than renewal used in Laplace transform at age t_0 ($NBRUL-t_0$) class. Hassan et al. [14] for new better than renewal used in moment generation function ($NBRU_{mgf}$) class. Abu-Youssef et al. [15] for used better than aged in increasing concave ordering ($UBAC(2)$) class. Mahmoud et al. [16,17] for new better than renewal used in Laplace transform ($NBRUL$) class.

Therefore, in this paper, our theme formulates a new class of life distribution ($NBRULC$), discuss its characterization and testing exponentiality versus $NBRULC$ class based on U_n-statistic.

The rest of this paper can be organized as follows. In Section 2, we discuss preservation under convolution, mixture, homogeneous poisson shock model for $NBRULC$ class of life distribution. In Section 3, we present testing exponentiality against NBRULC class based on U-statistic. The Pitman asymptotic for several common alternatives is obtained in Section 4. In Section 5, Monte Carlo null distribution critical points and the power estimates are simulated. In Section 6, a proposed test is presented for right censored data. Finally, we discuss some applications to demonstrate the utility of the proposed statistical test in Section 7.

Definition 1. *Let X and Y be non-negative random variables with distribution functions F(x) and G(x) and survival functions $\bar{F}(x)$ and $\bar{G}(x)$ respectively,. X is said to be smaller than Y in:*
(i) Usual stochastic order, denoted by $X \leq_{st} Y$ if

$$\bar{F}(x) \leq \bar{G}(x), \quad \text{for all } x$$

(ii) Increasing convex order, denoted by $X \leq_{icx} Y$ if

$$\int_x^\infty \bar{F}(u)\,du \leq \int_x^\infty \bar{G}(u)\,du, \quad \text{for all } x$$

Definition 2. *A random variable X is said to be*
(i) New better than renewal used, denoted by $X \in NBRU$, if

$$\overline{W_F}(x+t) \leq \overline{F}(x)\overline{W_F}(t), \quad x,t, \geq 0,$$

(ii) New better than renewal used in Laplace transform order, denoted by $X \in NBRUL$, if

$$\int_0^\infty e^{-kx}\overline{W_F}(x+t)\,dx \leq \overline{W_F}(t) \times \int_0^\infty e^{-kx}\overline{F}(x)\,dx, \quad x,t,k \geq 0,$$

or

$$\int_0^\infty \int_{x+t}^\infty e^{-kx}\overline{F}(u)dudx \leq \int_0^\infty \int_t^\infty e^{-kx}\overline{F}(x)\overline{F}(u)dudx,$$

where $\overline{W_F}(x+t) = \frac{1}{\mu}\int_{x+t}^\infty \overline{F}(u)du$.

Now, depending on Definition (1) and (2) a new class of life distribution named new better than renewal used in Laplace transform in increasing convex order is defined as follows.

Definition 3. *If X is a random variable with survival function $\bar{F}(x)$, then X is said to be new better (worse) than renewal used in Laplace transform in increasing convex order, denoted by $NBRULC$ ($NWRULC$), if*

$$\int_0^\infty \int_t^\infty e^{-kx}\overline{W_F}(x+u)\,dxdu \leq (\geq) \int_0^\infty \int_t^\infty e^{-kx}\overline{F}(x)\,\overline{W_F}(u)\,dxdu, \quad x,t,k \geq 0,$$

or

$$\int_0^\infty \int_{x+t}^\infty e^{-kx}\overline{W_F}(u)\,dxdu \leq (\geq) \int_0^\infty \int_t^\infty e^{-kx}\overline{F}(x)\,\overline{W_F}(u)\,dxdu,$$

this could be rewritten as

$$\int_0^\infty e^{-kx}\Gamma(x+t)dx \leq \int_0^\infty e^{-kx}\overline{F}(x)\Gamma(t)dx,$$

where $\Gamma(x+t) = \int_{x+t}^\infty \overline{W_F}(u)\,du$.

Remark. It is obvious that $NBRU \Rightarrow NBRUL \Rightarrow NBRULC$

2 Closure Properties

In this section we study the closure properties of the new better than renewal used in laplace transform in increasing convex order under some reliability operations such as convolution, mixture and the shock model in homogeneous case.

2.1 Convolution properties

The aim of this subsection is to discuss preservation under convolution properties of $NBRULC$ class.

Theorem 1. *The NBRULC class is preserved under convolution.*

Proof. Suppose that F_1 and F_2 are two independent $NBRULC$ lifetime distributions and their convolution is given by

$$\overline{F}(u) = \int_0^\infty \overline{F_1}(u-z)dF_2(z).$$

Therefore

$$\int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}(u)dudydx$$
$$= \int_0^\infty e^{-kx} \int_t^\infty \int_{x+y}^\infty \int_0^\infty \overline{F}_1(u-z)dF_2(z)dudydx$$
$$= \int_0^\infty \int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}_1(u-z)dudydxdF_2(z).$$

Since $\bar{F}_1$ is $NBRULC$, then

$$\int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}(u)dudydx$$
$$\leq \int_0^\infty \int_0^\infty \int_t^\infty \int_y^\infty e^{-kx}\overline{F}_1(x)\overline{F}_1(u-z)dudydxdF_2(z)$$
$$\leq \int_0^\infty e^{-kx}\overline{F}_1(x) \int_t^\infty \int_y^\infty \int_0^\infty \overline{F}_1(u-z)dF_2(z)dudydx$$
$$\leq \int_0^\infty e^{-kx}\overline{F}_1(x) \int_t^\infty \int_y^\infty \overline{F}(u)dudydx,$$

by using $\overline{F}_i(x) \leq \overline{F}(x)$ for $i = 1, 2$ we get

$$\int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}(u)dudydx$$
$$\leq \int_0^\infty \int_t^\infty \int_y^\infty e^{-kx}\overline{F}(x)\overline{F}(u)dudydx,$$

which complete the proof.

2.2 Mixture properties

The following theorem is stated and proved to show that the $NWRULC$ class is preserved under mixture.

Theorem 2. *The NWRULC class is preserved under mixture.*

Proof. Suppose that $F(u)$ is the mixture of F_γ, where each F_γ is $NWRULC$ since

$$\overline{F}(u) = \int_0^\infty \overline{F_\gamma}(u)dG(\gamma),$$

then

$$\int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}(u)dudydx$$
$$= \int_0^\infty \int_t^\infty \int_{x+y}^\infty \int_0^\infty e^{-kx}\overline{F}_\gamma(u)dG(\gamma)dudydx$$
$$= \int_0^\infty \int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}_\gamma(u)dudydxdG(\gamma),$$

since F_γ is $NWRULC$ then

$$\int_0^\infty \int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}_\gamma(u)dudydxdG(\gamma)$$
$$\geq \int_0^\infty \{\int_0^\infty \int_t^\infty \int_y^\infty e^{-kx}\overline{F}_\gamma(x)\overline{F}_\gamma(u)dudydx\}dG(\gamma).$$

Using Chebyshev's inequality for similarity ordered functions, we get

$$\int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{F}(u)dudydx$$
$$\geq \int_0^\infty \int_0^\infty e^{-kx}\overline{F}_\gamma(x)dG(\gamma)dx$$
$$\times \int_t^\infty \int_y^\infty \int_0^\infty \overline{F}_\gamma(u)dG(\gamma)dudy$$
$$\geq \int_0^\infty e^{-kx}\overline{F}(x)dx. \int_t^\infty \int_y^\infty \overline{F}(u)dudy$$
$$\geq \int_0^\infty \int_t^\infty \int_y^\infty e^{-kx}\overline{F}(x)\overline{F}(u)dudydx,$$

which complete the proof.

2.3 Homogeneous Poisson shock model

Suppose that a device is subjected to a sequence of shocks happening randomly in the time according to a Poisson process with intensity λ. Suppose further that the device has probability $\overline{P}_s$ of surviving the first s shocks, where $1 = \overline{P}_0 \geq \overline{P}_1 \geq ......$ Denote $p_j = \overline{P}_{j-1} - \overline{P}_j, j \geq 1$. Then the survival function of the device is given by

$$\overline{H}(t) = \sum_{s=0}^\infty \overline{P}_s \frac{(\lambda t)^s}{s!} e^{-\lambda t}, \qquad t \geq 0. \tag{1}$$

This shock model has been studied by Esary et al. [18] for different ageing properties such as IFR, $IFRA$, NBU, and $NBUE$, Klefsjo [19] for $HNBUE$ and Mahmoud et al. [20] for $NBURFR - t_0$.

Definition 4. *A discrete distribution $p_s, s = 0, 1, ..., \infty$ is said to have discrete new better (worse) than renewal used in laplace transform in increasing convex order (NBRULC) (NWRULC) if*

$$\sum_{i=0}^\infty \sum_{l=j}^\infty \sum_{r=i+l}^\infty z^i \overline{p}_r \leq (\geq) \sum_{i=0}^\infty \sum_{l=j}^\infty \sum_{r=l}^\infty z^i \overline{p}_i \overline{p}_r \tag{2}$$
$$, 0 \leq z \leq 1.$$

Theorem 3. *If $\overline{P}_s$ is discrete NBRULC, then $\overline{H}(t)$ given by (1) is NBRULC.*

Proof. It must be shown that

$$\int_0^\infty \int_t^\infty \int_{x+y}^\infty e^{-kx}\overline{H}(u)dudydx$$
$$\leq \int_0^\infty \int_t^\infty \int_y^\infty e^{-kx}\overline{H}(x)\overline{H}(u)dudydx.$$

Upon using (1), we get

$$\int_0^\infty\int_t^\infty\int_{x+y}^\infty e^{-kx}\overline{H}(u)dudydx$$
$$=\int_0^\infty\int_t^\infty\int_{x+y}^\infty e^{-kx}\sum_{s=0}^\infty\overline{P}_s\frac{(\lambda u)^s}{s!}e^{-\lambda u}dudydx$$
$$=\int_0^\infty\int_t^\infty e^{-kx}\sum_{s=0}^\infty\overline{P}_s\frac{1}{s!}\int_{x+y}^\infty(\lambda u)^s e^{-\lambda u}dudydx$$
$$=\frac{1}{\lambda}\int_0^\infty\int_t^\infty e^{-kx}\sum_{s=0}^\infty\overline{P}_s\frac{1}{s!}\sum_{r=0}^s\frac{s![\lambda(x+y)]^r}{r!}e^{-\lambda(x+y)}dydx,$$

where $\int_{x+y}^\infty(\lambda u)^s e^{-\lambda u}du=\frac{1}{\lambda}\sum_{r=0}^s\frac{s![\lambda(x+y)]^r}{r!}e^{-\lambda(x+y)}$,

$$\int_0^\infty\int_t^\infty\int_{x+y}^\infty e^{-kx}\overline{H}(u)dudydx$$
$$=\frac{1}{\lambda}\int_0^\infty\int_t^\infty e^{-kx}\sum_{s=0}^\infty\overline{P}_s\sum_{r=0}^s\frac{e^{-\lambda(x+y)}}{r!}\times$$
$$\sum_{j=0}^r\binom{r}{j}(\lambda y)^{r-j}(\lambda x)^j dydx$$
$$=\frac{1}{\lambda}\sum_{s=0}^\infty\overline{P}_s\sum_{r=0}^s\sum_{j=0}^r\binom{r}{j}\int_t^\infty\frac{e^{-\lambda y}}{r!}(\lambda y)^{r-j}$$
$$\int_0^\infty(\lambda x)^j e^{-x(k+\lambda)}dxdy$$
$$=\frac{1}{\lambda^2}\sum_{j=0}^\infty\sum_{r=j}^\infty\sum_{s=r}^\infty\overline{P}_s\frac{1}{(r-j)!}[\frac{\lambda}{k+\lambda}]^{j+1}\int_t^\infty(\lambda y)^{r-j}e^{-\lambda y}dy,$$

let $l=r-j$,

$$\int_0^\infty\int_t^\infty\int_{x+y}^\infty e^{-kx}\overline{H}(u)dudydx$$
$$=\frac{1}{\lambda^2}\sum_{j=0}^\infty\sum_{l=0}^\infty\sum_{s=l+j}^\infty\overline{P}_s\frac{1}{l!}[\frac{\lambda}{k+\lambda}]^{j+1}\int_t^\infty(\lambda y)^l e^{-\lambda y}dy$$
$$=\frac{1}{\lambda^3}\sum_{j=0}^\infty\sum_{s=l+j}^\infty\sum_{l=0}^\infty\sum_{n=0}^l\overline{P}_s[\frac{\lambda}{k+\lambda}]^{j+1}\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$
$$=\frac{1}{\lambda^3}\sum_{j=0}^\infty\sum_{s=l+j}^\infty\sum_{n=0}^\infty\sum_{l=n}^\infty\overline{P}_s[\frac{\lambda}{k+\lambda}]^{j+1}\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$
$$=\frac{1}{\lambda^3}\sum_{n=0}^\infty\sum_{j=0}^\infty\sum_{l=n}^\infty\sum_{s=l+j}^\infty\overline{P}_s[\frac{\lambda}{k+\lambda}]^{j+1}\frac{(\lambda t)^n}{n!}e^{-\lambda t},$$

since F is *NBRULC*

$$\int_0^\infty\int_t^\infty\int_{x+y}^\infty e^{-kx}\overline{H}(u)dudydx$$
$$\le\frac{1}{\lambda^3}\sum_{n=0}^\infty\sum_{j=0}^\infty\sum_{l=n}^\infty\sum_{s=l}^\infty\overline{P}_j\overline{P}_s\frac{(\lambda t)^n}{n!}e^{-\lambda t}\cdot[\frac{\lambda}{k+\lambda}]^{j+1}$$
$$\le\frac{1}{\lambda^2}\sum_{j=0}^\infty\sum_{s=l}^\infty\sum_{n=0}^\infty\sum_{l=n}^\infty\overline{P}_j\overline{P}_s\frac{(\lambda t)^n}{n!}e^{-\lambda t}\int_0^\infty\frac{(\lambda x)^j}{j!}e^{-x(k+\lambda)}dx$$
$$\le\frac{1}{\lambda^2}\int_0^\infty\sum_{j=0}^\infty\sum_{s=l}^\infty\sum_{l=0}^\infty\sum_{n=0}^l\overline{P}_j\overline{P}_s\frac{(\lambda t)^n}{n!}e^{-\lambda t}\frac{(\lambda x)^j}{j!}e^{-x(k+\lambda)}dx$$
$$\le\frac{1}{\lambda}\int_0^\infty\sum_{j=0}^\infty\sum_{l=0}^\infty\sum_{s=l}^\infty\overline{P}_j\overline{P}_s\int_t^\infty\frac{(\lambda y)^l}{l!}e^{-\lambda y}dy.\frac{(\lambda x)^j}{j!}e^{-x(k+\lambda)}dx$$
$$\le\frac{1}{\lambda}\int_0^\infty\sum_{j=0}^\infty\sum_{s=0}^\infty\sum_{l=0}^s\overline{P}_j\overline{P}_s\int_t^\infty\frac{(\lambda y)^l}{l!}e^{-\lambda y}dy.\frac{(\lambda x)^j}{j!}e^{-x(k+\lambda)}dx$$
$$\le\int_0^\infty\int_t^\infty\int_y^\infty e^{-kx}\sum_{j=0}^\infty\overline{P}_j\frac{(\lambda x)^j}{j!}e^{-\lambda x}\sum_{s=0}^\infty\overline{P}_s\frac{(\lambda u)^s}{s!}e^{-\lambda u}dudydx$$
$$\le\int_0^\infty\int_t^\infty\int_y^\infty e^{-kx}\overline{H}(x)\overline{H}(u)dudydx.$$

The proof for the NWRULC class is got by reversing the inequalities.

3 Testing Against NBRULC Alternatives

In this section, we test the null hypothesis $H_0: F$ is exponential versus the alternative hypothesis $H_1: F$ is *NBRULC* and not exponential. The following lemma is essential for the development of our test statistic.

Lemma 1. *If X is a random variable with distribution function F belongs to NBRULC class, then*

$$\frac{1}{2k^2}\mu_{(2)}-\frac{1}{k^4}\xi(k)-\frac{1}{k^3}\mu+\frac{1}{k^4}\ge\frac{1}{6k}\mu_{(3)}\xi(k),k\ge0 \quad (3)$$

where

$$\xi(k)=Ee^{-kX}=\int_0^\infty e^{-kx}dF(x),\mu_{(r)}=E(X^r)$$

Proof. since F is *NBRULC* then

$$\int_0^\infty e^{-kx}\Gamma(x+t)dx\le\int_0^\infty e^{-kx}\overline{F}(x)\Gamma(t)dx,\quad x,t\ge0.$$

Integrating both sides with respect to t over $[0,\infty)$, gives

$$\int_0^\infty\int_0^\infty e^{-kx}\Gamma(x+t)dxdt\le\int_0^\infty\int_0^\infty e^{-kx}\overline{F}(x)\Gamma(t)dxdt. \quad (4)$$

Setting

$$I_1=\int_0^\infty\int_0^\infty e^{-kx}\Gamma(x+t)dxdt$$
$$=\frac{1}{2}E\int_0^\infty\int_0^\infty e^{-kx}(X-x-t)^2I(X>x+t)I(X>t)dxdt$$
$$=\frac{1}{2}E\int_0^X\int_0^{X-t}e^{-kx}(X-x-t)^2dxdt.$$

So,

$$I_1 = \frac{1}{6k}\mu_{(3)} - \frac{1}{2k^2}\mu_{(2)} + \frac{1}{k^4}\xi(k) + \frac{1}{k^3}\mu - \frac{1}{k^4}. \quad (5)$$

Similary if we set

$$\begin{aligned} I_2 &= \int_0^\infty \int_0^\infty e^{-kx}\overline{F}(x)\Gamma(t)dxdt \\ &= E\int_0^\infty \Gamma(t)\int_0^\infty e^{-kx}I(X>x)dxdt \\ &= (\frac{1}{k} - \frac{1}{k}\xi(k))\int_0^\infty \Gamma(t)dt \\ &= \frac{1}{2}(\frac{1}{k} - \frac{1}{k}\xi(k))E\int_0^\infty (X-t)^2 I(X>t)dt, \end{aligned}$$

then

$$I_2 = \frac{1}{6k}\mu_{(3)} - \frac{1}{6k}\mu_{(3)}\xi(k). \quad (6)$$

Substituting (5) and (6) into (4), we get:

$$\frac{1}{2k^2}\mu_{(2)} - \frac{1}{k^4}\xi(k) - \frac{1}{k^3}\mu + \frac{1}{k^4} \geq \frac{1}{6k}\mu_{(3)}\xi(k),$$

which completes the proof.

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from a population with distribution F. Using Lemma 1 and $\gamma(k)$, as measure of departure from exponentiality, we get

$$\gamma(k) = \frac{1}{2k^2}\mu_{(2)} - \frac{1}{6k}\mu_{(3)}\xi(k) - \frac{1}{k^4}\xi(k) - \frac{1}{k^3}\mu + \frac{1}{k^4}. \quad (7)$$

One can notice that under $H_0, \gamma(k) = 0$, while it is positive under H_1. The empirical estimate $\gamma_n(k)$ of $\gamma(k)$ can be obtained as

$$\gamma_n(k) = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \left[\frac{1}{2k^2}X_i^2 - \frac{1}{6k}X_i^3 e^{-kX_j} - \frac{1}{k^4}e^{-kX_i} - \frac{1}{k^3}X_i + \frac{1}{k^4}\right].$$

To make the test invariant, let $\Lambda_n(k) = \frac{\gamma_n(k)}{\overline{X}^4}$ where $\overline{X} = \sum_{j=1}^n \frac{x_i}{n}$ is the sample mean. Then

$$\Lambda_n(k) = \frac{1}{n^2\overline{X}^4}\sum_{i=1}^n\sum_{j=1}^n \left[\frac{1}{2k^2}X_i^2 - \frac{1}{6k}X_i^3 e^{-kX_j} - \frac{1}{k^4}e^{-kX_i} - \frac{1}{k^3}X_i + \frac{1}{k^4}\right] (8)$$

It is easy to show that $E[\Lambda_n(k)] = \gamma(k)$.
Now, set

$$\phi(X_i, X_j) = \frac{1}{2k^2}X_i^2 - \frac{1}{6k}X_i^3 e^{-kX_j} - \frac{1}{k^4}e^{-kX_i} - \frac{1}{k^3}X_i + \frac{1}{k^4}. \quad (9)$$

The following theorem summarizes the asymptotic properties of the test statistic $\Lambda_n(k)$.

Theorem 4. *(i) As $n \to \infty, [\Lambda_n(k) - \gamma(k)]$ is asymptotically normal with mean 0 and variance $\sigma^2(k)/n$, where*

$$\sigma^2(k) = Var\{\frac{X}{2k^2} - \frac{X^3}{6k}E(e^{-kx}) - \frac{1}{k^4}e^{-kx} - \frac{X}{k^3} + \frac{1}{2k^2}\mu_{(2)} - \frac{1}{6k}\mu_{(3)}e^{-kx} - \frac{1}{k^4}E(e^{-kx}) - \frac{1}{k^3}\mu + \frac{2}{k^4}\}. \quad (10)$$

(ii) Under H_0, the variance tends to

$$\sigma_0^2(k) = \frac{19 + 14k + k^2}{(1+k)^4(1+2k)}. \quad (11)$$

Proof. Using standard U-statistics theory, see Lee [21]

$$\sigma^2 = Var\{E[\phi(X_1,X_2) \mid X_1] + E[\phi(X_1,X_2) \mid X_2]\}. \quad (12)$$

Using (9), we can find $E[\phi(X_1,X_2) \mid X_1]$ and $E[\phi(X_1,X_2) \mid X_2]$ as follows

$$E(\phi(X_1,X_2) \mid X_1) = \frac{X}{2k^2} - \frac{X^3}{6k}\int_0^\infty e^{-kx}dF(x) - \frac{1}{k^4}e^{-kx} - \frac{X}{k^3} + \frac{1}{k^4}, \quad (13)$$

and

$$E(\phi(X_1,X_2) \mid X_2) = \frac{1}{2k^2}\int_0^\infty x^2 dF(x) - \frac{1}{6k}e^{-kx}\int_0^\infty x^3 dF(x) - \frac{1}{k^4}\int_0^\infty e^{-kx}dF(x) - \frac{1}{k^3}\int_0^\infty x dF(x) + \frac{1}{k^4}. \quad (14)$$

Upon using (12), (13) and (14) Eq (10) is obtained.
Under H_0, (11) is obtained.

4 The PAE of $\hat{\Delta}(k)$

To judge on the quality of this procedure, The PAEs are computed and compared with some other tests for the following other choice distributions:

(i)The Weibull distribution:

$$\bar{F}_1(u) = e^{-u^\theta}, u \geq 0, \theta \geq 1.$$

(ii)The linear failure rate distribution (LFR):

$$\bar{F}_2(u) = e^{-u - \frac{\theta}{2}u^2}, u \geq 0, \theta \geq 0.$$

(iii)The Makeham distribution:

$$\bar{F}_3(u) = e^{-u - \theta(u + e^{-u} - 1)}, u \geq 0, \theta \geq 0.$$

Note that For $\theta = 1, \bar{F}_1(u)$ reduce to exponential distribution while for $\theta = 0, \bar{F}_2(u)$ and $\bar{F}_3(u)$ reduce to exponential distribution. The PAE is defined by

$$PAE(\Lambda_n(k)) = \frac{1}{\sigma_\circ(k)}\left|\frac{d}{d\theta}\gamma(k)\right|_{\theta \to \theta_\circ}.$$

Table 1: Comparison between the PAE of our test $\Lambda_n(5)$ and some other tests

Test	Weibull	LFR	Makeham
[22]	0.132	0.433	0.144
[23]	0.170	0.408	0.039
[24]	0.223	0.535	0.184
[25]	0.050	0.217	0.144
Our test	1.046	0.932	0.233

At $k=5$, this leads to:

$$PAE[\Lambda_n(5), Weibull] = 1.046,$$
$$PAE[\Lambda_n(5), LFR] = 0.932$$
$$PAE[\Lambda_n(5), Makeham] = 0.233$$
$$, \text{ where } \sigma_0(5) = 0.08942.$$

It is obvious that $\Lambda_n(5)$ is better than the other tests based on the PAEs.

5 Monte Carlo Null Distribution Critical Points

In this section, the upper percentile of $\Lambda_n(5)$ for $90\%, 95\%$ and 99% are calculated using Mathematica 8 program and based on 10000 generated samples of size $n=5(5)50, 39, 43$ and tabulated in **Table 2**.

Table 2: The upper percentile of $\Lambda_n(5)$ with 10000 replications

n	90%	95%	99%
5	0.040407	0.052467	0.083304
10	0.023907	0.029561	0.043785
15	0.019486	0.023540	0.032107
20	0.016798	0.020148	0.027142
25	0.015166	0.018055	0.023933
30	0.014189	0.016865	0.022148
35	0.013170	0.015410	0.019925
39	0.012662	0.014689	0.018743
40	0.012423	0.014672	0.018973
43	0.012176	0.014168	0.018113
45	0.011886	0.013867	0.017943
50	0.011310	0.013066	0.016792

It can be noticed from **Table 2** and **Fig 1** that the critical values are increasing as the confidence level increasing and decreasing as the sample size increasing.

5.1 *Power estimates of the test* $\Lambda_n(5)$

The power of proposed test will be estimated at $(1-\alpha)\%$ confidence level, $\alpha = 0.05$ with suitable parameters values of θ at $n = 10, 20$ and 30 for some commonly used distributions such as linear failure rate, Weibull and Gamma distributions based on 10000 samples tabulated in **Table 3**.

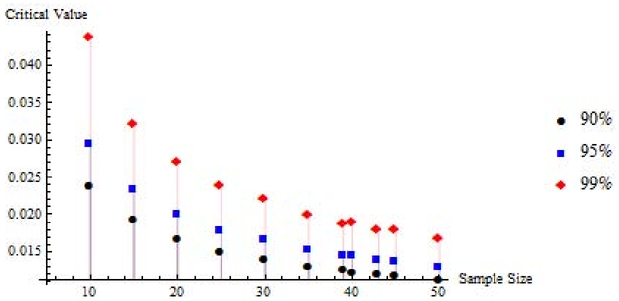

Fig. 1: Relation between critical values, sample size and confidence levels.

Table 3: Power estimates of $\Lambda_n(5)$.

n	θ	LFR	Weibull	Gamma
10	2	0.6543	0.9969	0.9797
	3	0.8501	0.9998	0.9978
	4	0.9313	1.0000	0.9995
20	2	0.9368	1.0000	0.9868
	3	0.9793	1.0000	0.9992
	4	0.9934	1.0000	1.0000
30	2	0.9775	1.0000	0.9897
	3	0.9944	1.0000	0.9999
	4	0.9983	1.0000	1.0000

From **Table 3,** it seen that our test $\Lambda_n(5)$ has good power for all other choices.

6 Testing Against NBRULC Class for Censored Data

A test statistic is proposed to test $H_\circ$ against H_1 with randomly right-censored data. Such a censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can formally be modeled as follows. Suppose n objects are put on test, and $X_1, X_2, ..., X_n$ denote their true life time.

We let that $X_1, X_2, ..., X_n$ be independent and identically distributed (i.i.d.) according to a continuous life distribution F. Let $Y_1, Y_2, ..., Y_n$ be (i.i.d.) according to a continuous life distribution G.

Also we assume that X's and Y's are independent. In the randomly right-censored model, we observe the pairs (Z_j, δ_j), $j = 1, ..., n$, where $Z_j = \min(X_i, Y_j)$ and

$$\delta_j = \begin{cases} 1, \text{ if } Z_j = X_j \text{ (j-th observation is uncensored)} \\ 0, \text{ if } Z_j = Y_j \text{ (j-th observation is censored)} \end{cases}.$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < ..... < Z_{(n)}$ denote the orderd Z's and $\delta_{(j)}$ is δ_j corresponding to $Z_{(j)}$. Using the censored data (Z_j, δ_j), $j = 1, ..., n$. Kaplan and Meier [26] proposed the product limit estimator,

$$\bar{F}_n(X) = \prod_{[j:Z(j)\leq X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}}, X \in [0, Z_{(n)}].$$

Now, for testing $H_\circ : \Lambda_c(k) = 0$ against $H_1 : \Lambda_c(k) > 0$, using the randomly right censored data, we propose the following test statistic

$$\Lambda_c(k) = \frac{1}{2k^2}\mu_{(2)} - \frac{1}{6k}\mu_{(3)}\xi(k) - \frac{1}{k^4}\xi(k) - \frac{1}{k^3}\mu + \frac{1}{k^4},$$

where $\xi(k) = \int_0^\infty e^{-kx} dF_n(x)$. For computational purpose, $\Lambda_c(k)$ may be rewritten as

$$\Lambda_c(k) = \frac{1}{2k^2}\tau - \frac{1}{6k}\Phi\eta - \frac{1}{k^4}\eta - \frac{1}{k^3}\Omega + \frac{1}{k^4},$$

where

$$\Omega = \sum_{s=1}^{n}[\prod_{m=1}^{s-1} C_m^{\delta(m)} (Z_{(s)} - Z_{(s-1)})],$$

$$\eta = \sum_{j=1}^{n} e^{-kZ_{(j)}} [\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}],$$

$$\tau = 2\sum_{i=1}^{n}[\prod_{v=1}^{i-1} Z_{(i)} C_v^{\delta(v)} (Z_{(i)} - Z_{(i-1)})],$$

$$\Phi = 3\sum_{i=1}^{n}[\prod_{v=1}^{i-1} Z_{(i)}^2 C_v^{\delta(v)} (Z_{(i)} - Z_{(i-1)})],$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j), c_s = [n-s][n-s+1]^{-1}.$$

To make the test invarient, let

$$\hat{\Lambda}_c = \frac{\Lambda_c}{\bar{Z}^4}, \text{ where } \bar{Z} = \sum_{i=1}^{n} \frac{Z_{(i)}}{n}.$$

Table (4), gives the critical values percentiles of $\hat{\Lambda}_c$ test for sample sizes $n = 10(10)80, 51, 81$.

The Monte Carlo null distribution critical values of $\hat{\Lambda}_c$ at $k = 5$ with 10000 replications are simulated from the standard exponential distribution by using *Mathematica 8.* program.

Table 4: The upper percentile of $\hat{\Lambda}_c$ at $k = 5$

n	90%	95%	99%
10	0.021622	0.023954	0.028387
20	0.020642	0.022766	0.025955
30	0.019498	0.021407	0.024428
40	0.018657	0.020462	0.023348
50	0.018167	0.019948	0.022538
51	0.018013	0.019788	0.022624
60	0.017580	0.019379	0.022254
70	0.017290	0.019157	0.021718
80	0.016735	0.018629	0.021143
81	0.016724	0.018550	0.021280

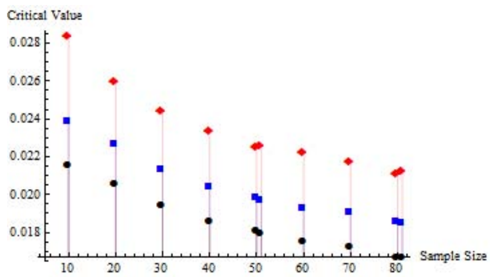

Fig. 2: Relation between critical values, sample size and confidence levels.

From **Table 4** and **Fig 2** It can be watched that the critical values decrease as the samples sizes increase and they increase as the confidence level increases.

6.1 Power estimates of the test $\hat{\Lambda}_c(k)$

The power of our test $\hat{\Lambda}_c(k)$ is considered at the significant level $\alpha = 0.05$ with occasion parameters values of θ at $n = 10, 20$ and 30 with respect to three choices Weibull, LFR and Gamma distributions based on 10000 samples.

Table 5: Power estimates of $\hat{\Lambda}_c(5)$

n	θ	Weibull	LFR	Gamma
10	2	0.9614	0.9865	0.9980
	3	0.9983	0.9970	1.0000
	4	1.0000	0.9993	1.0000
20	2	0.9189	0.9746	0.9988
	3	0.9948	0.9945	1.0000
	4	1.0000	0.9987	1.0000
30	2	0.9005	0.9751	0.9993
	3	0.9956	0.9928	1.0000
	4	0.9999	0.9975	1.0000

Table 5 shows that the power estimates of our test $\hat{\Lambda}_c(5)$ are good power for all other choices.

7 Applications to Real Data

In this section, we apply our test to some real data-sets in the both non censored and censored data at 95% confidence level.

7.1 Non censored data

*Example 1.*We consider a classical real data in Keating et al. [27] set on the times, in operating days, between successive failures of air conditioning equipment in an aircraft. Theses data are recorded.

3.750	0.417	2.500	7.750	2.542
2.042	0.583	1.000	2.333	0.833
3.292	3.500	1.833	2.458	1.208
4.917	1.042	6.500	12.917	3.167
1.083	1.833	0.958	2.583	5.417
8.667	2.917	4.208	8.667	

The $\hat{\Lambda}(5) = 0.0023287$ is gotten which is less than the tabulated value in Table 2. It is evident at the significant level $\alpha = 0.05$. This means that this kined of data doesn't fit with *NBRULC* property.

*Example 2.*The following data set consists of 16 intervals in operating days between successive failures of air conditioning equipment in a Boeing 720 aircraft. (See Edgeman et al. [28]).

4.250	8.708	0.583	2.375	2.250
1.333	2.792	2.458	5.583	6.333
1.125	0.583	9.583	2.750	2.542
1.417				

In this case, $\hat{\Lambda}(5) = 0.0023405$ is gotten which is less than the corresponding critical value in Table 2, then, H_0 the null hypotheses are accepted which show that the data set has exponential property.

*Example 3.*Consider the data in Abouammoh et al. [29]. These data represent set of 40 patients suffering from blood cancer (Leukemia) from one ministry of health hospital in Saudi Arabia and order values in years are:

0.315	0.496	0.699	1.145	1.208
1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805
2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.718
3.751	3.858	3.986	4.049	4.244
4.323	4.323	4.381	4.392	4.397
4.647	4.753	4.929	4.973	5.074

The $\hat{\Lambda}(5) = 0.0020425$ is gotten which is less than the corresponding critical value in Table 2. It is evident at the significant level $\alpha = 0.05$. This means that this kined of data doesn't fit with *NBRULC* property.

*Example 4.*Consider the following data set is from Kotz and Johnson [30] and represents the survival times (in years) after diagnosis of 43 patient with certain kind of leukemia.

0.019	0.129	0.159	0.203	0.485
0.636	0.748	0.781	0.869	1.175
1.206	1.219	1.219	1.282	1.356
1.362	1.458	1.564	1.586	1.592
1.781	1.923	1.959	2.134	2.413
2.466	2.548	2.652	2.951	3.038
3.6	3.655	3.754	4.203	4.690
4.888	5.143	5.167	5.603	5.633
6.192	6.655	6.874		

We get $\hat{\Lambda}(5) = 0.0066037$ which is less than the critical value of the Table 2. Then, H_0 the null hypotheses are accepted which show that the data set has exponential property.

7.2 Censored data

*Example 5.*The following data sets are associated with 101 patients with advanced acute myelogenous leukemia reported to the International Bone Marrow Transplant Registry (see Ghitany and Al-Awadhi [31]). Fifty of these patients had an allogeneic bone marrow transplant where marrow from an HLA (Histocompatibility Leukocyte Antigen) matched sibling was used to replenish their immune systems. Fifty patients had an autologous bone marrow transplant in which, after high doses of chemotherapy, their own marrow was reinfused to replace their destroyed immune system. The leukemia free-survival times (in months) for the 50 allogeneic transplant patients (+ indicates censored observations) are:

0.030	0.493	0.855	1.184
1.283	1.480	1.776	2.138
2.500	2.763	2.993	3.224
3.421	4.178	4.441+	5.691
5.855+	6.941+	6.941	7.993+
8.882	8.882	9.145+	11.480
11.513	12.105+	12.796	12.993+
13.849+	16.612+	17.138+	20.066
20.329+	22.368+	26.776+	28.717+
28.717+	32.928+	33.783+	34.221+
34.770+	39.539+	41.118+	45.033+
46.053+	46.941+	48.289+	57.401+
58.322+	60.625+		

Taking into account the whole set of survival data (both censored and uncensored).

We get $\hat{\Lambda}_c(5) = -1.49361 \times 10^{27}$ which is less than the critical value of the Table 4. Then, H_1 is rejected and data set has exponential property.

The leukemia free-survival times (in months) for the 51 autologous transplant patients are:

0.658	0.822	1.414	2.500
3.322	3.816	4.737	4.836+
4.934	5.033	5.757	5.855
5.987	6.151	6.217	6.447+
8.651	8.717	9.441+	10.329
11.480	12.007	12.007+	12.237
12.401+	13.059+	14.474+	15.000+
15.461	15.757	16.480	16.711
17.204+	17.237	17.303+	17.664+
18.092	18.092+	18.750+	20.625+
23.158	27.730+	31.184+	32.434+
35.921+	42.237+	44.638+	46.480+
47.467+	48.322+	56.086	

Taking into account the whole set of survival data (both censored and uncensored).

In this case, $\hat{\Lambda}_c(5) = -1.33033 \times 10^{38}$ is gotten which is less than the critical value inTable 4. Then, H_1 which states that the set of data have $NBRULC$ property is rejected.

*Example 6.*Consider the data in Susarla and Vanryzin [32]. These data represent 81 survival times of patients of melanoma. Of them 46 represent whole life times(non-censored data) and the observed values are:

13	14	19	19	20	21	23
23	25	26	26	27	27	31
32	34	34	37	38	38	40
46	50	53	54	57	58	59
60	65	65	66	70	85	90
98	102	103	110	118	124	130
136	138	141	234			

The ordered censored observations are:

16	21	44	50	55
67	73	76	80	81
86	93	100	108	114
120	124	125	129	130
132	134	140	147	148
151	152	152	158	181
190	193	194	213	215

Taking into account the whole set of survival data (both censored and uncensored).

We get $\hat{\Lambda}_c(5) = -1.58905 \times 10^{40}$ which is less than the critical value of the Table 4.Then, H_0 the null hypotheses are accepted which show that the data set has exponential property.

8 Conclusion

Sometimes the classes follows an exponential distribution (constant faliure rate) and sometimes it is increasing or decreasing faliure rate, so class of life distribution is added to the family of renewal classes of life distribution and called NBRULC. This class is more general and comprehensive than some other classes so it is used when the failure rate is increasing. In this paper, the closure properties under different reliability operations is studied. A testing hypothesis is proposed to test exponentiality against this new class based on censored and non-censored data. Our test statistic is found to be more efficient than other tests and have good power for other alternative classes of life distributions. Finally, our test's critical values are given. Censored and non-censored real data sets are used to show the test's usefulness.

Conflict of interests:

The author declare that there is no conflict of interest regarding the publication of this paper.

References

[1] D. Stoyan, *Comparison methods for queues and other stochastic models*. Wiley, New York, (1983)

[2] S. M. Ross, *Stochastic processes*. Wiley, New York, (1983)

[3] R. E. Barlow, and F. Proschan, *Statistical theory of reliability and life testing*. To Begin with Silver Spring, M D, (1981).

[4] J. Cao, and Y. Wang, The NBUC and NWUC classes of life distributions. J. Appl. Prob., **28**, 473–479, (1991)

[5] F. Pellerey, Shock models with underlying counting process. J. Appl. Prob., **31**, 156-166, (1994)

[6] S. E. Abu-Youssef, E. M. A. Hassan, and Silvana T. Gerges, A new nonparametric class of life distributions based on ordering moment generating approach. Stat. Appl. Prob. Lett, **7**(3), 151-162, (2020)

[7] S. E. Abu-Youssef, and Silvana T. Gerges, Based on the goodness of fit approach, a new test statistics for testing $NBUC_{mgf}$ class of life distributions. Pak. J. Statist., **38**(1), 129-144, (2022)

[8] M. A.W. Mahmoud, L. S. Diab, and D. M. Radi, A nonparametric test for testing NBUCL class of life distributions with applications. IOSR-JM., **15**(1), 15-21, (2019)

[9] M. A.W. Mahmoud, L. S. Diab, and D. M. Radi, Testing exponentiality against exponential better than equilibrium life in convex based on Laplace transformation. Int. J. Comp. Appl., **182**(33), 6-10, (2018)

[10] F. H. Al-Gashgari, A. I. Shawky, and M. A. W. Mahmoud, A nonparametric test for testing exponentiality against NBUCA class of life distributions based on Laplace transform. Qual. Reliab. Eng. Int., **32**(1), 29–36, (2016)

[11] A. M. Abouammoh, and I. S. Qamber, New better than renewal-used classes of life distributions. IEEE Trans. Reliab., **52**(2), 150-153, (2003)

[12] R. M. EL-Sagheer, S. E. Abu-Youssef, A. Sadek, K. M. Omar, and W. B. H. Etman, Characterizations and testing NBRUL class of life distributions based on Laplace transform technique. Stat. Appl. Prob., **11**(1), 75-88, (2022)
[13] R. M. EL-Sagheer, M. A.W. Mahmoud, and W. B. H. Etman, Characterizations and testing hypotheses for $NBRUL-t_0$ class of life distributions. J. Stat. Theory Pract., (2022). To appear
[14] E. M. A. Hassan, and M. M. Said, A new class of life distributions based on moment inequalities. Asian J. Prob. Stat., **13**(4), 47-57, (2021)
[15] S. E. Abu-Youssef, and A. A. El-Toony, A new non-parametric statistical test for testing exponentiality based on Kernel method with applications on hypothesis testing. Appl. Math. Info. Sci., **15** (3), 285-292, (2021)
[16] M. A.W. Mahmoud, R. M. EL-Sagheer, and W. B. H. Etman, Testing exponentiality against new better than renewal used in Laplace transform order. Stat. Appl. Prob., **5**(2), 279-285, (2016)
[17] M. A.W. Mahmoud, R. M. EL-Sagheer, and W. B. H. Etman, Moments inequalities for NBRUL distributions with hypotheses testing applications. Aust. J. Stat., **47**(1), 95-104, (2018)
[18] J. D. Esary, A. W. Marshal, and F. Proschan, Shock models and wear processes. Ann. Prob., **1**(4), 627-649, (1973).
[19] B. Klefsjo, HNBUE survival under some shock models. Scand. J. Stat., **8**, 39-47, (1981)
[20] M. A. W. Mahmoud, L. S. Diab, and N. A. Abdul Alim, On the new better than used renewal failure rate at specified time. Econ. Quality Control,. **24**(1), 87-99, (2009)
[21] A. J. Lee, *U-Statistics*. Marcel Dekker, New York, (1989)
[22] A. I. Kango, Testing for new is better than used. Commun. Statist. Theor. Meth., **12,** 311-321, (1993).
[23] A. R. Mugdadi, and I. A. Ahmad, Moment inequalities derived from comparing life with its equilibrium form. J. Stat. Plann. Inference, **134,** 303-317, (2005)
[24] A. A. Abdel Aziz, On testing exponentiality against RNBRUE alternatives. Appl. Math. Sci., **35,** 1725-1736, (2007).
[25] M. A. W. Mahmoud, and N. A. Abdul Alim, A goodness of fit approach to for testing NBUFR (NWUFR) and NBAFR (NWAFR) properties. Int. J. Reliab. Appl., **9,** 125-140, (2008)
[26] E. L. Kaplan, and P. Meier, Nonparametric estimation from incomplete observation. J. Amer. Stat. Ass., **53**(282), 457-481, (1958)
[27] J. P. Keating, R. E. Glaser, and N. S. Ketchum, Testing hypotheses about the shape of a gamma distribution. Technometrics, **32**, 67-82, (1990)
[28] R. L. Edgeman, R. C. Scott, and R. J. Pavur, A modified Kolmogorov-Smirnov test for the inverse Gaussian density with unknown parameters. Commun. Stat. Simul. Comput, **17**(4), 1203-1212, (1988)
[29] A. M. Abouammoh, S. A. Abdulghani, and I. S. Qamber, On partial orderings and testing of new better than renewal used classes. Reliab. Eng. Syst. Saf., **43,** 37-41, (1994)
[30] Kotz, S. and Johnson, N. L. *Encyclopedia of statistical sciences*.Wiley,New York, (1983)
[31] M. E. Ghitany, and S. Al-Awadhi, Maximum likelihood estimation of Burr XII distribution parameters under random censoring. Appl. Stat., **29**(7), 955-965, (2002)
[32] V. Susarla, and J. Vanryzin, Empirical Bayes estimations of a survival function right censored observation. Ann. Stat., **6**, 710-755, (1978)