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A Note on the Spectra of Forward Difference Operator on the Sequence Space ℓ_1

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Abstract: In this work, we compute the fine spectra of the forward difference operator Δ_μ^+ on ℓ_1, where $\mu = (\mu_k)$ is a positive real sequence of non-increasing terms satisfying certain conditions. The point spectrum, the residual spectrum, the continuous spectrum, the spectrum and some fine spectra of the operator Δ_μ^+ on the Banach space ℓ_1 are computed which give a natural modifications of the results obtained in [7] and [11]. In this context, some illustrative examples are also provided.

Keywords: Sequence spaces; Forward difference operator; Regular values, Resolvent set; Spectrum and fine spectra of an operator.

1 Introduction

The spectrum of a bounded linear operator usually generalizes the idea of eigenvalues associated with that operator. Its applications are useful and more apparent in various fields of functional analysis, numerical analysis and operator theory. Many prominent researchers have been continuously providing their valuable contributions in the field of spectral theory using different operators. For instances, the idea was initially studied by [1] for the Cesàro operator on the sequence space c and was further examined by [2]. Subsequently, these results were generalized by [3] on the sequence space $\ell_p,\ 1 < p < \infty$. Then it was further investigated by [4] and [5] on the sequence spaces c_0 and bv, respectively. The theory was developed for the backward difference operator Δ by [6, 7] on the sequence spaces $\ell_p,\ 0 < p < 1$ and c, c_0. The results were extended over the sequence spaces ℓ_1 and bv by [8]. Later on, the problem has been studied for the generalized difference operator Δ_μ by [9] on the space c_0. Recently, the theory was examined for the upper triangle double band matrix Δ^+ and the generalized forward difference operator Δ^μ by [10] and [11] over the sequence spaces ℓ_1 and c_0, respectively. The idea was further developed for higher order difference operators Δ^2 and $\Delta_\mu^r,\ r \in \mathbb{N}$ by [12, 13] over the sequence spaces c_0 and ℓ_1 respectively. Also, it was extended for the operator defined by a lambda matrix over the sequence spaces c_0 and c by [14]. For more details on the spectrum of various difference operators and their subdivisions, we may refer to [15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Note that throughout we use the notations $\ell_1, \ell_p, \ell_\infty, c, c_0$ and bv for the spaces of all absolutely summable, $p-$summable, bounded, convergent, null and bounded variation sequences, respectively.

Let $\mu = (\mu_k)$ be either a constant sequence (CS) or strictly decreasing sequence (SDS) of positive real numbers satisfying

$$\lim_{k\to\infty} \mu_k = L > 0, \tag{1}$$

$$\sup_k \mu_k \le 2L. \tag{2}$$

Then, the forward difference operator $\Delta_\mu^+ : \ell_1 \to \ell_1$ was defined by [10] as follows:

$$(\Delta_\mu^+ x)_k = \mu_k x_k - \mu_{k+1} x_{k+1},$$

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where $x \in \ell_1$ and $k \in \mathbb{N}_0$. It is noted that the operator Δ_μ^+ represents an upper triangle of the form

$$\Delta_\mu^+ = \begin{pmatrix} \mu_0 & -\mu_1 & 0 & 0 & \dots \\ 0 & \mu_1 & -\mu_2 & 0 & \dots \\ 0 & 0 & \mu_2 & -\mu_3 & \dots \\ 0 & 0 & 0 & \mu_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously, the operator Δ_μ^+ is the natural extension of the difference operator Δ of order one which was introduced by [25]. The spectrum and fine spectrum of the difference operator Δ_μ^+ on the sequence space ℓ_1 have been computed by [10]. In this note, we have demonstrated that these results can be made more sharper and pin-pointed. Now we mention some primary definitions and results which are closed to our investigation.

2 Spectrum of a bounded linear operator

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $\mathscr{R}(T)$, we denote the range of T, i.e.

$$\mathscr{R}(T) = \{y \in Y : y = Tx\ ;\ x \in X\}.$$

By $B(X)$, we denote the space all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\mathbf{0}\}$ be a normed linear space over the complex field and $T : D(T) \to X$ be a linear operator, where $D(T)$ denotes the domain of T. With T, for a complex number ξ, we associate an operator $T_\xi = T - \xi I$, where I is the identity operator on $D(T)$ and if T_ξ has an inverse, we denote it by T_ξ^{-1} i.e.

$$T_\xi^{-1} = (T - \xi I)^{-1}$$

and is called the *resolvent* operator of T. Many properties of T_ξ and T_ξ^{-1} depend on ξ and the spectral theory is concerned with those properties. We are interested in the set of all ξ's in the complex plane such that T_ξ^{-1} exists/ T_ξ^{-1} is bounded/ domain of T_ξ^{-1} is dense in X. For our investigation, we need some basic concepts in spectral theory which are given as some definitions and lemmas.

Definition 1.([26, p. 371]). *Let X be a Banach space and T, defined by $T : X \to X$ be a bounded linear operator. A complex number ξ is said be a regular value of T if*

(A)$(T - \xi I)^{-1}$ exists;

(B)$(T - \xi I)^{-1}$ is bounded;

(C)$(T - \xi I)^{-1}$ is defined on a set which is dense in X.

The set $\rho(T,X)$ consisting of such regular values, is called the resolvent set of T.The complement set $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ is known as the spectrum of the operator T. Moreover, the spectrum $\sigma(T,X)$ is subdivided into three disjoint sets such as

(i)The point spectrum, denoted by $\sigma_p(T,X)$ (i.e., $\{\xi \in \mathbb{C} : \xi$ does not satisfy $(A)\}$),

(ii)The continuous spectrum, denoted by $\sigma_c(T,X)$ (i.e., $\{\xi \in \mathbb{C} : \xi$ satisfies (A) and (C) but does not $(B)\}$), and

(iii)The residual spectrum, denoted by $\sigma_r(T,X)$ (i.e.,$\{\xi \in \mathbb{C} : \xi$ satisfies (A) but does not $(C)\}$.

3 Main Results

This section deals with the determination of the subdivisions of the spectrum such as point spectrum, the continuous spectrum, the residual spectrum and some fine spectra of the operator Δ_μ^+ on ℓ_1.

Theorem 1.*The difference operator $\Delta_\mu^+ : \ell_1 \to \ell_1$ is a bounded linear operator and satisfies*

$$\| \Delta_\mu^+ \|_{(\ell_1,\ell_1)} = \begin{cases} 2L, & \textit{for CS } (\mu_k) \\ 2\mu_0, & \textit{for SDS } (\mu_k). \end{cases}$$

*Proof.*To prove this we use the definition given in [27, p. 126].

Theorem 2.*The point spectrum of Δ_μ^+ over ℓ_1 is given by*

$$\sigma_p(\Delta_\mu^+,\ell_1) = \begin{cases} \left\{\xi \in \mathbb{C} : \left|1 - \frac{\xi}{L}\right| < 1\right\}, & \textit{for CS } (\mu_k) \\ \left\{\xi \in \mathbb{C} : \left|1 - \frac{\xi}{L}\right| < 1\right\} \cup D, & \textit{for SDS } (\mu_k), \end{cases}$$

where

$$D = \left\{\xi \in \mathbb{C} : \sum_k \left|\prod_{i=1}^{k} \frac{\mu_{i-1} - \xi}{\mu_i}\right| < \infty\right\}.$$

In particular,

$$\{L, \mu_0\} \subset \sigma_p(\Delta_\mu^+,\ell_1).$$

*Proof.*First, we consider that (μ_k) is a constant sequence, i.e., $\mu_k = L$ for all $k \in \mathbb{N}_0$. Then the system of linear equations $\Delta_\mu^+ x = \xi x$ for $x \neq \mathbf{0} = (0,0,0,\dots)$ in ℓ_1, becomes

$$\begin{aligned} Lx_0 - Lx_1 &= \xi x_0 \\ Lx_1 - Lx_2 &= \xi x_1 \\ &\vdots \\ Lx_{n-1} - Lx_n &= \xi x_{n-1} \\ &\vdots \end{aligned}$$

Therefore, we have

$$x_n = \frac{L-\xi}{L}x_{n-1} = \left(\frac{L-\xi}{L}\right)^n x_0$$

Clearly, it is obtained that the sequence $(x_k) \in \ell_1$, being a solution of the above system of linear equations if and only if $\left|1-\frac{\xi}{L}\right| < 1$. Also, it is observed that $\xi = L$ is an eigenvalue corresponding to the eigenvector $(1,0,0,\dots)$. Hence, $\{L\} \subset \sigma_p(\Delta_\mu^+, \ell_1)$.

Secondly, if (μ_k) is a strictly decreasing sequence. Then the system of linear equations $\Delta_\mu^+ x = \xi x$ for $x \neq \mathbf{0}$ in ℓ_1, leads to

$$\begin{aligned} \mu_0 x_0 - \mu_1 x_1 &= \xi x_0 \\ \mu_1 x_1 - \mu_2 x_2 &= \xi x_1 \\ &\vdots \\ \mu_{n-1}x_{n-1} - \mu_n x_n &= \xi x_{n-1} \\ &\vdots \end{aligned}$$

On solving, we have

$$x_n = \frac{\mu_{n-1}-\xi}{\mu_n}x_{n-1} = \prod_{i=1}^{n}\frac{\mu_{i-1}-\xi}{\mu_i}x_0,$$

and

$$\lim_{n\to\infty}\left|\frac{x_n}{x_{n-1}}\right| = \lim_{n\to\infty}\left|\frac{\mu_{n-1}-\xi}{\mu_n}\right| = \left|\frac{L-\xi}{L}\right|.$$

This shows that the above system of linear equations has a solution $x \in \ell_1$ if and only if $\left|1-\frac{\xi}{L}\right| < 1$. Also, one of the solutions of the system of linear equations is of the form $\xi = \mu_0$ which is an eigenvalue corresponding to the eigenvector $(1,0,0,\dots)$. Therefore, $\{\mu_0\} \subset \sigma_p(\Delta_\mu^+, \ell_1)$.

Furthermore, if we consider $\Delta_\mu^+ x = \xi x$, then we obtain that $x_k = \prod_{i=1}^{k}\left(\frac{\mu_{i-1}-\xi}{\mu_i}\right)x_0$ for all $k \in \mathbb{N}$. If $x_0 = 0$, then $x = \mathbf{0}$, a contradiction, otherwise if $x_0 \neq 0$, then

$$\begin{aligned} \sum_{k=0}^{\infty}|x_k| &= |x_0| + |x_0|\sum_{k=1}^{\infty}\prod_{i=1}^{k}\left|\frac{\mu_{i-1}-\xi}{\mu_i}\right| \\ &= |x_0| + |x_0|\lim_{s\to\infty}\sum_{k=1}^{s}\left|\frac{\mu_0-\xi}{\mu_k}\right|\left|\frac{\mu_1-\xi}{\mu_1}\right|\left|\frac{\mu_2-\xi}{\mu_2}\right|\dots \\ &\qquad \times\left|\frac{\mu_{k-1}-\xi}{\mu_{k-1}}\right|. \end{aligned}$$

Now, we conclude that the above system of linear equations has a nonzero solution if and only if $\xi \in D$.

For more clarifications, we have an example:

Example: Suppose $\mu = (\mu_k)$ is a strictly decreasing sequence and

$$\mu_k = \frac{(k+3)^2}{(k+2)^2+(k+3)^2} \text{ for all } k \in \mathbb{N}_0.$$

Clearly, (μ_k) satisfies the conditions (1.1) and (1.2), in fact

$$L = \lim_{k\to\infty}\mu_k = \frac{1}{2}, \mu_0 = \frac{9}{13} < 1 = 2L.$$

For $\xi = 1$, we consider $\Delta_\mu^+ x = (1)x$, then we obtain that $x_k = \prod_{i=1}^{k}\left(\frac{\mu_{i-1}-1}{\mu_i}\right)x_0$ for all $k \in \mathbb{N}$. If $x_0 = 0$, then $x = \mathbf{0}$ which is a contradiction, otherwise if $x_0 \neq 0$, then

$$\begin{aligned} \sum_{k=0}^{\infty}|x_k| &= |x_0| + |x_0|\sum_{k=1}^{\infty}\prod_{i=1}^{k}\left|\frac{\mu_{i-1}-1}{\mu_i}\right| \\ &= |x_0| + |x_0|\sum_{k=1}^{\infty}\left|\frac{\mu_0-1}{\mu_k}\right|\left|\frac{\mu_1-1}{\mu_1}\right|\left|\frac{\mu_2-1}{\mu_2}\right|\dots\left|\frac{\mu_{k-1}-1}{\mu_{k-1}}\right| \\ &= |x_0| + \frac{9\times 4}{13}|x_0|\sum_{k=1}^{\infty}\left(\frac{1}{(k+2)^2}+\frac{1}{(k+3)^2}\right). \end{aligned}$$

This concludes that $1 \in D \subset \sigma_p(\Delta_\mu^+, \ell_1)$..

Theorem 3. *The point spectrum of the adjoint operator $[\Delta_\mu^+]^*$ over ℓ_∞ is given by*

$$\sigma_p([\Delta_\mu^+]^*, \ell_1^*) = \sigma_p([\Delta_\mu^+]^*, \ell_\infty) = \emptyset.$$

Proof. We may divide the proof as follows:

First, we consider that (μ_k) is a constant sequence, say $\mu_k = L$ for all $k \in \mathbb{N}_0$. Then we have the system of linear equations $[\Delta_\mu^+]^* f = \xi f$ for $\mathbf{0} \neq f \in \ell_\infty$, where

$$[\Delta_\mu^+]^* = \begin{pmatrix} \mu_0 & 0 & 0 & 0 & \dots \\ -\mu_1 & \mu_1 & 0 & 0 & \dots \\ 0 & -\mu_2 & \mu_2 & 0 & \dots \\ 0 & 0 & -\mu_3 & \mu_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

Now, consider

$$\left.\begin{aligned} \mu_0 f_0 &= \xi f_0 \\ -\mu_1 f_0 + \mu_1 f_1 &= \xi f_1 \\ -\mu_2 f_1 + \mu_2 f_2 &= \xi f_2 \\ -\mu_3 f_2 + \mu_3 f_3 &= \xi f_3 \\ &\vdots \end{aligned}\right\} \tag{3}$$

Since (μ_k) is a constant sequence, then on solving the above system of linear equation, it is clear that $\mu_0 = L = \xi$ and the only corresponding solution is $f = \mathbf{0}$, a contradiction.

Secondly, suppose (μ_k) is a strictly decreasing sequence. Then the system of linear equations (3)

$[\Delta_\mu^+]^* f = \xi f$ for $\mathbf{0} \neq f \in \ell_\infty$, has a solution of the form $\mu_0 = \xi$ and

$$\begin{aligned} f_1 &= \frac{\mu_1}{\mu_1-\mu_0} f_0 = \frac{\mu_1}{\mu_1-\xi} f_0, \\ f_2 &= \Big[\frac{\mu_1\mu_2}{(\mu_1-\xi)(\mu_2-\xi)}\Big] f_0, \\ f_3 &= \Big[\frac{\mu_1\mu_2\mu_3}{(\mu_1-\xi)(\mu_2-\xi)(\mu_3-\xi)}\Big] f_0, \\ &\vdots \\ f_n &= \Big[\prod_{i=1}^{n} \frac{\mu_i}{(\mu_i-\xi)}\Big] f_0, \\ &\vdots \end{aligned}$$

Clearly, for $\xi \notin \{\mu_1, \mu_2, \mu_3 \ldots\}$, we get $f_k = 0$ for all $k \in \mathbb{N}_0$ which implies that

$$\sigma_p([\Delta_\mu^+]^*, \ell_\infty) \subseteq \{\mu_1, \mu_2, \mu_3, \ldots\}.$$

Again for $\xi \in \{\mu_1, \mu_2, \mu_3 \ldots\}$ we get the system of equations (3) which has a solution $f = \mathbf{0}$, a contradiction. Finally, for $\xi = \mu_0$,

$$\begin{aligned} \left|\frac{f_k}{f_{k-1}}\right| &= \left|\frac{\mu_k}{\mu_k-\mu_0}\right| > 1, \\ \lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right| &= \lim_{k\to\infty}\left|\frac{\mu_k}{\mu_k-\mu_0}\right| = \left|\frac{L}{L-\mu_0}\right| > 1. \end{aligned}$$

This shows that the sequence $(f_0, f_1, f_2, \ldots)$ is increasing and therefore, $f \notin \ell_\infty$. Hence $\sigma_p([\Delta_\mu^+]^*, \ell_\infty) = \emptyset$.

Theorem 4. *The residual spectrum of Δ_μ^+ over ℓ_1 is given by*

$$\sigma_r(\Delta_\mu^+, \ell_1) = \emptyset.$$

Proof. To prove this theorem we use Theorem 11.3.7 of [28] with the following equality

$$\sigma_p([\Delta_\mu^+]^*, \ell_\infty) = \sigma_r(\Delta_\mu^+, \ell_1) = \emptyset.$$

Theorem 5. *The spectrum of Δ_μ^+ over ℓ_1 is given by*

$$\sigma(\Delta_\mu^+, \ell_1) = \left\{\xi \in \mathbb{C} : \left|1 - \frac{\xi}{L}\right| \leq 1\right\} \cup D.$$

Proof. The entire proof is divided into two parts as follows:

Part 1:

In this part, w show that

$$\sigma(\Delta_\mu^+, \ell_1) \subseteq \left\{\xi \in \mathbb{C} : \left|1 - \frac{\xi}{L}\right| \leq 1\right\}.$$

Or, we need to show if $\xi \in \mathbb{C}$ with $|1 - \frac{\xi}{L}| > 1$, then it implies $\xi \notin \sigma(\Delta_\mu^+, \ell_1)$.

Suppose $\xi \in \mathbb{C}$ with $|1 - \frac{\xi}{L}| > 1$. Therefore $\xi \neq L$ and $\xi \neq \mu_k$, for each $k \in \mathbb{N}_0$. Now $(\Delta_\mu^+ - \xi I) = (a_{nk})$ is an upper triangular matrix and has an inverse $(\Delta_\mu^+ - \xi I)^{-1} = (b_{nk})$, as below

$$\begin{pmatrix} \frac{1}{(\mu_0-\xi)} & \frac{\mu_1}{(\mu_0-\xi)(\mu_1-\xi)} & \frac{\mu_1\mu_2}{(\mu_0-\xi)(\mu_1-\xi)(\mu_2-\xi)} & \cdots \\ 0 & \frac{1}{(\mu_1-\xi)} & \frac{\mu_2}{(\mu_1-\xi)(\mu_2-\xi)} & \cdots \\ 0 & 0 & \frac{1}{(\mu_2-\xi)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The general form of b_{nk} is given by

$$b_{nk} = \begin{cases} \frac{1}{(\mu_n-\xi)}, & (k=n) \\ \prod_{i=n}^{k} \frac{\mu_{i+1}}{(\mu_i-\xi)}, & (k>n) \\ 0, & (k<n) \end{cases}$$

Consider

$$\begin{aligned} S_k &= \sum_{n=0}^{\infty} |b_{nk}| \\ &= |b_{0k}| + |b_{1k}| + |b_{2k}| + \ldots + |b_{k-1,k}| + |b_{kk}| \\ &= \left|\frac{1}{(\mu_k-\xi)}\right| + \left|\frac{\mu_k}{(\mu_{k-1}-\xi)(\mu_k-\xi)}\right| \\ &\quad + \left|\frac{\mu_{k-1}\mu_k}{(\mu_{k-2}-\xi)(\mu_{k-1}-\xi)(\mu_k-\xi)}\right| + \cdots + \left|\prod_{i=0}^{k} \frac{\mu_{i+1}}{(\mu_i-\xi)}\right| \end{aligned}$$

Now, taking limit as $k \to \infty$,

$$\begin{aligned} &\lim_{k\to\infty} S_k \\ &= \lim_{k\to\infty}\Bigg[\left|\frac{1}{(\mu_k-\xi)}\right| + \left|\frac{\mu_k}{(\mu_{k-1}-\xi)(\mu_k-\xi)}\right| \\ &\quad + \left|\frac{\mu_{k-1}\mu_k}{(\mu_{k-2}-\xi)(\mu_{k-1}-\xi)(\mu_k-\xi)}\right| + \cdots + \left|\prod_{i=0}^{k} \frac{\mu_{i+1}}{(\mu_i-\xi)}\right|\Bigg] \\ &= \left|\frac{1}{(L-\xi)}\right| + \left|\frac{L}{(L-\xi)(L-\xi)}\right| \\ &\quad + \left|\frac{L^2}{(L-\xi)(L-\xi)(L-\xi)}\right| + \cdots + \lim_{k\to\infty}\Bigg[\left|\prod_{i=0}^{k} \frac{\mu_{i+1}}{(\mu_i-\xi)}\right|\Bigg] \\ &\leq \frac{1}{|L-\xi|}\Bigg[1 + \left|\frac{L}{L-\xi}\right|^2 + \left|\frac{L}{L-\xi}\right|^3 + \cdots + \left|\frac{L}{L-\xi}\right|^k + \ldots\Bigg] \\ &= \frac{1}{|L-\xi| - L} < \infty. \end{aligned}$$

This follows from the fact that $0 < \left|\frac{L}{L-\xi}\right| < 1$. As (S_k) is a sequence of positive real numbers and $\lim_k S_k < \infty$, $(\Delta_\mu^+ - \xi I)^{-1} \in B(\ell_1)$ with the condition that $|1 - \frac{\xi}{L}| > 1$, and therefore,

$$\sigma(\Delta_\mu^+, \ell_1) \subseteq \left\{\xi \in \mathbb{C} : \left|1 - \frac{\xi}{L}\right| \leq 1\right\}.$$

Part 2:

To show $\left\{\xi \in \mathbb{C} : \left|1 - \frac{\xi}{L}\right| \leq 1\right\} \subseteq \sigma(\Delta_\mu^+, \ell_1)$, we take

$\xi \neq L$ and also $\xi \neq \mu_k$ for each $k \in \mathbb{N}_0$. Suppose $\xi \in \mathbb{C}$ with the condition $|1-\frac{\xi}{L}| \leq 1$. Then $(\Delta_\mu^+ - \xi I)$ is a triangle and has an inverse $(\Delta_\mu^+ - \xi I)^{-1}$. But $(\Delta_\mu^+ - \xi I)^{-1}$ is not bounded inℓ_1. This is because for $n < k$

$$b_{nk} = \prod_{i=n}^{k} \frac{\mu_{i+1}}{(\mu_i - \xi)}, (n \geq 1).$$

Finally, we have

$$\lim_{n\to\infty} b_{nk} = \frac{1}{L-\xi} \left| \frac{L}{L-\xi} \right|^k > 1.$$

As a result, it is seen that $(\Delta_\mu^+ - \xi I)^{-1} \notin B(\ell_1)$ if $|1-\frac{\xi}{L}| \leq 1$. Finally, we can prove that the matrix $\Delta_\mu^+ - \xi I$ in not invertible under the assumption that $\xi = L$ and $\xi = \mu_k$ for all $k \in \mathbb{N}_0$. Thus

$$\left\{ \xi \in \mathbb{C} : \left| 1 - \frac{\xi}{L} \right| \leq 1 \right\} \subseteq \sigma(\Delta_\mu^+, \ell_1).$$

Theorem 6.*The continuous spectrum of Δ_μ^+ over ℓ_1 is given by*

$$\sigma_c(\Delta_\mu^+, \ell_1) = \begin{cases} \left\{ \xi \in \mathbb{C} : \left| 1 - \frac{\xi}{L} \right| = 1 \right\}, & \textit{for CS } (\mu_k) \\ \left\{ \xi \in \mathbb{C} : \left| 1 - \frac{\xi}{L} \right| = 1 \right\} \setminus D, & \textit{for SDS } (\mu_k) \end{cases}$$

*Proof.*The proof follows from Theorems 2, 4, and 5 by combining the fact that

$$\sigma(\Delta_\mu^+, \ell_1) = \sigma_p(\Delta_\mu^+, \ell_1) \cup \sigma_r(\Delta_\mu^+, \ell_1) \cup \sigma_c(\Delta_\mu^+, \ell_1).$$

Now, using proposed theorems and the Goldberg's classifications of the difference operator Δ_μ^+, the following results are given:

Theorem 7.*(i)*

$$III_1(\Delta_\mu^+, \ell_1) = III_2(\Delta_\mu^+, \ell_1) = \emptyset.$$

(ii)

$$III_3(\Delta_\mu^+, \ell_1) = \{\xi : |\xi - L| < L\}.$$

(iii)

$$II_2(\Delta_\mu^+, \ell_1) = \{\xi : |\xi - L| = L\}.$$

(iv)

$$I_3(\Delta_\mu^+, \ell_1) \cup II_3(\Delta_\mu^+, \ell_1) \cup III_3(\Delta_\mu^+, \ell_1) = \{\xi : |\xi - L| \leq L\}$$

(v)

$$III(\Delta_\mu^+, \ell_1) = \{\xi : |\xi - L| \leq L\}.$$

*Proof.*Proofs are obtained by using the Goldberg's classifications for bounded linear operators(see, Theorems 3.2-3.5, [28]).

4 Conclusion:

Spectra of forward difference operators have been computed in different recent works of [10] and [25]. This work has taken an attempt to unify those results and provide some new additional ideas in order to get improved and sharper estimations on them. Spectral subdivisions such as the point spectrum, the continuous spectrum, the residual spectrum and some fine spectra of the operator Δ_μ^+ on ℓ_1 have been determined. As an application, this work provides some new idea and pin-pointed estimations on fine spectra of forward difference operator of higher orders on various sequence space such as c, c_0 and $\ell_p, (1 \leq p < \infty)$.

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Conflict of Interest The authors declare that they have no conflict of interest

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