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# Higher Order Moments, Cumulants, Spectral and Bispectral Density Functions of the ZTPINAR(1) Process

*Mohammed H. El-Menshawy*[1,*], *Abd El-Moneim A. M. Teamah*[2], *S. E. Abu-Youssef*[1] *and Hasnaa M. Faied*[3]

[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
[2]Department of Mathematics, Faculty of Science Tanta University, Tanta, Egypt
[3]Department of Mathematics, Faculty of Science, Al-Azhar University (Girls Branch), Nasr City, Cairo, Egypt

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**Abstract:** In this paper, the higher order moments, cumulants, spectral, bispectral and normalized bispectral density functions of zero truncated Poisson first-order integer-valued autoregressive (ZTPINAR(1)) model are calculated. We estimated the spectrum, bispectrum and normalized bispectrum using the smoothed periodogram method with different lag windows. Finally, we used the bispectral density function and normalized bispectral density function in order for studying the linearity of integer valued time series models.

**Keywords:** INAR(1); ZTPINAR(1); Moments; Cumulants; Spectrum; Bispectrum; Normalized bispectrum; Daniell lag window; Parzen lag window; Tukey Hamming lag window; 2-dimensional Subba Rao and Gabr lag window.

## 1 Introduction

Integer-valued time series play an important role in statistical researches in the last few decades. This series are fairly common, such as the number of births at a hospital in successive months, count of chromosome interchanges in cells, count of accidents, number of transmitted messages, count of patients and so on. There have been many attempts in modeling such series through history. Particularly attractive is integer-valued autoregressive (INAR) model introduced by [1]. INAR model was based on a binomial thinning operator generated by Bernoulli distributed counting series. It considered present number of data as the sum of those that remained from the previous period, and those that entered in the observed period. This model was further developed by several authors, for example [2,3,4,5,6,7,8, 9,10,11,12,13,14]. Models based on negative binomial thinning operator generated by geometric distributed counting series were also considered [15,16,17,18]. Integer-valued time series generated by mixtures of binomial and negative binomial thinning operators are considered in [19] and [20]. Also, random coefficient integer-valued time series are introduced [21,22,23]. The common feature of all these models is the assumption of independence of count variables. Some generalizations concerning relaxing the assumption of independence can be found in [24]. [25] introduce a new stationary first-order INAR process with geometric marginals on the basis of the generalized binomial thinning operator, which contains dependent Bernoulli counting series. He relax the assumption of independence underlying the basic INAR model and make the model more readily available for applications in practise. [26] study some higher order moments, spectral and bispectral density functions for some integer autoregressive of order one (INAR(1)) models. These models are the new skew INAR(1) (NSINAR(1)), the shifted geometric INAR(1) typeII (SGINAR(1)-II) and the dependent counting geometric INAR(1) (DCGINAR(1)).

In this paper, we study some higher order moments, spectral and bispectral density functions for a new stationary first-order zero truncated Poisson integer valued autoregressive process which symbolized by ZTPINAR(1). So, motivation and our interest of this process is of such process arises from its ability in modelling and analysis of counting point processes and positive integer-valued time series. An example of a situation in which the zeros often are not observed (truncated) is the number of weekly bus trips where the analyst in this case do not observe the entire distribution of counts.

* Corresponding author e-mail: mwa_t95@yahoo.com

As in literature, the integer-valued autoregressive time series of the first-order (INAR(1)) is given by

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t, t \in Z, \tag{1}$$

where $X_t$ is a non-negative integer-valued random variable and the compound random variable $\alpha \circ X_{t-1}$ is defined by

$$\alpha \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_i, \alpha \in (0,1) \tag{2}$$

where $\{Y_i\}$ is a sequence of i.i.d. Bernoulli r.v.'s with parameter $\alpha$ independant of $X_{t-1}$ and $\{\varepsilon_t\}$ is a sequence of i.i.d. non negative integer valued r.v.'s and independent of $X_{t-i}$ for i≥1 with mean $\mu_\varepsilon$ and finite variance $\sigma_\varepsilon^2$. $\{X_t\}$ is stationary if $0 < \alpha < 1$. The operation in (2) is known as thinning and hence $\circ$ is called (binomial) thinning operator. For more properties of $\circ$ see [9].

This paper is organized as follows. In Section 2, the construction of the zero truncated Poisson first-order integer-valued autoregressive (ZTPINAR(1)) process and its properties are mentioned. In Section 3, the higher order moments and cumulants up to order three are calculated. In Section 4, the spectral, bispectral and normalized bispectral density functions are computed. In Section 5, the spectrum, bispectrum and normalized bispectrum are estimated using the smoothed periodogram based on different lag windows (Daniell, Tukey Hamming and Parzen lag window) and using a generated realizations of size N=500 from the ZTPINAR(1).

## 2 The ZTPINAR(1) Process

The ZTPINAR(1) model was presented by [14]. The sequence $\{X_t\}$ is called ZTPINAR(1) process, which is given by

$$X_t = \begin{cases} \varepsilon_t, & \text{with probability } e^{-\lambda} \\ \alpha \circ X_{t-1} + \varepsilon_t, & \text{with probability } 1 - e^{-\lambda} \end{cases} \tag{3}$$

where $\alpha \in (0, \frac{1}{2}], \lambda > 0, \{\varepsilon_t\}$ is a sequence of i.i.d. random variables and the marginal distribution of $\{X_t\}$ is zero truncated Poisson($\lambda$) distribution (denoted as ZTP($\lambda$)), $P(X_t = k) = \lambda^k e^{-\lambda}/(k!(1-e^{-\lambda})), k = 1, 2, \ldots$. The probability generating function of $X_t$ and $\varepsilon_t$ are respectively given by

$$\phi_X(s) = \frac{e^{\lambda s} - 1}{e^{\lambda} - 1},$$

$$\phi_\varepsilon(s) = \frac{\phi_X(s)}{e^{-\lambda} + (1 - e^{-\lambda})\phi_X(1 - \alpha + \alpha s)} = \frac{e^{\lambda s} - 1}{(1 - e^{-\lambda})e^{\lambda(1-\alpha+\alpha s)}}.$$

Therefore, the probability mass function of the random variable $\varepsilon_t$ is given by

$$P(\varepsilon_t = k) = \frac{[(1-\alpha)^k - (-\alpha)^k]\lambda^k e^{\lambda\alpha}}{k!(e^{\lambda} - 1)}, k = 1, 2, \ldots$$

The mean and variance of $X_t$ and $\varepsilon_t$ are then given by respectively

$$\mu_X = \frac{\lambda}{1 - e^{-\lambda}}, \qquad \sigma_X^2 = \frac{\lambda e^{\lambda}(e^{\lambda} - \lambda - 1)}{(e^{\lambda} - 1)^2},$$

$$\mu_\varepsilon = \frac{\lambda(\alpha + e^{\lambda} - \alpha e^{\lambda})}{e^{\lambda} - 1},$$

$$\sigma_\varepsilon^2 = \frac{\lambda(e^{2\lambda} - \alpha - e^{\lambda} + 2\alpha e^{\lambda} - \lambda e^{\lambda} - \alpha e^{2\lambda})}{(e^{\lambda} - 1)^2}.$$

The ZTPINAR(1) that defined by (3) may be rewritten as

$$X_t = \alpha_t \circ X_{t-1} + \varepsilon_t, \tag{4}$$

where $\{\alpha_t\}$ is a sequance of i.i.d r.v.'s independant of $\{X_t\}$ and $\{\varepsilon_t\}$ with the following distribution

$$P(\alpha_t = 0) = 1 - P(\alpha_t = \alpha) = \frac{1}{1 + \mu}.$$

For more information about the model see [14] and [27].

## 3 Higher Order Joint Moments and Cumulants Up to Order Three

**Theorem 1.** *Let $\{X_t\}$ be a stationary process satisfying (3), then:*
*The second-order joint moment are calculated as*

$$\mu_{(0)} = \frac{\lambda e^{\lambda}(1+\lambda)}{e^{\lambda} - 1},$$

$$\mu_{(s)} = (\alpha(1 - e^{-\lambda}))^s[\mu_{(0)} - \mu_X^2] + \mu_X^2 = \frac{\lambda e^{\lambda}(1+\lambda)(\alpha - \alpha e^{-\lambda})^s}{e^{\lambda} - 1} - \frac{\lambda^2 e^{2\lambda}[\alpha^s(1-e^{-\lambda})^s - 1]}{(e^{\lambda} - 1)^2}.$$

*The second order joint centeral moment (cumulant) is calculated as*

$$C_2(s) = (\alpha(1 - e^{-\lambda}))^s C_2(0) = (\alpha(1 - e^{-\lambda}))^s \frac{\lambda e^{\lambda}(e^{\lambda} - \lambda - 1)}{(e^{\lambda} - 1)^2}.$$

*The third-order joint moments are*

$$\mu_{(0,0)} = \frac{\lambda e^{\lambda}(\lambda^2 + 3\lambda + 1)}{e^{\lambda} - 1},$$

$$\mu_{(0,s)} = (\alpha(1 - e^{-\lambda}))^s[\mu_{(0,0)} - \mu_{(0)}\mu_X] + \mu_{(0)}\mu_X = \frac{\lambda^2 e^{2\lambda}(\lambda+1)}{(e^{\lambda} - 1)^2} - \frac{\lambda e^{\lambda}(\alpha - \alpha e^{-\lambda})^s(3\lambda - e^{\lambda} - 2\lambda e^{\lambda} + \lambda^2 + 1)}{(e^{\lambda} - 1)^2},$$

$$\mu_{(s,s)} = \frac{\lambda e^{\lambda}(\lambda^2 + 3\lambda + 1)(-\alpha^2(e^{-\lambda} - 1))^s}{e^{\lambda} - 1} - \left[\frac{\lambda e^{\lambda}(\lambda - e^{\lambda} + 1)(\alpha + e^{\lambda} + 2\alpha\lambda - \alpha e^{\lambda} + 2\lambda e^{\lambda} - 2\alpha\lambda e^{\lambda} - 1)}{(e^{\lambda} - 1)^3(\alpha - 1)} \times ((-\alpha^2(e^{-\lambda} - 1))^s - (\alpha - \alpha e^{-\lambda})^s)\right]$$

$-\frac{\lambda^2 e^{2\lambda}(\lambda+1)((\alpha^2 e^{-\lambda}(e^{\lambda}-1))^s-1)}{(e^{\lambda}-1)^2}$,
$\mu_{(s,\tau)} = (\alpha(1-e^{-\lambda}))^{\tau-s}(\mu_{(s,s)} - \mu_{(s)}\mu_X) + \mu_{(s)}\mu_X$.
*The third-order joint central moments(cumulants) are*
$C_3(0,0) = \frac{\lambda e^{\lambda}(3\lambda+e^{2\lambda}-2e^{\lambda}-3\lambda e^{\lambda}+\lambda^2+\lambda^2 e^{\lambda}+1)}{(e^{\lambda}-1)^3}$,
$C_3(0,s) = (\alpha(1-e^{-\lambda}))^s C_3(0,0)$
$= \frac{\lambda e^{\lambda}(\alpha(1-e^{-\lambda}))^s(3\lambda+e^{2\lambda}-2e^{\lambda}-3\lambda e^{\lambda}+\lambda^2+\lambda^2 e^{\lambda}+1)}{(e^{\lambda}-1)^3}$,
$C_3(s,s) =$
$(\alpha^2(1-e^{-\lambda}))^s C_3(0,0) + gC_2(0)\frac{(\alpha(1-e^{-\lambda}))^s-(\alpha^2(1-e^{-\lambda}))^s}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}$
$= \frac{\lambda e^{\lambda}(-\alpha^2(e^{-\lambda-1}))^s(3\lambda+e^{2\lambda}-2e^{\lambda}-3\lambda e^{\lambda}+\lambda^2+\lambda^2 e^{\lambda}+1)}{(e^{\lambda}-1)^3} -$
$\frac{\lambda e^{\lambda}(e^{\lambda}+2\alpha\lambda-1)(\lambda-e^{\lambda}+1)[(-\alpha^2(e^{-\lambda}-1))^s-(\alpha-\alpha e^{-\lambda})^s]}{(e^{\lambda}-1)^3(\alpha-1)}$,
$C_3(s,\tau) = \alpha^{\tau-s} C_3(s,s)$.

*Proof.* $\mu_{(0)}$ is computed by

$$\begin{aligned}
\mu_{(0)} &= E(X_t^2) = E[(\alpha_t \circ X_{t-1} + \varepsilon_t)^2] \\
&= E(\alpha_t \circ X_{t-1})^2 + 2E(\alpha_t \circ X_{t-1})\mu_\varepsilon + E(\varepsilon_t^2) \\
&= \alpha^2(1-e^{-\lambda})E(X_{t-1}^2) + \alpha(1-e^{-\lambda})E(X_{t-1}) \\
&\quad - \alpha^2(1-e^{-\lambda})E(X_{t-1}) + 2\alpha(1-e^{-\lambda})E(X_{t-1})\mu_\varepsilon \\
&\quad + E(\varepsilon_t^2) \\
&= \alpha^2(1-e^{-\lambda})\mu_{(0)} + \alpha(1-e^{-\lambda})\mu_X - \alpha^2(1-e^{-\lambda})\mu_X \\
&\quad + 2\alpha(1-e^{-\lambda})\mu_X\mu_\varepsilon + E(\varepsilon_t^2) \\
&= \frac{\alpha(1-e^{-\lambda})\mu_X - \alpha^2(1-e^{-\lambda})\mu_X}{1-\alpha^2(1-e^{-\lambda})} \\
&\quad + \frac{2\alpha(1-e^{-\lambda})\mu_X\mu_\varepsilon + E(\varepsilon_t^2)}{1-\alpha^2(1-e^{-\lambda})} \\
&= \frac{\lambda e^{\lambda}(1+\lambda)}{e^{\lambda}-1},
\end{aligned}$$

then,
$C_2(0) = \mu_{(0)} - \mu_X^2 = \frac{\lambda e^{\lambda}(1+\lambda)}{e^{\lambda}-1} - \frac{\lambda^2 e^{2\lambda}}{(e^{\lambda}-1)^2} = \frac{\lambda e^{\lambda}(e^{\lambda}-\lambda-1)}{(e^{\lambda}-1)^2}$.
$\mu_{(s)}$ is obtained by

$$\begin{aligned}
\mu_{(s)} &= E(X_t X_{t+s}) = E(X_t(\alpha_{t+s} \circ X_{t+s-1} + \varepsilon_{t+s})) \\
&= \alpha(1-e^{-\lambda})E(X_t X_{t+s-1}) + \mu_X\mu_\varepsilon \\
&= \alpha(1-e^{-\lambda})\mu_{(s-1)} + \mu_X\mu_\varepsilon \\
&= (\alpha(1-e^{-\lambda}))^s\mu_{(0)} + \mu_X\mu_\varepsilon\frac{1-(\alpha(1-e^{-\lambda}))^s}{1-(\alpha(1-e^{-\lambda}))} \\
&= (\alpha(1-e^{-\lambda}))^s\mu_{(0)} + \mu_X(1-\alpha(1-e^{-\lambda}))\mu_X \times \\
&\quad \frac{1-(\alpha(1-e^{-\lambda}))^s}{1-(\alpha(1-e^{-\lambda}))} \\
&= (\alpha(1-e^{-\lambda}))^s\mu_{(0)} + \mu_X^2(1-(\alpha(1-e^{-\lambda}))^s) \\
&= (\alpha(1-e^{-\lambda}))^s[\mu_{(0)} - \mu_X^2] + \mu_X^2 \\
&= \frac{\lambda e^{\lambda}(1+\lambda)(\alpha-\alpha e^{-\lambda})^s}{e^{\lambda}-1} - \frac{\lambda^2 e^{2\lambda}[\alpha^s(1-e^{-\lambda})^s-1]}{(e^{\lambda}-1)^2},
\end{aligned}$$

then,

$$\begin{aligned}
C_2(s) &= \mu_{(s)} - \mu_X^2 \\
&= \frac{\lambda e^{\lambda}(1+\lambda)(\alpha-\alpha e^{-\lambda})^s}{e^{\lambda}-1} - \frac{\lambda^2 e^{2\lambda}[\alpha^s(1-e^{-\lambda})^s-1]}{(e^{\lambda}-1)^2} \\
&\quad - \frac{\lambda^2}{(1-e^{-\lambda})^2} \\
&= \frac{\lambda e^{\lambda}(\alpha-\alpha e^{-\lambda})^s(e^{\lambda}-\lambda-1)}{(e^{\lambda}-1)^2} \\
&= (\alpha-\alpha e^{-\lambda})^s\frac{\lambda e^{\lambda}(e^{\lambda}-\lambda-1)}{(e^{\lambda}-1)^2} \\
&= \alpha^s(1-e^{-\lambda})^s C_2(0).
\end{aligned}$$

$\mu_{(0,0)}$ is calculated by

$$\begin{aligned}
\mu_{(0,0)} &= E(X_t^3) = E[(\alpha_t \circ X_{t-1} + \varepsilon_t)^3] \\
&= E(\alpha_t \circ X_{t-1})^3 + E(\varepsilon_t^3) + 3E((\alpha_t \circ X_{t-1})^2\varepsilon_t) \\
&\quad + 3E[(\alpha_t \circ X_{t-1})\varepsilon_t^2] \\
&= \alpha^3(1-e^{-\lambda})E(X_{t-1}^3) + 3\alpha^2(1-e^{-\lambda})E(X_{t-1}^2) \\
&\quad - 3\alpha^3(1-e^{-\lambda})E(X_{t-1}^2) \\
&\quad + [\alpha(1-e^{-\lambda}) - 3\alpha^2(1-e^{-\lambda}) + 2\alpha^3(1-e^{-\lambda})] \times \\
&\quad E(X_{t-1}) + E(\varepsilon_t^3) + 3\mu_\varepsilon[\alpha^2(1-e^{-\lambda})E(X_{t-1}^2) \\
&\quad + \alpha(1-e^{-\lambda})E(X_{t-1}) - \alpha^2(1-e^{-\lambda})E(X_{t-1})] \\
&\quad + 3\alpha(1-e^{-\lambda})E(X_{t-1})E(\varepsilon_t^2) \\
&= \alpha^3(1-e^{-\lambda})\mu_{(0,0)} + 3\alpha^2(1-e^{-\lambda})\mu_{(0)} \\
&\quad - 3\alpha^3(1-e^{-\lambda})\mu_{(0)} \\
&\quad + [\alpha(1-e^{-\lambda}) - 3\alpha^2(1-e^{-\lambda}) \\
&\quad + 2\alpha^3(1-e^{-\lambda})]E(X_{t-1}) + E(\varepsilon_t^3) \\
&\quad + 3\mu_\varepsilon[\alpha^2(1-e^{-\lambda})\mu_{(0)} \\
&\quad + \alpha(1-e^{-\lambda})\mu_X - \alpha^2(1-e^{-\lambda})\mu_X] \\
&\quad + 3\alpha(1-e^{-\lambda})E(X_{t-1})E(\varepsilon_t^2) \\
&= \frac{\lambda e^{\lambda}(\lambda^2+3\lambda+1)}{e^{\lambda}-1},
\end{aligned}$$

then,

$$\begin{aligned}
C_3(0,0) &= \mu_{(0,0)} - 3\mu_X\mu_{(0)} + 2\mu_X^3 \\
&= \frac{\lambda e^{\lambda}(\lambda^2+3\lambda+1)}{e^{\lambda}-1} - \frac{3\lambda^2 e^{2\lambda}(1+\lambda)}{(e^{\lambda}-1)^2} + 2(\frac{\lambda e^{\lambda}}{e^{\lambda}-1})^3 \\
&= \frac{\lambda e^{\lambda}(3\lambda+e^{2\lambda}-2e^{\lambda}-3\lambda e^{\lambda}+\lambda^2+\lambda^2 e^{\lambda}+1)}{(e^{\lambda}-1)^3},
\end{aligned}$$

$\mu_{(0,s)}$ is proved by

$$\begin{aligned}
\mu_{(0,s)} &= E(X_tX_tX_{t+s}) \\
&= E(X_tX_t[\alpha_{t+s}\circ X_{t+s-1}+\varepsilon_{t+s}]) \\
&= \alpha(1-e^{-\lambda})E(X_tX_tX_{t+s-1})+E(X_t^2)E(\varepsilon_{t+s}) \\
&= \alpha(1-e^{-\lambda})\mu_{(0,s-1)}+\mu_{(0)}\mu_\varepsilon \\
&= (\alpha(1-e^{-\lambda}))^s\mu_{(0,0)}+\mu_{(0)}\mu_\varepsilon\frac{1-(\alpha(1-e^{-\lambda}))^s}{1-\alpha(1-e^{-\lambda})} \\
&= (\alpha(1-e^{-\lambda}))^s\mu_{(0,0)}+\mu_{(0)}\mu_X[1-(\alpha(1-e^{-\lambda}))^s] \\
&= (\alpha(1-e^{-\lambda}))^s[\mu_{(0,0)}-\mu_{(0)}\mu_X]+\mu_{(0)}\mu_X.
\end{aligned}$$

After substituting about $\mu_{(0,0)}$, $\mu_{(0)}$ and $\mu_X$ and simplifying, we get
$\mu_{(0,s)}=\frac{\lambda^2e^{2\lambda}(\lambda+1)}{(e^\lambda-1)^2}-\frac{\lambda e^\lambda(\alpha-\alpha e^{-\lambda})^s(3\lambda-e^\lambda-2\lambda e^\lambda+\lambda^2+1)}{(e^\lambda-1)^2}$,
hence,

$$\begin{aligned}
C_3(0,s) &= E[(X_t-\mu)^2(X_{t+s}-\mu)] \\
&= \mu_{(0,s)}-2\mu_X\mu_{(s)}-\mu_X\mu_{(0)}+2\mu_X^3 \\
&= (1-e^{-\lambda})\mu_{(0,s-1)}+\mu_{(0)}\mu_\varepsilon \\
&\quad -2\mu_X[\alpha(1-e^{-\lambda})\mu_{(s-1)}+\mu_X\mu_\varepsilon]-\mu_X\mu_{(0)}+2\mu_X^3 \\
&= \alpha(1-e^{-\lambda})\mu_{(0,s-1)}+\mu_{(0)}(\mu_X-\alpha\lambda) \\
&\quad -2\alpha(1-e^{-\lambda})\mu_X\mu_{(s-1)}-2\mu_X^2\mu_\varepsilon-\mu_X\mu_{(0)}+2\mu_X^3 \\
&= \alpha(1-e^{-\lambda})\mu_{(0,s-1)}-2\alpha(1-e^{-\lambda})\mu_X\mu_{(s-1)} \\
&\quad -\alpha\lambda\mu_{(0)}-2\mu_X^2\mu_\varepsilon+2\mu_X^3 \\
&= \alpha(1-e^{-\lambda})\mu_{(0,s-1)}-2\alpha(1-e^{-\lambda})\mu_X\mu_{(s-1)} \\
&\quad -\alpha\lambda\mu_{(0)}-2\mu_X^2(\mu_X-\alpha\lambda)+2\mu_X^3 \\
&= \alpha(1-e^{-\lambda})\mu_{(0,s-1)}-2\alpha(1-e^{-\lambda})\mu_X\mu_{(s-1)} \\
&\quad -\alpha\lambda\mu_{(0)}-2\mu_X^3+2\alpha\lambda\mu_X^2+2\mu_X^3 \\
&= \alpha(1-e^{-\lambda})[\mu_{(0,s-1)}-2\mu_X\mu_{(s-1)}-\frac{\lambda\mu_{(0)}}{1-e^{-\lambda}} \\
&\quad +\frac{2\lambda\mu_X^2}{1-e^{-\lambda}}] \\
&= \alpha(1-e^{-\lambda})[\mu_{(0,s-1)}-2\mu_X\mu_{(s-1)}-\frac{\lambda e^\lambda}{e^\lambda-1}\mu_{(0)} \\
&\quad +\frac{2\lambda e^\lambda}{e^\lambda-1}\mu_X^2] \\
&= \alpha(1-e^{-\lambda})[\mu_{(0,s-1)}-2\mu_X\mu_{(s-1)}-\mu_X\mu_{(0)}+2\mu_X^3] \\
&= \alpha(1-e^{-\lambda})C_3(0,s-1).
\end{aligned}$$

$\mu_{(s,s)}$ is proved by

$$\begin{aligned}
\mu_{(s,s)} &= E(X_tX_{t+s}^2)=E(X_t[\alpha_{t+s}\circ X_{t+s-1}+\varepsilon_{t+s}]^2) \\
&= \alpha^2(1-e^{-\lambda})E(X_tX_{t+s-1}^2) \\
&\quad +\alpha(1-\alpha)(1-e^{-\lambda})E(X_tX_{t+s-1}) \\
&\quad +E(X_t)E(\varepsilon_{t+s}^2) \\
&\quad +2\alpha(1-e^{-\lambda})E(X_tX_{t+s-1})E(\varepsilon_{t+s}) \\
&= \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+\alpha(1-\alpha)(1-e^{-\lambda})\mu_{(s-1)} \\
&\quad +\mu_XE(\varepsilon^2)+2\alpha(1-e^{-\lambda})\mu_{(s-1)}\mu_\varepsilon \\
&= \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-\alpha)(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon]\mu_{(s-1)}+\mu_XE(\varepsilon^2) \\
&= \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-\alpha)(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon][(\alpha(1-e^{-\lambda}))^{s-1}[\mu_{(0)}-\mu_X^2] \\
&\quad +\mu_X^2]+\mu_XE(\varepsilon^2) \\
&= \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-\alpha)(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon](\alpha(1-e^{-\lambda}))^{s-1}[\mu_{(0)}-\mu_X^2] \\
&\quad +[\alpha(1-\alpha)(1-e^{-\lambda})+2\alpha(1-e^{-\lambda})\mu_\varepsilon]\mu_X^2 \\
&\quad +\mu_XE(\varepsilon^2)
\end{aligned}$$

By using the iterations, we get

$$\begin{aligned}
\mu_{(s,s)} &= (\alpha^2(1-e^{-\lambda}))^s\mu_{(0,0)}+[\alpha(1-\alpha)(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon][\mu_{(0)}-\mu_X^2]\sum_{i=0}^{s-1}(\alpha^2(1-e^{-\lambda}))^i \\
&\quad \times(\alpha(1-e^{-\lambda}))^{s-(i+1)}+[[\alpha(1-\alpha)(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon]\mu_X^2+\mu_XE(\varepsilon^2)]\sum_{j=0}^{s-1}(\alpha^2(1-e^{-\lambda}))^j \\
&= (\alpha^2(1-e^{-\lambda}))^s\mu_{(0,0)}+[\alpha(1-\alpha)(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon][\mu_{(0)}-\mu_X^2]\times \\
&\quad \frac{(\alpha(1-e^{-\lambda}))^s-(\alpha^2(1-e^{-\lambda}))^s}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})} \\
&\quad +([\alpha(1-\alpha)(1-e^{-\lambda})+2\alpha(1-e^{-\lambda})\mu_\varepsilon]\mu_X^2 \\
&\quad +\mu_XE(\varepsilon^2))\frac{1-(\alpha^2(1-e^{-\lambda}))^s}{1-\alpha^2(1-e^{-\lambda})},
\end{aligned}$$

hence,

$$\begin{aligned}
C_3(s,s) &= E[(X_t-\mu)(X_{t+s}-\mu)^2] \\
&= \mu_{(s,s)}-2\mu_X\mu_{(s)}-\mu_XE(X_{t+s}^2)+2\mu_X^3 \\
&= \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-e^{-\lambda}) \\
&\quad -\alpha^2(1-e^{-\lambda})+2\alpha(1-e^{-\lambda})\mu_\varepsilon]\mu_{(s-1)}+\mu_XE(\varepsilon^2) \\
&\quad -2\mu_X[\alpha(1-e^{-\lambda})\mu_{(s-1)}+\mu_X\mu_\varepsilon] \\
&\quad -\mu_XE[(\alpha\circ X_{t+s-1}+\varepsilon_{t+s})^2]+2\mu_X^3 \\
&= \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda}) \\
&\quad +2\alpha(1-e^{-\lambda})\mu_\varepsilon-2\mu_X\alpha(1-e^{-\lambda})]\mu_{(s-1)} \\
&\quad +\mu_XE(\varepsilon^2)+2\mu_X^3-2\mu_X^2\mu_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&-\mu_X[\alpha^2(1-e^{-\lambda})E(X_{t+s-1}^2)+\alpha(1-e^{-\lambda})E(X_{t+s-1})\\
&-\alpha^2(1-e^{-\lambda})E(X_{t+s-1})+E(\varepsilon^2)\\
&+2\alpha(1-e^{-\lambda})E(X_{t+s-1})E(\varepsilon_{t+s})]\\
=&\,\alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})\\
&+2\alpha(1-e^{-\lambda})(1-\alpha(1-e^{-\lambda}))\mu_X\\
&-2\mu_X\alpha(1-e^{-\lambda})]\mu_{(s-1)}+\mu_X E(\varepsilon^2)+2\mu_X^3\\
&-2\mu_X^3(1-\alpha(1-e^{-\lambda}))-\alpha^2(1-e^{-\lambda})\mu_X E(X_{t+s-1}^2)\\
&-\alpha(1-e^{-\lambda})\mu_X^2+\alpha^2(1-e^{-\lambda})\mu_X^2-\mu_X E(\varepsilon^2)\\
&-2\alpha(1-e^{-\lambda})(1-\alpha(1-e^{-\lambda}))\mu_X^3\\
=&\,\alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)}+[\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})\\
&-2\alpha^2(1-e^{-\lambda})\mu_X+2\alpha\mu_X-2\alpha e^{-\lambda}\mu_X\\
&+2\alpha^2e^{-\lambda}(1-e^{-\lambda})\mu_X-2\mu_X\alpha(1-e^{-\lambda})]\mu_{(s-1)}\\
&+\mu_X E(\varepsilon^2)+2\mu_X^3-2\mu_X^3+2\alpha(1-e^{-\lambda})\mu_X^3\\
&-\alpha^2(1-e^{-\lambda})\mu_X E(X_{t+s-1}^2)-\alpha(1-e^{-\lambda})\mu_X^2\\
&+\alpha^2(1-e^{-\lambda})\mu_X^2-\mu_X E(\varepsilon^2)-2\alpha\mu_X^3\\
&+2\alpha^2(1-e^{-\lambda})\mu_X^3+2\alpha e^{-\lambda}\mu_X^3-2\alpha^2e^{-\lambda}(1-e^{-\lambda})\mu_X^3\\
=&\,\alpha^2(1-e^{-\lambda})[\mu_{(s-1,s-1)}-2\mu_X\mu_{(s-1)}-\mu_X E(X_{t+s-1}^2)\\
&+2\mu_X^3]+\alpha(1-e^{-\lambda})(1-\alpha)[\mu_{(s-1)}-\mu_X^2]\\
&+2\alpha^2\lambda e^{-\lambda}[\mu_{(s-1)}-\mu_X^2\\
=&\,\alpha^2(1-e^{-\lambda})C_3(s-1,s-1)+[\alpha(1-\alpha)(1-e^{-\lambda})\\
&+2\alpha^2\lambda e^{-\lambda}]C_2(s-1)\\
=&\,\alpha^2(1-e^{-\lambda})C_3(s-1,s-1)+[\alpha(1-\alpha)(1-e^{-\lambda})\\
&+2\alpha^2\lambda e^{-\lambda}](\alpha(1-e^{-\lambda}))^{s-1}C_2(0)
\end{aligned}
$$

For simplicity, take $g=\alpha(1-\alpha)(1-e^{-\lambda})+2\alpha^2\lambda e^{-\lambda}$ and by using the iterations, we get

$$
\begin{aligned}
&C_3(s,s)=(\alpha^2(1-e^{-\lambda}))^sC_3(0,0)+gC_2(0)\times\\
&\sum_{i=0}^{s-1}(\alpha(1-e^{-\lambda}))^{s-(i+1)}.(\alpha^2(1-e^{-\lambda}))^i\\
&=(\alpha^2(1-e^{-\lambda}))^sC_3(0,0)+gC_2(0)\times\\
&\frac{(\alpha(1-e^{-\lambda}))^s-(\alpha^2(1-e^{-\lambda}))^s}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}.
\end{aligned}
$$

$\mu_{(s,\tau)}$ is calculated as

$$
\begin{aligned}
&\mu_{(s,\tau)}=E(X_tX_{t+s}X_{t+\tau})=E(X_tX_{t+s}(\alpha_{t+\tau}\circ X_{t+\tau-1}+\varepsilon_{t+\tau}))\\
&=\alpha(1-e^{-\lambda})E(X_tX_{t+s}X_{t+\tau-1})+E(X_tX_{t+s})\mu_\varepsilon\\
&=\alpha(1-e^{-\lambda})\mu_{(s,\tau-1)}+\mu_{(s)}\mu_\varepsilon=(\alpha(1-e^{-\lambda}))^{\tau-s}\mu_{(s,s)}\\
&+\mu_{(s)}\mu_\varepsilon\frac{1-(\alpha(1-e^{-\lambda}))^{\tau-s}}{1-\alpha(1-e^{-\lambda})}\\
&=(\alpha(1-e^{-\lambda}))^{\tau-s}\mu_{(s,s)}+\mu_{(s)}\mu_X[1-(\alpha(1-e^{-\lambda}))^{\tau-s}]\\
&=(\alpha(1-e^{-\lambda}))^{\tau-s}(\mu_{(s,s)}-\mu_{(s)}\mu_X)+\mu_{(s)}\mu_X,
\end{aligned}
$$

and then,

$$
\begin{aligned}
&C_3(s,\tau)\\
&=\mu_{(s,\tau)}-\mu_X\mu_{(s)}-\mu_X\mu_{(\tau)}-\mu_XE(X_{t+\tau}X_{t+s})+2\mu_X^3\\
&=\alpha(1-e^{-\lambda})\mu_{(s,\tau-1)}+\mu_{(s)}\mu_\varepsilon-\mu_X\mu_{(s)}\\
&-\mu_X[\alpha(1-e^{-\lambda})\mu_{(\tau-1)}+\mu_X\mu_\varepsilon]\\
&-\mu_XE[(\alpha\circ X_{t+\tau-1})X_{t+s}]+2\mu_X^3\\
&=\alpha(1-e^{-\lambda})\mu_{(s,\tau-1)}+\mu_{(s)}(1-\alpha(1-e^{-\lambda}))\mu_X\\
&-\mu_X\mu_{(s)}-\alpha(1-e^{-\lambda})\mu_X\mu_{(\tau-1)}-\mu_X^3(1-\alpha(1-e^{-\lambda}))\\
&-\alpha(1-e^{-\lambda})\mu_XE(X_{t+\tau-1}X_{t+s})-\mu_X^2\mu_\varepsilon+2\mu_X^3\\
&=\alpha(1-e^{-\lambda})\mu_{(s,\tau-1)}-\alpha(1-e^{-\lambda})\mu_{(s)}\mu_X\\
&-\alpha(1-e^{-\lambda})\mu_X\mu_{(\tau-1)}+\alpha(1-e^{-\lambda})\mu_X^3\\
&-\alpha(1-e^{-\lambda})\mu_XE(X_{t+\tau-1}X_{t+s})+\alpha(1-e^{-\lambda})\mu_X^3\\
&=\alpha(1-e^{-\lambda})[\mu_{(s,\tau-1)}-\mu_X\mu_{(s)}-\mu_X\mu_{(\tau-1)}\\
&-\mu_XE(X_{t+\tau-1}X_{t+s})+2\mu_X^3]\\
&=\alpha(1-e^{-\lambda})C_3(s,\tau-1)\\
&=(\alpha(1-e^{-\lambda}))^{\tau-s}C_3(s,s).
\end{aligned}
$$

## 4 Spectral and Bispectral Density Functions

The non-normalized spectral density function $f_{XX}(\omega)$ of ZTPINAR(1) is calculated as (see [14])

$$
f_{XX}(\omega)=\frac{1-\alpha^2(1-e^{-\lambda})^2}{2\pi(1+\alpha^2(1-e^{-\lambda})^2-2\alpha(1-e^{-\lambda})\cos\omega)}. \tag{5}
$$

The normalized spectral density function $g_{XX}(\omega)$ is calculated as

$$
g_{XX}(\omega)=\frac{\lambda e^{\lambda}(e^{\lambda}-\lambda-1)(1-\alpha^2(1-e^{-\lambda})^2)}{2\pi(e^{\lambda}-1)^2(1+\alpha^2(1-e^{-\lambda})^2-2\alpha(1-e^{-\lambda})\cos\omega)}. \tag{6}
$$

**Theorem 2.** *The bispectral density function $f_{XXX}(\omega_1,\omega_2)$ of ZTPINAR(1) is calculated as*

$$
\begin{aligned}
&f_{XXX}(\omega_1,\omega_2)=\frac{1}{(2\pi)^2}[C_3(0,0)+C_3(0,0)\times\\
&\{H_1(-\omega_1)+H_1(-\omega_2)+H_1(\omega_1+\omega_2)\}\\
&+(C_3(0,0)-\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})})\times\\
&\{H_2(\omega_1)+H_2(\omega_2)+H_2(-\omega_1-\omega_2)\}\\
&+(\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})})\times
\end{aligned}
$$

$$\{H_1(\omega_1)+H_1(\omega_2)+F_1(-\omega_1-\omega_2)\}$$
$$+(C_3(0,0)-\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})})\times$$
$$\{H_2(-\omega_1-\omega_2)H_1(-\omega_2)+H_2(-\omega_1-\omega_2)H_1(-\omega_1)$$
$$+H_2(\omega_2)H_1(-\omega_2)+H_2(\omega_2)H_1(-\omega_1)$$
$$+H_2(\omega_1)H_1(\omega_1+\omega_2)+H_2(\omega_2)H_1(\omega_1+\omega_2)\}$$
$$+(\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})})$$
$$\times\{H_1(-\omega_{-}\omega_2)H_1(-\omega_2)+H_1(-\omega_1-\omega_2)H_1(-\omega_1)$$
$$+H_1(\omega_1)H_1(-\omega_2)+H_1(\omega_2)H_1(-\omega_1)$$
$$+H_1(\omega_1)H_1(\omega_1+\omega_2)+H_1(\omega_2)H_1(\omega_1+\omega_2)\}], \tag{7}$$

*where*

$$H_1(\omega_k)=\frac{\alpha(1-e^{-\lambda})e^{i\omega_k}}{1-\alpha(1-e^{-\lambda})e^{i\omega_k}}$$

*and*

$$H_2(\omega_k)=\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_k}}{1-(\alpha^2(1-e^{-\lambda})e^{i\omega_k}}, \qquad k=1,2.$$

*Proof.*We can write $f_{XXX}(\omega_1,\omega_2)$ as (see [14]):

$$f_{XXX}(\omega_1,\omega_2)=\frac{1}{(2\pi)^2}[\sum_{t_1=0}^{\infty}\sum_{t_2=t_1}^{\infty}C_3(t_1,t_2)e^{-i(t_1\omega_1+t_2\omega_2)}$$
$$+\sum_{t_2=0}^{\infty}\sum_{t_1=t_2+1}^{\infty}C_3(t_2,t_1)e^{-i(t_1\omega_1+t_2\omega_2)}$$
$$+\sum_{t_1=0}^{\infty}\sum_{t_2=-\infty}^{-1}C_3(-t_2,t_1-t_2)e^{-i(t_1\omega_1+t_2\omega_2)}$$
$$+\sum_{t_1=-\infty}^{-1}\sum_{t_2=-\infty}^{t_1-1}C_3(t_1-t_2,-t_2)e^{-i(t_1\omega_1+t_2\omega_2)}$$
$$+\sum_{t_2=-\infty}^{-1}\sum_{t_1=-\infty}^{-t_2}C_3(t_2-t_1,-t_1)e^{-i(t_1\omega_1+t_2\omega_2)}$$
$$+\sum_{t_1=-\infty}^{-1}\sum_{t_2=0}^{\infty}C_3(-t_1,t_2-t_1)e^{-i(t_1\omega_1+t_2\omega_2)}]$$

using the symmetry properties of the third-order cumulants (see [9]) in the equation above, then

$$f_{XXX}(\omega_1,\omega_2)=\frac{1}{(2\pi)^2}[C_3(0,0)$$
$$+\sum_{\tau=1}^{\infty}C_3(0,\tau)\{e^{-i\tau\omega_1}+e^{-i\tau\omega_2}+e^{i\tau(\omega_1+\omega_2)}\}$$
$$+\sum_{\tau=1}^{\infty}C_3(\tau,\tau)\{e^{i\tau\omega_1}+e^{i\tau\omega_2}+e^{-i\tau(\omega_1+\omega_2)}\}$$
$$+\sum_{s=1}^{\infty}\sum_{\tau=1}^{\infty}C_3(s,s+\tau)\{e^{-is\omega_1-i(s+\tau)\omega_2}$$
$$+e^{-is\omega_2-i(s+\tau)\omega_1}+e^{is\omega_1-i\tau\omega_2}e^{is\omega_2-i\tau\omega_1}$$
$$+e^{i\tau\omega_1+i(s+\tau)\omega_2}+e^{i\tau\omega_2+i(s+\tau)\omega_1}\}]$$

Using expressions of $C_3(0,\tau),C_3(\tau,\tau)$ and $C_3(s,s+\tau)=(\alpha(1-e^{-\lambda}))^{\tau}C_3(s,s)$ given by Theorem 1, we get

$$f_{XXX}(\omega_1,\omega_2)=\frac{1}{(2\pi)^2}[C_3(0,0)+\sum_{\tau=1}^{\infty}(\alpha(1-e^{-\lambda}))^{\tau}\times$$
$$C_3(0,0)\{e^{-i\tau\omega_1}+e^{-i\tau\omega_2}+e^{i\tau(\omega_1+\omega_2)}\}$$
$$+\sum_{\tau=1}^{\infty}[(\alpha^2(1-e^{-\lambda}))^{\tau}C_3(0,0)+gC_2(0)\times$$
$$\frac{(\alpha(1-e^{-\lambda}))^{\tau}-(\alpha^2(1-e^{-\lambda}))^{\tau}}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}]$$
$$\{e^{i\tau\omega_1}+e^{i\tau\omega_2}+e^{-i\tau(\omega_1+\omega_2)}\}$$
$$+\sum_{s=1}^{\infty}\sum_{\tau=1}^{\infty}(\alpha(1-e^{-\lambda}))^{\tau}[(\alpha^2(1-e^{-\lambda}))^{s}C_3(0,0)$$
$$+gC_2(0)\frac{(\alpha(1-e^{-\lambda}))^{s}-(\alpha^2(1-e^{-\lambda}))^{s}}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}]\times$$
$$\{e^{-is\omega_1-i(s+\tau)\omega_2}+e^{-is\omega_2-i(s+\tau)\omega_1}+e^{is\omega_1-i\tau\omega_2}$$
$$+e^{is\omega_2-i\tau\omega_1}+e^{i\tau\omega_1+i(s+\tau)\omega_2}+e^{i\tau\omega_2+i(s+\tau)\omega_1}\}],$$

All these summations can be evaluated as follows, for example

$$\sum_{\tau=1}^{\infty}(\alpha(1-e^{-\lambda})^{\tau}e^{-i\tau\omega_1}=\sum_{\tau=1}^{\infty}(\alpha(1-e^{-\lambda})e^{-i\omega_1})^{\tau}$$
$$=\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}},$$

so after some calculations and compuations for all summations, we have

$$f_{XXX}(\omega_1,\omega_2)=\frac{1}{(2\pi)^2}[C_3(0,0)+C_3(0,0)\times$$
$$\{\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}}+\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}}$$
$$+\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}\}$$
$$+(C_3(0,0)-\frac{gc_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})})\times$$
$$\{\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_1}}+\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_2}}$$
$$+\frac{\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}$$
$$+\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}\{\frac{\alpha(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{i\omega_1}}\}$$
$$+\frac{\alpha(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{i\omega_2}}$$
$$+\frac{\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}\}+(C_3(0,0)$$
$$-\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})})\times$$
$$\{\frac{\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}}$$
$$+\frac{\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}}+\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_1}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}}+\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_2}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}}+\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_1}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}+\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_2}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}\}+\frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}\times$$
$$\{\frac{\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})^{-i(\omega_1+\omega_2)}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}}+\frac{\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}}+\frac{\alpha(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha e^{i\omega_1}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}}+\frac{\alpha(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{i\omega_2}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}}+\frac{\alpha(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{i\omega_1}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}+\frac{\alpha(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{i\omega_2}}\times$$
$$\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}\}],$$

by taking

$$H_1(\omega_k)=\frac{\alpha(1-e^{-\lambda})e^{i\omega_k}}{1-\alpha(1-e^{-\lambda})e^{i\omega_k}}$$

and

$$H_2(\omega_k)=\frac{\alpha^2(1-e^{-\lambda})e^{i\omega_k}}{1-(\alpha^2(1-e^{-\lambda})e^{i\omega_k}}, \qquad k=1,2.,$$

the proof is complete.

The normalized bispectral density function $g_{XXX}(\omega_1,\omega_2)$ is calculated as

$$g_{XXX}(\omega_1,\omega_2)=\frac{f_{XXX}(\omega_1,\omega_2)}{\sqrt{f_{XX}(\omega_1)f_{XX}(\omega_2)f_{XX}(\omega_1+\omega_2)}}, \quad (8)$$

where $f_{XXX}(\omega_1,\omega_2)$ and $f_{XX}(\omega_1)$ are defined in (5) and (7).

The bispectral density function provides us useful information about the non-linearity of the process. For continuous non- Gaussian time series the modulus of the normalized bispectrum is flat. The bispectral density function is a complex valued function takes the form

$$f_{XXX}(\omega_1,\omega_2)=r(\omega_1,\omega_2)+iq(\omega_1,\omega_2),$$

The modulus and phase of the bispectral density function are given respectively, by

$$|f_{XXX}(\omega_1,\omega_2)|=\sqrt{r^2(\omega_1,\omega_2)+q^2(\omega_1,\omega_2)}, \quad (9)$$

$$phase=tan^{-1}(\frac{q(\omega_1,\omega_2)}{r(\omega_1,\omega_2)}). \quad (10)$$

Fig.1 illustrate the simulated series of the ZTPINAR(1) model at $\lambda=1$ and $\alpha=0.35$. From Fig.1, we conclude that the process is stationary and values of the simulated series are non negative integers, so the plot satisfy the definition of the ZTPINAR(1) model. The theoretical spectrum $f_{XX}(\omega)$, theoretical bispectrum $f_{XXX}(\omega_1,\omega_2)$ and normalized bispectrum modulus $g_{XXX}(\omega_1,\omega_2)$ are respectively computed by setting $\lambda=1$ and $\alpha=0.35$ in (5), (7) and (8), and they are represented by Fig.2, Fig.3 and Fig.4, respectively.

## 5 Estimation of Spectrum and Bispectrum

Estimates of the spectral, bispectral and normalized bispectral density functions are calculated using the smoothed periodogram method with different lag windows (as Daniell, Tukey Hamming and Parzen lag window) and simulated series $\{X_t, t=1,2,...,500\}$ from the ZTPINAR(1) model that defined by (3). Generally if $X_1,X_2,...,X_N$ be a relizations of a real valued third order stationary process $\{X_t\}$ with mean $\mu$, autocovariance

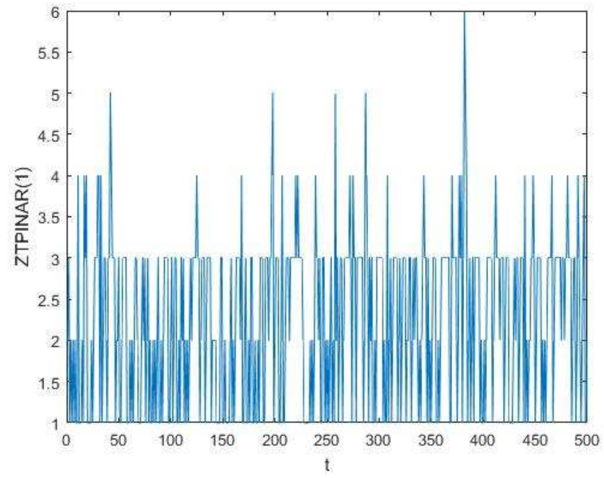


**Fig. 1:** The simulated series of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

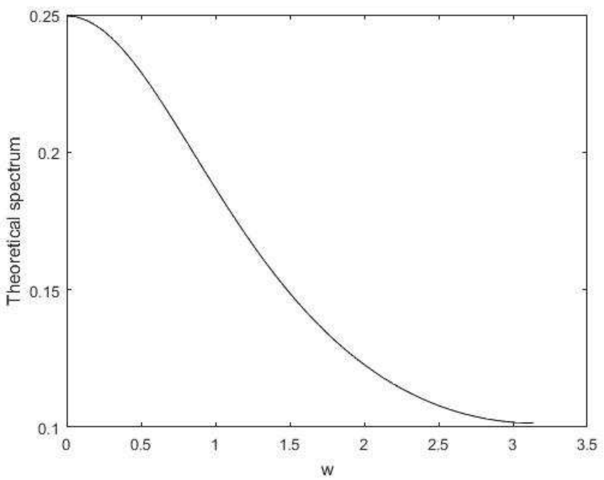


**Fig. 2:** The theoretical spectraum of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

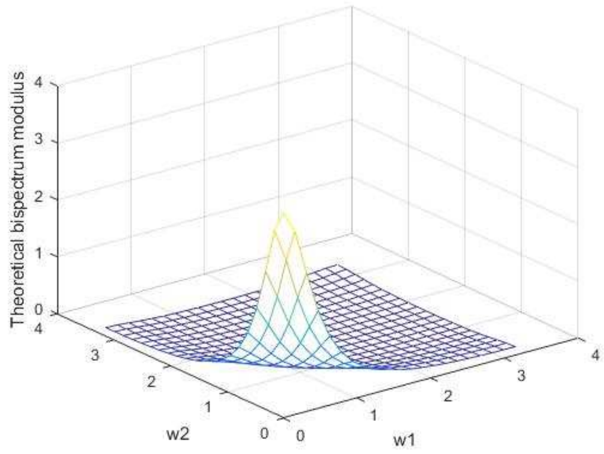


**Fig. 3:** The theoretical bispectral of the ZTPNAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

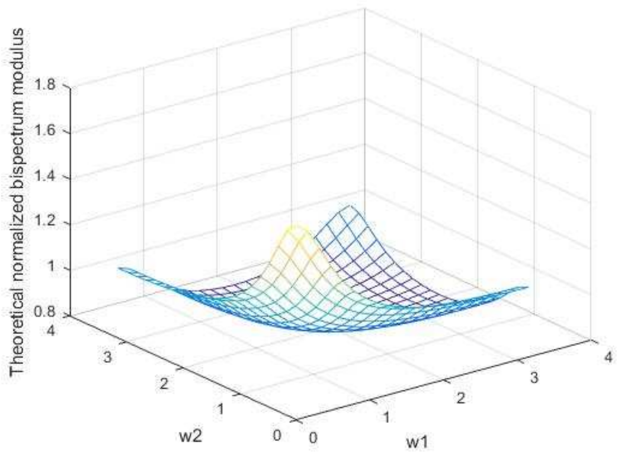


**Fig. 4:** The theoretical normalized bispectral of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

$C_2(s)$ and third cumulant $C_3(s_1, s_2)$. The smoothed spectral and bispectral density functions are respectively given by (see [26])

$$\hat{f}(\omega) = \frac{1}{2\pi} \sum_{s=-(N-1)}^{N-1} \lambda(s)\hat{C}_2(s)e^{-isw} = \frac{1}{2\pi} \sum_{s=-(N-1)}^{N-1} \lambda(s)\hat{C}_2(s)cos\omega s, \quad (11)$$

$$\hat{f}(\omega_1, \omega_2) = \frac{1}{4\pi^2} \sum_{s_1=-(N-1)}^{N-1} \sum_{s_2=-(N-1)}^{N-1} \lambda(s_1, s_2)\hat{C}_3(s_1, s_2)e^{-is_1\omega_1 - is_2\omega_2}, \quad (12)$$

where $\hat{C}_2(s)$ and $\hat{C}_3(s_1, s_2)$ the natural estimators for estimators for $C_2(s)$ and $C_3(s_1, s_2)$ are, respectively, given by

$$\hat{C}_2(s) = \frac{1}{N-s} \sum_{t=1}^{N-|s|} (X_t - \bar{X})(X_{t+|s|} - \bar{X}), \quad (13)$$

$$\bar{X} = \frac{1}{N}\sum_{t=1}^{N} X_t,$$

$$\hat{C}_3(s_1,s_2) = \frac{1}{N}\sum_{t=1}^{N-\gamma}(X_t-\bar{X})(X_{t+s_1}-\bar{X})(X_{t+s_2}-\bar{X}) \quad (14)$$

where $s_1,s_2 \geq 0, \gamma = max(0,s_1,s_2), s = 0,\pm1,\pm2,...,\pm(N-1), -\pi \leq \omega_1,\omega_2 \leq \pi$,"$\lambda(.)$"is one-dimensional lag window and "$\lambda(s_1,s_2)$" is two-dimensional lag window.

The normalized bispectrum is estimated by

$$\hat{g}(\omega_1,\omega_2) = \frac{\hat{f}(\omega_1,\omega_2)}{\sqrt{\hat{f}(\omega_1)\hat{f}(\omega_2)\hat{f}(\omega_1+\omega_2)}} \quad (15)$$

To compare the spectral estimates with different lag windows With each other and with the theoretical spectrum, we used the sample mean square errors criterion for measuring the accuracy of $\hat{f}(\omega)$ as an estimate of $f_{XX}(\omega)$. This sample mean square error (M.S.E) is defined as

$$M.S.E = \frac{1}{k}\sum_{i}^{k}(\hat{f}(\omega_i) - f_{XX}(\omega_i))^2$$

where $\hat{f}(\omega_i)$ and $f_{XX}(\omega_i)$ are given by (11) and (6) respectively. k is the number of frequencies $\omega_i$. As for the bispectral case, the M.S.E is defined as

$$M.S.E = \frac{1}{K}\sum_{i=1}^{k}\sum_{j=1}^{k}(|\hat{f}(\omega_i,\omega_j)| - |f_{XX}(\omega_i,\omega_j)|)^2,$$

where $|\hat{f}(\omega_i,\omega_j)|$ is the modulus of the bispectra1 density estimate and $|f_{XX}(\omega_i,\omega_j)|$ is the theoretical bispectral modulus. k is the number of frequencies $\omega_i$ or $\omega_j$ and $K = k^2$ is the total number of these frequencies $(\omega_i,\omega_j)$.

**Firstly** using the Daniell lag window, [28] introduced Daniell lag window as

$$\lambda(s) = \frac{sin(\frac{s\pi}{M})}{\frac{s\pi}{M}} \quad (16)$$

where M is window parmeter or number of frequencies used in smoothed. In this paper we choose M=7 for all different lag windows. The two dimensional lag window $\lambda(s_1,s_2)$, given by [29] is given by

$$\lambda(s_1,s_2) = \lambda(s_1)\lambda(s_2)\lambda(s_1-s_2) \quad (17)$$

Fig.5 represent the theoretical spectrum and the estimated spectral density function using Daniell window with M=7 from (11) and (16). Fig.6 and Fig.7 represents the estimate of the bispectrum and normalized bispectrum modulus using Daniell window at M=7 as in (12), (17) (16) and (15).

**Secondly** using the Parzen lag window, [30] proposed the Parzen lag window

$$\lambda(s) = \begin{cases} 1-6s^2+6|s|^3 & |s| \leq \frac{1}{2} \\ 2(1-|s|)^3 & \frac{1}{2} < |s| \leq 1, \\ 0 & |s| > 1 \end{cases} \quad (18)$$

and $\lambda(s_1,s_2)$ is given by (17). Fig.8 represent the theoretical spectrum and the estimated spectral density function using Parzen window with M=7 from (11) and (18). Fig.9 and Fig.10, represents the estimate of the bispectrum and normalized bispectrum modulus using Parzen window at M=7 as in (12), (17) (18) and (15).

**Thirdly and finally** using the Tukey Hamming window which reduced from [31] and given by

$$\lambda(s) = \begin{cases} 0.54+0.46\cos(\frac{\pi s}{M}) & |s| \leq M \\ 0 & |s| > M \end{cases}, \quad (19)$$

and $\lambda(s_1,s_2)$ is given by equation(17). Fig.11 represent the theoretical spectrum and the estimated spectral density function using Tukey Hamming window with M=7 from (11) and (19). Fig.12 and Fig.13, represents the estimate of the bispectrum and normalized bispectrum modulus using Tukey Hamming window at M=7 as in (12), (17) (19) and (15).

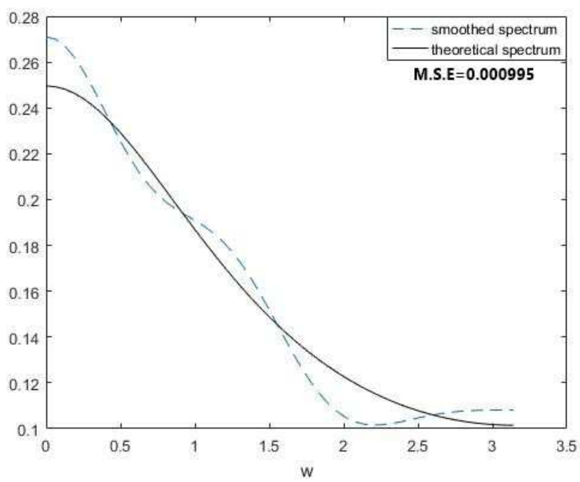


**Fig. 5:** Estimated spectrum using Daniell window at M=7 and theoretical spectrum.

**Table 1:** Theoretical bispectrum modulus of ZTPINAR(1) with $\lambda = 1$ and $\alpha = 0.35$.

| $\omega_1$ \ $\omega_2$ | **0.00π** | **0.05π** | **0.10π** | **0.15π** | **0.20π** | **0.25π** | **0.30π** | **0.35π** | **0.40π** | **0.45π** | **0.50π** | **0.55π** | **0.60π** | **0.65π** | **0.70π** | **0.75π** | **0.80π** | **0.85π** | **0.90π** | **0.95π** | π |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.00π** | 3.604 | 3.221 | 2.433 | 1.719 | 1.215 | 0.884 | 0.667 | 0.521 | 0.420 | 0.348 | 0.296 | 0.258 | 0.229 | 0.206 | 0.189 | 0.176 | 0.166 | 0.159 | 0.155 | 0.152 | 0.151 |
| **0.05π** | 3.221 | 2.646 | 1.933 | 1.365 | 0.979 | 0.725 | 0.556 | 0.441 | 0.361 | 0.303 | 0.261 | 0.229 | 0.205 | 0.186 | 0.172 | 0.161 | 0.153 | 0.148 | 0.144 | 0.142 | 0.142 |
| **0.10π** | 2.433 | 1.933 | 1.411 | 1.010 | 0.736 | 0.554 | 0.432 | 0.347 | 0.287 | 0.244 | 0.212 | 0.187 | 0.169 | 0.155 | 0.144 | 0.136 | 0.130 | 0.126 | 0.124 | 0.123 | 0.124 |
| **0.15π** | 1.719 | 1.365 | 1.010 | 0.735 | 0.545 | 0.416 | 0.329 | 0.267 | 0.224 | 0.191 | 0.168 | 0.150 | 0.136 | 0.125 | 0.117 | 0.112 | 0.107 | 0.105 | 0.103 | 0.103 | 0.105 |
| **0.20π** | 1.215 | 0.979 | 0.736 | 0.545 | 0.410 | 0.317 | 0.253 | 0.208 | 0.176 | 0.152 | 0.134 | 0.120 | 0.110 | 0.102 | 0.096 | 0.092 | 0.089 | 0.087 | 0.087 | 0.087 | 0.089 |
| **0.25π** | 0.884 | 0.725 | 0.554 | 0.416 | 0.317 | 0.248 | 0.200 | 0.166 | 0.141 | 0.123 | 0.109 | 0.099 | 0.091 | 0.085 | 0.081 | 0.078 | 0.076 | 0.075 | 0.075 | 0.076 | 0.078 |
| **0.30π** | 0.667 | 0.556 | 0.432 | 0.329 | 0.253 | 0.200 | 0.163 | 0.136 | 0.117 | 0.103 | 0.092 | 0.084 | 0.077 | 0.073 | 0.069 | 0.067 | 0.066 | 0.065 | 0.066 | 0.067 | 0.069 |
| **0.35π** | 0.521 | 0.441 | 0.347 | 0.267 | 0.208 | 0.166 | 0.136 | 0.115 | 0.099 | 0.088 | 0.079 | 0.072 | 0.067 | 0.064 | 0.061 | 0.060 | 0.059 | 0.059 | 0.060 | 0.061 | 0.064 |
| **0.40π** | 0.420 | 0.361 | 0.287 | 0.224 | 0.176 | 0.141 | 0.117 | 0.099 | 0.086 | 0.077 | 0.070 | 0.064 | 0.060 | 0.057 | 0.055 | 0.054 | 0.054 | 0.054 | 0.055 | 0.057 | 0.060 |
| **0.45π** | 0.348 | 0.303 | 0.244 | 0.191 | 0.152 | 0.123 | 0.103 | 0.088 | 0.077 | 0.069 | 0.063 | 0.058 | 0.055 | 0.052 | 0.051 | 0.050 | 0.050 | 0.051 | 0.052 | 0.055 | 0.058 |
| **0.50π** | 0.296 | 0.261 | 0.212 | 0.168 | 0.134 | 0.109 | 0.092 | 0.079 | 0.070 | 0.063 | 0.057 | 0.054 | 0.051 | 0.049 | 0.048 | 0.048 | 0.048 | 0.049 | 0.051 | 0.054 | 0.057 |
| **0.55π** | 0.258 | 0.229 | 0.187 | 0.150 | 0.120 | 0.099 | 0.084 | 0.072 | 0.064 | 0.058 | 0.054 | 0.050 | 0.048 | 0.047 | 0.046 | 0.046 | 0.047 | 0.048 | 0.050 | 0.054 | 0.058 |
| **0.60π** | 0.229 | 0.205 | 0.169 | 0.136 | 0.110 | 0.091 | 0.077 | 0.067 | 0.060 | 0.055 | 0.051 | 0.048 | 0.046 | 0.045 | 0.045 | 0.045 | 0.046 | 0.048 | 0.051 | 0.055 | 0.060 |
| **0.65π** | 0.206 | 0.186 | 0.155 | 0.125 | 0.102 | 0.085 | 0.073 | 0.064 | 0.057 | 0.052 | 0.049 | 0.047 | 0.045 | 0.044 | 0.044 | 0.045 | 0.047 | 0.049 | 0.052 | 0.057 | 0.064 |
| **0.70π** | 0.189 | 0.172 | 0.144 | 0.117 | 0.096 | 0.081 | 0.069 | 0.061 | 0.055 | 0.051 | 0.048 | 0.046 | 0.045 | 0.044 | 0.045 | 0.046 | 0.048 | 0.051 | 0.055 | 0.061 | 0.069 |
| **0.75π** | 0.176 | 0.161 | 0.136 | 0.112 | 0.092 | 0.078 | 0.067 | 0.060 | 0.054 | 0.050 | 0.048 | 0.046 | 0.045 | 0.045 | 0.046 | 0.048 | 0.050 | 0.054 | 0.060 | 0.067 | 0.078 |
| **0.80π** | 0.166 | 0.153 | 0.130 | 0.107 | 0.089 | 0.076 | 0.066 | 0.059 | 0.054 | 0.050 | 0.048 | 0.047 | 0.046 | 0.047 | 0.048 | 0.050 | 0.054 | 0.059 | 0.066 | 0.076 | 0.089 |
| **0.85π** | 0.159 | 0.148 | 0.126 | 0.105 | 0.087 | 0.075 | 0.065 | 0.059 | 0.054 | 0.051 | 0.049 | 0.048 | 0.048 | 0.049 | 0.051 | 0.054 | 0.059 | 0.065 | 0.075 | 0.087 | 0.105 |
| **0.90π** | 0.155 | 0.144 | 0.124 | 0.103 | 0.087 | 0.075 | 0.066 | 0.060 | 0.055 | 0.052 | 0.051 | 0.050 | 0.051 | 0.052 | 0.055 | 0.060 | 0.066 | 0.075 | 0.087 | 0.103 | 0.124 |
| **0.95π** | 0.152 | 0.142 | 0.123 | 0.103 | 0.087 | 0.076 | 0.067 | 0.061 | 0.057 | 0.055 | 0.054 | 0.054 | 0.055 | 0.057 | 0.061 | 0.067 | 0.076 | 0.087 | 0.103 | 0.123 | 0.142 |
| π | 0.151 | 0.142 | 0.124 | 0.105 | 0.089 | 0.078 | 0.069 | 0.064 | 0.060 | 0.058 | 0.057 | 0.058 | 0.060 | 0.064 | 0.069 | 0.078 | 0.089 | 0.105 | 0.124 | 0.142 | 0.151 |

**Table 2:** Theoretical normalized bispectrum modulus of ZTPINAR(1) with $\lambda = 1$ and $\alpha = 0.35$.

| $\omega_1$ \ $\omega_2$ | **0.00π** | **0.05π** | **0.10π** | **0.15π** | **0.20π** | **0.25π** | **0.30π** | **0.35π** | **0.40π** | **0.45π** | **0.50π** | **0.55π** | **0.60π** | **0.65π** | **0.70π** | **0.75π** | **0.80π** | **0.85π** | **0.90π** | **0.95π** | π |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.00π** | 1.654 | 1.615 | 1.526 | 1.434 | 1.356 | 1.297 | 1.252 | 1.218 | 1.192 | 1.172 | 1.156 | 1.144 | 1.135 | 1.127 | 1.121 | 1.117 | 1.113 | 1.110 | 1.109 | 1.108 | 1.107 |
| **0.05π** | 1.615 | 1.551 | 1.461 | 1.377 | 1.309 | 1.257 | 1.218 | 1.188 | 1.165 | 1.147 | 1.133 | 1.122 | 1.114 | 1.107 | 1.102 | 1.098 | 1.095 | 1.093 | 1.091 | 1.090 | 1.090 |
| **0.10π** | 1.526 | 1.461 | 1.381 | 1.307 | 1.247 | 1.201 | 1.165 | 1.139 | 1.118 | 1.102 | 1.089 | 1.079 | 1.072 | 1.066 | 1.061 | 1.057 | 1.055 | 1.053 | 1.052 | 1.051 | 1.052 |
| **0.15π** | 1.434 | 1.377 | 1.307 | 1.241 | 1.187 | 1.145 | 1.113 | 1.089 | 1.070 | 1.055 | 1.043 | 1.034 | 1.027 | 1.021 | 1.017 | 1.014 | 1.011 | 1.010 | 1.009 | 1.009 | 1.010 |
| **0.20π** | 1.356 | 1.309 | 1.247 | 1.187 | 1.138 | 1.099 | 1.069 | 1.046 | 1.028 | 1.014 | 1.004 | 0.995 | 0.988 | 0.983 | 0.979 | 0.976 | 0.974 | 0.973 | 0.973 | 0.973 | 0.974 |
| **0.25π** | 1.297 | 1.257 | 1.201 | 1.145 | 1.099 | 1.063 | 1.034 | 1.013 | 0.996 | 0.982 | 0.972 | 0.964 | 0.958 | 0.953 | 0.949 | 0.947 | 0.945 | 0.944 | 0.944 | 0.945 | 0.947 |
| **0.30π** | 1.252 | 1.218 | 1.165 | 1.113 | 1.069 | 1.034 | 1.007 | 0.986 | 0.970 | 0.958 | 0.948 | 0.940 | 0.934 | 0.930 | 0.927 | 0.924 | 0.923 | 0.923 | 0.923 | 0.924 | 0.927 |
| **0.35π** | 1.218 | 1.188 | 1.139 | 1.089 | 1.046 | 1.013 | 0.986 | 0.966 | 0.951 | 0.939 | 0.929 | 0.922 | 0.916 | 0.912 | 0.909 | 0.907 | 0.906 | 0.906 | 0.907 | 0.909 | 0.912 |
| **0.40π** | 1.192 | 1.165 | 1.118 | 1.070 | 1.028 | 0.996 | 0.970 | 0.951 | 0.936 | 0.924 | 0.915 | 0.908 | 0.903 | 0.899 | 0.896 | 0.895 | 0.894 | 0.895 | 0.896 | 0.899 | 0.903 |
| **0.45π** | 1.172 | 1.147 | 1.102 | 1.055 | 1.014 | 0.982 | 0.958 | 0.939 | 0.924 | 0.913 | 0.904 | 0.897 | 0.892 | 0.889 | 0.887 | 0.886 | 0.886 | 0.887 | 0.889 | 0.892 | 0.897 |
| **0.50π** | 1.156 | 1.133 | 1.089 | 1.043 | 1.004 | 0.972 | 0.948 | 0.929 | 0.915 | 0.904 | 0.896 | 0.889 | 0.885 | 0.882 | 0.880 | 0.879 | 0.880 | 0.882 | 0.885 | 0.889 | 0.896 |
| **0.55π** | 1.144 | 1.122 | 1.079 | 1.034 | 0.995 | 0.964 | 0.940 | 0.922 | 0.908 | 0.897 | 0.889 | 0.883 | 0.879 | 0.877 | 0.875 | 0.875 | 0.877 | 0.879 | 0.883 | 0.889 | 0.897 |
| **0.60π** | 1.135 | 1.114 | 1.072 | 1.027 | 0.988 | 0.958 | 0.934 | 0.916 | 0.903 | 0.892 | 0.885 | 0.879 | 0.876 | 0.873 | 0.873 | 0.873 | 0.876 | 0.879 | 0.885 | 0.892 | 0.903 |
| **0.65π** | 1.127 | 1.107 | 1.066 | 1.021 | 0.983 | 0.953 | 0.930 | 0.912 | 0.899 | 0.889 | 0.882 | 0.877 | 0.873 | 0.872 | 0.872 | 0.873 | 0.877 | 0.882 | 0.889 | 0.899 | 0.912 |
| **0.70π** | 1.121 | 1.102 | 1.061 | 1.017 | 0.979 | 0.949 | 0.927 | 0.909 | 0.896 | 0.887 | 0.880 | 0.875 | 0.873 | 0.872 | 0.873 | 0.875 | 0.880 | 0.887 | 0.896 | 0.909 | 0.927 |
| **0.75π** | 1.117 | 1.098 | 1.057 | 1.014 | 0.976 | 0.947 | 0.924 | 0.907 | 0.895 | 0.886 | 0.879 | 0.875 | 0.873 | 0.873 | 0.875 | 0.879 | 0.886 | 0.895 | 0.907 | 0.924 | 0.947 |
| **0.80π** | 1.113 | 1.095 | 1.055 | 1.011 | 0.974 | 0.945 | 0.923 | 0.906 | 0.894 | 0.886 | 0.880 | 0.877 | 0.876 | 0.877 | 0.880 | 0.886 | 0.894 | 0.906 | 0.923 | 0.945 | 0.974 |
| **0.85π** | 1.110 | 1.093 | 1.053 | 1.010 | 0.973 | 0.944 | 0.923 | 0.906 | 0.895 | 0.887 | 0.882 | 0.879 | 0.879 | 0.882 | 0.887 | 0.895 | 0.906 | 0.923 | 0.944 | 0.973 | 1.010 |
| **0.90π** | 1.109 | 1.091 | 1.052 | 1.009 | 0.973 | 0.944 | 0.923 | 0.907 | 0.896 | 0.889 | 0.885 | 0.883 | 0.885 | 0.889 | 0.896 | 0.907 | 0.923 | 0.944 | 0.973 | 1.009 | 1.052 |
| **0.95π** | 1.108 | 1.090 | 1.051 | 1.009 | 0.973 | 0.945 | 0.924 | 0.909 | 0.899 | 0.892 | 0.889 | 0.889 | 0.892 | 0.899 | 0.909 | 0.924 | 0.945 | 0.973 | 1.009 | 1.051 | 1.090 |
| π | 1.107 | 1.090 | 1.052 | 1.010 | 0.974 | 0.947 | 0.927 | 0.912 | 0.903 | 0.897 | 0.896 | 0.897 | 0.903 | 0.912 | 0.927 | 0.947 | 0.974 | 1.010 | 1.052 | 1.090 | 1.107 |

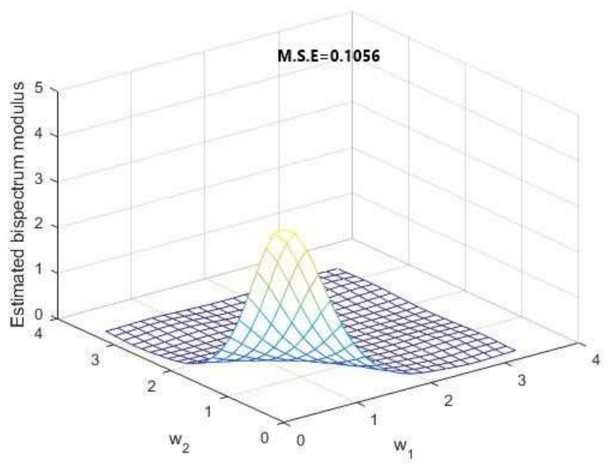


**Fig. 6:** Estimated bispectrum modulususing Daniell window at M=7.

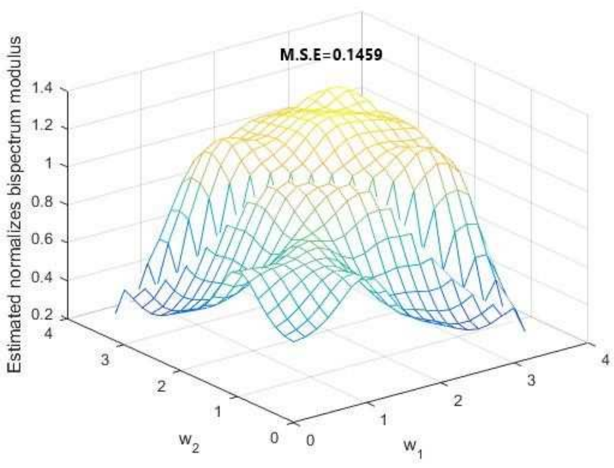


**Fig. 7:** Estimated normalized bispectrum modulus using Daniell window at M=7.

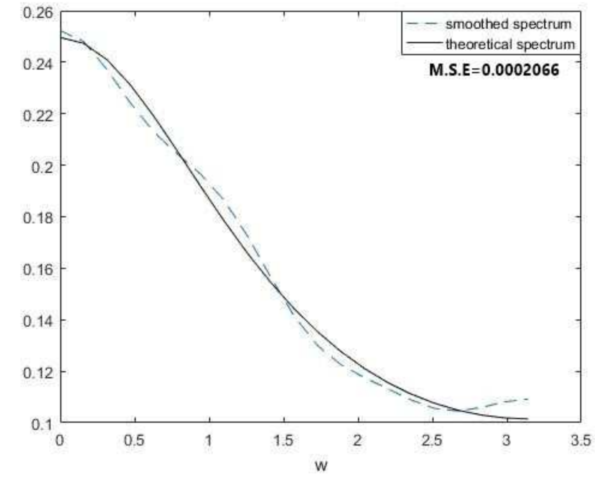


**Fig. 8:** Estimated spectrum using Parzen window at M=7 and theoretical spectrum.

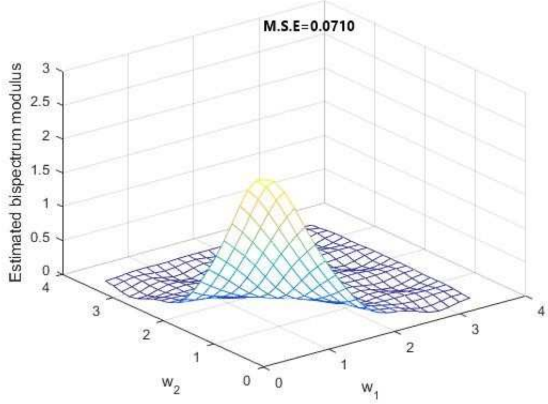


**Fig. 9:** Estimated bispectrum modulus using Parzen window at M=7.

- Depending on M.S.E which appears on each image, we find that
  - When comparing the smoothed spectrum based on Daniell, Parzen, Tukey Hamming lag windows with each other and with the theoretical spectrum that calculated at $\lambda = 1$ and $\alpha = 0.35$, we conclude that using the Parzen window is the approriate window among the other lag windows in Figures 5, 8 and 11, since the smoothed spectrum using Parzen window is closer to the theoretical spectrum than the smoothed spectrum based on Daniell or Tukey Hamming window.
  - Also, Figures 6, 9 and 12 show that the parzen window is the appropriate window among the

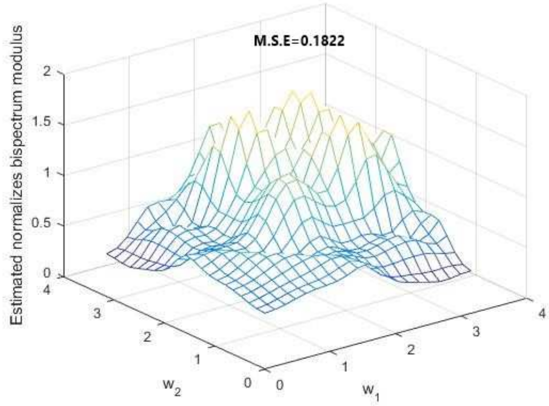


**Fig. 10:** Estimated normalized bispectrum modulus using Parzen window at M=7.

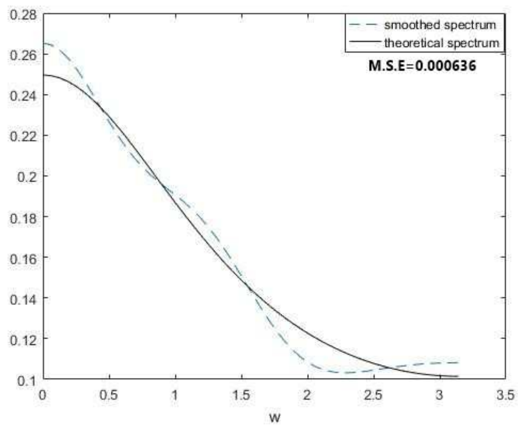


**Fig. 11:** Estimated spectrum using Tukey Hamming window at M=7 and theoretical spectrum.

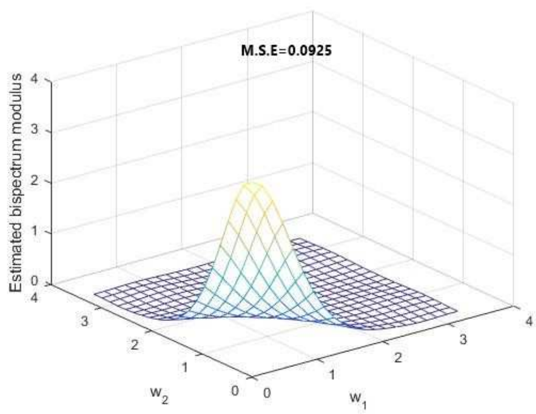


**Fig. 12:** Estimated bispectrum modulus using Tukey Hamming window at M=7.

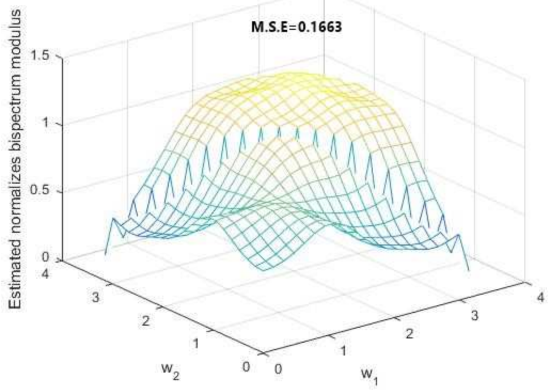


**Fig. 13:** Estimated normalized bispectrum modulus using Tukey Hamming window at M=7.

other lag windows for estimating the bispectrum modulus, scince the smoothed bispectrum using Parzen window to the theoretical bispectrum modulus than the smoothed bispectrum based on Daniell or Tukey Hamming windows.

–Moreover, Figures 7, 10 and 13 show that the Daniell window is the appropriate window among the other lag windows for estimating the normalized bispectrum modulus, scince the smoothed bispectrum using Daniell window to the theoretical bispectrum modulus than the smoothed bispectrum based on Daniell or Tukey Hamming windows.

–From Fig.3 and Fig.4 and Tabel 1 and Tabel 2, the normalized bispectrum modulus of the ZTPINAR(1) is more flat than the non-normalized bispectrum modulus, since the values of the normalized bispectrum modulus lies between (0.8,1.7) and the non-normalized bispectrum modulus lies between (0,4). This indicates that the test of linearity given by [32] can be used for integer valued time series models.

## 6 Conclusions

We studied in this paper, some higher order moments, cumulants, spectrum, bispectrum and normalized bispectrum density functions of the ZTPINAR(1) model. The spectrum, bispectrum and normalized bispectrum are estimated using a smoothed periodogram based on the different lag windows (Daniell, parzen and Tukey Hamming lag windows) and using a simulated series from this process. The normalized bispectrum modulus of the mentioned model is more flat than the non-normalized bispectrum modulus, so the test of linearity given by [32] can be used for integer valued time series models.

## Conflict of Interest

The author declare that there is no conflict of interest regarding the publication of this article.

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**Mohammed H. El-Menshawy** Assistant Lecturer of Mathematical Statistics at Mathematics Department Faculty of Science Al-Azhar University Cairo Egypt. He received MSc from Faculty of Science AL -Azhar University Egypt in 2018. His research interests are in the areas of Stochastic Process and its Application, Time Series, Multidimensional data analysis. He has published research articles in reputed international and national journals.

**Abd El-Moneim A. M. Teamah** Emeritus Professor of Mathematical statistics in Department of Mathematics Tanta University Tanta Egypt. He received his PhD in Mathematical statistics in (1986) from Tanta University Egypt. His research interests are in the areas of Stochastic Process and its Application, Time Series, SQC, Biostatistics, Multidimensional data analysis. He has published many papers in international refereed journals. He has supervised and discussed many of thesis in Egypt.

**S. E. Abu-Youssef** is presently employed as a professor of Mathematical statistics and Head of the Department of Mathematics of Faculty of Science, Al-Azhar University, Cairo, Egypt. He received his PhD in Mathematical statistics in 1994 from Faculty of Science, Al-Azhar University, Cairo, Egypt. His research interests include: Theory of reliability, ordered data, characterization, statistical inference, distribution theory, discriminant analysis and classes of life distributions. He published and Coauthored more than 80 papers in reputed international journals. He supervised and discussed many of thesis in Egypt.

**Hasnaa M. Faied** Lecturer of Mathematical Statistics at Mathematics Department Faculty of Science (Girls Branch) Al-Azhar University Cairo Egypt. She received PhD from Faculty of Science AL -Azhar University Egypt in 2010. Her research interests are in the areas of statistics including the mathematical statistics and applied statistics. she has published research articles in reputed international journals of Statistics.