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A Generalized Theorem on Double Absolute Factorable Matrix Summability

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Abstract: In this paper, we generalize a new result on absolute index double matrix summability. Dealing with $|A|_k$-summability, Savaş and Rhoades [E. Savaş and B. E. Rhoades, Nonlinear Anal. **69**, 189–200 (2008)], established a result on absolute indexed double matrix summability of infinite series which was generalized by Jena *et al.* [B. B. Jena, S. K. Paikray and U. K. Misra, Tbilisi Math. J. **11** , 1–18 (2018)], for $|A,\delta|_k$-summability. Here, we derive a new and more generalized result on $|U,\delta,\gamma|_q$-summability. Finally, we also highlight some important new and well-known results in the line of our findings in the conclusion section. We also suggest a direction for future researches on this subject towards application areas of science like a rectification of signals in FIR filter and IIR filter to speed of the rate of convergence.

Keywords: Absolute matrix summability, Hölder's inequality, Abel's theorem, matrix transformation, quasi-monotone sequences

1 Introduction and Motivation

Let $\{s_n\}=\sum_{k=0}^{n} a_k$ be the sequence of partial sums of the series $\sum a_n$, and let $T=(u_{nk})$ be an infinite matrix, then the n^{th} matrix transform $\{u_n\}$ of $\{s_n\}$ is given by

$$u_n=\sum_{k=0}^{\infty} u_{nk}s_k. \tag{1}$$

Definition 1.(*see [1]*) *If*

$$\lim_{n\to\infty} u_n = s,$$

then $\sum a_n$, is said to be matrix summable (or T- summable) to s, and if

$$\sum_{n=1}^{\infty} |u_n-u_{n-1}|<\infty. \tag{2}$$

then, $\sum a_n$ is absolute matrix summable (or $|T|$-summable).

Moreover, the matrix $T=(u_{nk})$ is regular if,

$$\lim_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} u_n = s.$$

Definition 2.(*see [2]*) *If*

$$\sum_{n=1}^{\infty} n^{k-1}|t_n-t_{n-1}|^k<\infty, \tag{3}$$

where t_n is the sequence of $(C,1)$-mean of the series, then $\sum a_n$ is summable $|C,1|_k$.

Definition 3.*Let $\{p_s\}$ be of positive numbers and*

$$P_s=\sum_{r=0}^{s} p_r\to\infty, \tag{4}$$

where $(P_{-s}=p_{-s}=0, s\geq 1)$.

If σ_s defines the $(\overline{N},p_s)$ mean [3] with

$$\sigma_s=\frac{1}{P_s}\sum_{q=0}^{s} p_q s_q,\ P_s\neq 0,\ s\in N \tag{5}$$

and $\lim_{s\to\infty}\sigma_s=k$, then $\sum a_s$ is $(\overline{N},p_s)$ summable generated by $\{p_s\}$.

Furthermore, if $\{\sigma_s\}$ is of bounded variation with index $q\geqslant 1$ [4] with

$$\sum_{n=1}^{\infty}\left(\frac{P_s}{p_s}\right)^{q-1}|\sigma_s-\sigma_{s-1}|^q<\infty, \tag{6}$$

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then $\sum a_s$ is $|\overline{N}, p_s|_q$-summable.

Let $U = (u_{nv})$ be a normal matrix. Then the transformation of sequence $s = \{s_n\}$ to $U(s) = \{U_n(s)\}$ by U is given by:

$$U_n(s) = \sum_{v=0}^{n} u_{nv} s_v, \quad n = 0, 1, \cdots. \tag{7}$$

If

$$\sum_{n=1}^{\infty} |u_{nn}|^{1-q} |\overline{\Delta} U_n(s)|^q < \infty, \tag{8}$$

then $\sum a_n$ is $|U|_q$ summable, $q \geq 1$, and if

$$\sum_{n=1}^{\infty} |u_{nn}|^{1-q-\delta q} |\overline{\Delta} U_n(s)|^q < \infty, \tag{9}$$

then $\sum a_n$ is $|U, \delta|_q$ summable, $q \geq 1$.

Also, if

$$\sum_{n=1}^{\infty} |u_{nn}|^{\gamma(1-q-\delta q)} |\overline{\Delta} U_n(s)|^q < \infty, \tag{10}$$

where γ is a real number, $q \geq 1$, $0 \leq \delta \leq 1/q$ and

$$\overline{\Delta} U_n(s) = U_n(s) - U_{n-1}(s),$$

then $\sum a_n$ is said to be $|U, \delta; \gamma|_q$-summable.

Taking $U = (\overline{N}, p_n)$ in condition (9), then $|U, \delta|_q$ changes to $|\overline{N}, p_n; \delta|_q$ summability. Also, if we take $\delta = 0$ in condition (9), then $|U, \delta|_q$ changes to $|U|_q$ summability.

Now, we use the following notations in the main result as below.

We are given with a normal matrix $U = (u_{nv})$. Two lower semi-matrices $\overline{U} = (\overline{u}_{nv})$ and $\hat{U} = (\hat{u}_{nv})$ are defined as

$$\overline{u}_{nv} = \sum_{i=v}^{n} u_{ni}, \quad n, v = 0, 1, 2, \cdots \tag{11}$$

and

$$\hat{u}_{00} = \overline{u}_{00} = u_{00}, \quad \hat{u}_{nv} = \overline{u}_{nv} - \overline{u}_{n-1,v} \quad n = 1, 2, \cdots. \tag{12}$$

Then, we have

$$U_n(s) = \sum_{v=0}^{n} u_{nv} s_v = \sum_{v=0}^{n} \overline{u}_{nv} a_v \tag{13}$$

and

$$\overline{\Delta} U_n(s) = \sum_{v=0}^{n} \hat{u}_{nv} a_v. \tag{14}$$

Similarly, let $U = (u_{mnjk})$ be a lower-triangular matrix and the partial sum's sequence of $\sum\sum a_{mn}$ is denoted by $\{s_{mn}\}$. The mn th U-transform of the sequence $\{s_{mn}\}$ is defined as,

$$T_{mn} = \sum_{\mu=0}^{m} \sum_{v=0}^{n} u_{mn\mu v} s_{\mu v}.$$

Note that, a doubly infinite matrix $U = (u_{mnjk})$ is doubly triangular if, u_{mnjk}=0 for $j>m$ or $k>n$. Also, for any double sequence $\{v_{xy}\}$, Δ_{11} is defined as:

$$\Delta_{11} v_{xy} = v_{xy} - v_{x+1,y} - v_{x,y+1} + v_{x+1,y+1}.$$

Similarly, for any fourfold sequence $\{v_{xyrs}\}$,

$$\Delta_{11} v_{xyrs} = v_{xyrs} - v_{x+1,y,r,s} - v_{x,y+1,r,s} + v_{x+1,y+1,r,s};$$

$$\Delta_{rs} v_{xyrs} = v_{xyrs} - v_{x,y,r+1,s} - v_{x,y,r,s+1} + v_{x,y,r+1,s+1};$$

$$\Delta_{0s} v_{xyrs} = v_{xyrs} - v_{x,y,r,s+1};$$

$$\Delta_{r0} v_{xyrs} = v_{xyrs} - v_{x,y,r+1,s}. \tag{15}$$

Let $\{s_{kl}\}$ denotes the partial sum of the series $\sum\sum b_{kl}$. If [5]

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (kl)^{q-1} |\Delta_{11} T_{k-1,l-1}|^q < \infty, \tag{16}$$

then $\sum\sum b_{kl}$ is $|U|_q$ summable, $q \geq 1$ and if [3]

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (kl)^{\delta q+q-1} |\Delta_{11} T_{k-1,l-1}|^q < \infty, \tag{17}$$

then $\sum\sum b_{kl}$ is $|U, \delta|_q$ summable, $q \geq 1$ and $\delta \geq 0$.

Also, if

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (kl)^{\gamma(\delta q+q-1)} |\Delta_{11} T_{k-1,l-1}|^q < \infty, \tag{18}$$

then $\sum\sum b_{kl}$ is $|U, \delta; \gamma|_q$ summable, $q \geq 1$, $0 \leq \delta \leq 1/q$, $\gamma \in \mathscr{R}$.

Let $\overline{U}$ and $\hat{U}$ be two doubly triangular matrices defined as follows

$$\overline{u}_{mn\rho\eta} = \sum_{\mu=\rho}^{m} \sum_{v=\eta}^{n} u_{mn\mu v}$$

and

$$\hat{u}_{m,n,\rho,\eta} = \Delta_{11} \overline{u}_{m-1,n-1,i\rho,\eta} \quad (m, n \in \mathbb{N}_0 =: \{0\} \cup \mathbb{N}). \tag{19}$$

Note that,

$$\hat{u}_{0000} = \overline{u}_{0000} = a_{0000}.$$

Let y_{kl} represents the $(kl)^{th}$ term of U-transform of $\sum_{\mu=0}^{k}\sum_{v=0}^{l} b_{\mu v}\lambda_{\mu v}$, then we can write,

$$\begin{aligned} y_{kl} &= \sum_{\mu=0}^{k}\sum_{v=0}^{l} u_{kl\mu v}\sum_{\rho=0}^{\mu}\sum_{\eta=0}^{v} b_{\rho\eta}\lambda_{\rho\eta} \\ &= \sum_{\rho=0}^{k}\sum_{\eta=0}^{l} b_{\rho\eta}\lambda_{\rho\eta}\sum_{\mu=\rho}^{k}\sum_{v=\eta}^{l} u_{kl\mu v} \\ &= \sum_{\rho=0}^{k}\sum_{\eta=0}^{l} b_{\rho\eta}\lambda_{\rho\eta}\overline{u}_{kl\rho\eta}. \end{aligned}$$

Thus,

$$\begin{aligned} &\Delta_{11}y_{k-1,l-1} \\ &= \sum_{\rho=0}^{k}\sum_{\eta=0}^{l} b_{\rho\eta}\lambda_{\rho\eta}\hat{u}_{k,l,\rho,\eta} - \sum_{\eta=0}^{l-1} b_{k\eta}\lambda_{k\eta}\overline{u}_{k-1,l-1,k,\eta} \\ &- \sum_{\rho=0}^{k-1} b_{\rho l}\lambda_{\rho l}\overline{u}_{k-1,l-1,\rho,l} + \sum_{\rho=0}^{k} b_{\rho l}\lambda_{\rho l}\overline{u}_{k,l-1,\rho,l} \\ &+ \sum_{\eta=0}^{l} b_{kl}\lambda_{k\eta}\overline{u}_{k-1,l,k,\eta} \\ &= \sum_{\rho=0}^{k}\sum_{\eta=0}^{l} b_{\rho\eta}\lambda_{\rho\eta}\hat{u}_{kl\rho\eta}. \end{aligned}$$

Since,

$$\overline{u}_{k-1,l-1,k,\eta} = \overline{u}_{k-1,l-1,\rho,l} = \overline{u}_{k,l-1,\rho,l} = \overline{u}_{k-1,l,k,l} = 0$$

and

$$b_{kl} = s_{k-1,l-1} - s_{k-1,l} - s_{k,l-1} + s_{kl},$$

so,

$$\begin{aligned} &\Delta_{11}y_{k-1,l-1} \\ &= \sum_{\rho=0}^{k}\sum_{\eta=0}^{l} \hat{u}_{kl\rho\eta}\lambda_{\rho\eta}(s_{\rho-1,\eta-1} - s_{\rho-1,\eta} - s_{\rho,\eta-1} + s_{\rho\eta}) \\ &= \sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1} \Delta_{\rho\eta}(\hat{u}_{kl\rho\eta}\lambda_{\rho\eta})s_{\rho\eta} - \sum_{\rho=0}^{k-1}\hat{u}_{k,l,\rho+1,l}\lambda_{\rho+1,l}s_{\rho l} \\ &- \sum_{\eta=0}^{l-1}\hat{u}_{k,l,k,\eta+1}\lambda_{k,\eta+1}s_{k\eta} + \sum_{\rho=0}^{l}\hat{u}_{klk\eta}\lambda_{k,\eta}s_{k\eta} \\ &\quad + \sum_{\rho=0}^{k-1}\hat{u}_{kl\rho l}\lambda_{\rho l}s_{\rho l} \\ &= \sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1} \Delta_{\rho\eta}(\hat{u}_{kl\rho\eta}\lambda_{\rho\eta})s_{\rho\eta} + \sum_{\rho=0}^{k-1}(\Delta_{\eta 0}\hat{u}_{kl\rho l}\lambda_{\rho l})s_{\rho l} \\ &+ \sum_{\eta=0}^{l-1}(\Delta_{0\eta}\hat{u}_{klk\eta}\lambda_{k\eta})s_{k\eta} + \hat{u}_{klkl}\lambda_{kl}s_{kl}. \end{aligned} \tag{20}$$

Also, we have

$$\Delta_{\rho 0}\hat{u}_{kl\rho l}\lambda_{\rho l} = \lambda_{\rho l}\Delta_{\rho 0}\hat{u}_{kl\rho l} + \hat{u}_{k,l,\rho+1,l}\Delta_{\rho 0}\lambda_{\rho l}$$

and

$$\Delta_{0\eta}\hat{u}_{klk\eta}\lambda_{k\eta} = \lambda_{k\eta}\Delta_{0\eta}\hat{u}_{klk\eta} + \hat{u}_{k,l,k,\eta+1}\Delta_{0\eta}\lambda_{k\eta}.$$

Clearly,

$$\begin{aligned} &\sum_{\rho=0}^{k-1}(\Delta_{\rho 0}\hat{u}_{kl\rho l}\lambda_{\rho l})s_{\rho l} + \sum_{\eta=0}^{l-1}(\Delta_{0\eta}\hat{u}_{klk\eta}\lambda_{k\eta})s_{k\eta} \\ &= \sum_{\rho=0}^{k-1}[\lambda_{\rho l}\Delta_{\rho 0}\hat{u}_{kl\rho l} + \hat{u}_{k,l,\rho+1,l}\Delta_{\rho 0}\lambda_{\rho l}]s_{\rho l} \\ &\quad + \sum_{\eta=0}^{l-1}[\lambda_{k\eta}\Delta_{0\eta}\hat{u}_{klk\eta} + \hat{u}_{k,l,k,\eta+1}\Delta_{0\eta}\lambda_{k\eta}]s_{k\eta}. \end{aligned} \tag{21}$$

Next, we present the following Lemma for two dimensional case, which is similar to the one dimensional formula helpful in proving our main result.

Lemma 1.*(see [5]) Let $(v_{\rho\eta})$ and $(w_{\rho\eta})$ be two double sequences. Then*

$$\begin{aligned} &\Delta_{\rho\eta}(v_{\rho\eta}w_{\rho\eta}) \\ &= w_{\rho\eta}\Delta_{\rho\eta}v_{\rho\eta} + (\Delta_{0\eta}v_{\rho+1,\eta})(\Delta_{\rho 0}w_{\rho\eta}) \\ &\quad + (\Delta_{\rho 0}v_{\rho,\eta+1})(\Delta_{0\eta}w_{\rho\eta}) + v_{\rho+1,\eta+1}\Delta_{\rho\eta}w_{\rho\eta}. \end{aligned} \tag{22}$$

In the year 2008, Savaş [1] has proved a theorem for generalized absolute summability factors. Subsequently, Savaş and Rhoades [5] has proved some inclusion theorems based on double absolute summability factor theorems and applications. Furthermore, in 2018, Jena *et al.* [6] has established a result on $|A;\delta|_k$-summability. Also, many interesting results related to matrix summability were provided by many researchers in [7,8,9,10].

Motivated essentially by the above-mentioned works, here based on $|U,\delta,\gamma|_q$-summability of double infinite lower triangular matrix, we have proved a new theorem that generalizes the result of Jena *et al.* [3]. Finally, at the concluding section we have presented some remarks in support of our result.

2 Main Result

The purpose of the article is to generalize the result of Jena *et al.* [3] for $|U,\delta,\gamma|_q$-summability, where $q \geq 1$.

Theorem 1.*Let U be a doubly triangular matrix with non-negative terms satisfying*

$$\Delta_{11}u_{k-1,l-1,\rho,\eta} \geq 0, \tag{23}$$

$$\sum_{v=0}^{l} u_{kl\rho v} = \sum_{v=0}^{l-1} u_{k,l-1,\rho,v} := b(k,\rho)$$

$$\text{and} \quad \sum_{\mu=0}^{k} u_{kl\mu\eta} = \sum_{\mu=0}^{k-1} u_{k-1,l,\mu,\eta} := u(l,\eta), \quad (24)$$

$$klu_{klkl} = \mathscr{O}(1), \quad (25)$$

$$u_{kl\rho\eta} \geq max\{u_{k,l+1,\rho,\eta}, u_{k+1,l,\rho,\eta}\}, \quad (26)$$

where $(k \geq \rho, l \geq \eta;\ \rho,\eta = 0,1,...)$,

$$\sum_{\rho=0}^{k}\sum_{\eta=0}^{l} u_{kl\rho\eta} = \mathscr{O}(1), \quad (27)$$

$$\sum_{k=\rho+1}^{K+1}\sum_{l=\eta+1}^{L+1} (kl)^{\gamma\delta q}|\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}| = \mathscr{O}((\rho\eta)^{\gamma\delta q}u_{\rho\eta\rho\eta}), \quad (28)$$

$$\sum_{k=\rho+1}^{K+1}\sum_{l=\eta+1}^{L+1} (kl)^{\gamma\delta q}\hat{u}_{k,l,\rho+1,\eta+1} = \mathscr{O}\left((\rho\eta)^{\gamma\delta q}\right). \quad (29)$$

Also, let (χ_{kl}) *be a given double sequence of positive numbers and suppose that* $(s_{kl}) = \mathscr{O}(\chi_{kl})$ $(k,l \to \infty)$. *If* $(\lambda_{kl}) \in \mathbb{R}$ *satisfying*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(kl)^{\gamma\delta q}u_{klkl}(|\lambda_{kl}|\chi_{kl})^q<\infty, \quad (30)$$

$$\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}(\rho\eta)^{\gamma\delta q}|\Delta_{0\eta}\lambda_{\rho\eta}|\chi_{\rho\eta} = \mathscr{O}(1), \quad (31)$$

$$\sum_{\rho=0}^{\infty}\sum_{\eta=0}^{\infty}(\rho\eta)^{\gamma\delta q}|\Delta_{\rho 0}\lambda_{\rho\eta}|\chi_{\rho\eta}<\infty, \quad (32)$$

$$\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}(\rho\eta)^{\gamma\delta q}|\Delta_{\rho\eta}\lambda_{\rho\eta}|\chi_{\rho\eta} = \mathscr{O}(1), \quad (33)$$

and

$$\sum_{\rho=0}^{k}\sum_{\eta=0}^{l}(\rho\eta)^{\gamma\delta q}(|\lambda_{\rho\eta}|\chi_{\rho\eta})^q = \mathscr{O}(1), \quad (34)$$

then the series $\sum\sum b_{kl}\lambda_{kl}$ *is summable* $|U,\delta,\gamma|_q$ $(q \geq 1;\ 0 \leq \delta \leq 1/q)$.

Proof. To prove our main result, it is enough to show that

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(kl)^{\gamma(\delta q+q-1)}|\Delta_{11}y_{kl}|<\infty.$$

By using Lemma 1, we have

$$\Delta_{\rho\eta}(\hat{u}_{kl\rho\eta}\lambda_{\rho\eta}) = \lambda_{\rho\eta}\Delta_{\rho\eta}(\hat{u}_{kl\rho\eta}) + (\Delta_{0\eta}\hat{u}_{k,l,\rho+1,\eta})(\Delta_{\rho 0}\lambda_{\rho\eta})$$
$$+ (\Delta_{rho0}\hat{u}_{k,l,\rho,\eta+1})(\Delta_{0\eta}\lambda_{\rho\eta}) + \hat{u}_{k,l,\rho+1,\eta+1}\Delta_{\rho\eta}\lambda_{\rho\eta}. \quad (35)$$

Now, using the above condition

$$\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}\Delta_{\rho\eta}(\hat{u}_{kl\rho\eta}\lambda_{\rho\eta})s_{\rho\eta}$$
$$= \sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}[\lambda_{\rho\eta}(\Delta_{\rho\eta}\hat{u}_{kl\rho\eta})(\Delta_{0\eta}\hat{u}_{k,l,\rho+1,\eta})(\Delta_{\rho 0}\lambda_{\rho\eta}) + (\Delta_{\rho 0}\hat{u}_{k,l,\rho,\eta+1})(\Delta_{0\eta}\lambda_{\rho\eta}) + \hat{u}_{k,l,\rho+1,\eta+1}(\Delta_{\rho\eta}\lambda_{\rho\eta})]s_{\rho\eta}. \quad (36)$$

Next, using (20), (21) and (36), we may write

$$\Delta_{11}y_{k-1,l-1} = \sum_{r=1}^{9}\mathscr{T}_{klr}.$$

Now using Minkowski's inequality, it is suffices to show,

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(kl)^{\gamma(\delta q+q-1)}|\mathscr{T}_{klr}|^q := J_r<\infty \quad (r = 1,2,\cdots,9).$$

For r=1, we have

$$J_1 = \mathscr{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}\left(\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}||\lambda_{\rho\eta}|^q|\chi_{\rho\eta}|^q\right) \times \left(\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}|\right)^{q-1}.$$

Also, from (19)

$$\hat{u}_{kl\rho\eta} = \Delta_{11}\overline{u}_{k-1,l-1,\rho,\eta} = \sum_{\mu=\rho}^{k-1}\sum_{v=\eta}^{l-1}u_{k-1,l-1,\rho,\eta} - \sum_{\mu=\rho}^{k}\sum_{v=\eta}^{l-1}u_{k,l-1,\rho,\eta} - \sum_{\mu=\rho}^{k-1}\sum_{v=\eta}^{l}u_{k-1,l,\rho,\eta} - \sum_{\mu=\rho}^{k}\sum_{v=\eta}^{l}u_{kl\rho\eta}.$$

Again since,

$$u_{k-1,l,k,v} = u_{k,l-1,\mu,l} = 0,$$

so, by using (15) and (24),

$$\hat{u}_{kl\rho\eta} = \sum_{\mu=\rho}^{k-1}[b(k-1,\mu) - \sum_{v=0}^{\eta-1}u_{k-1,l-1,\mu,v} - b(k,\mu) + \sum_{v=0}^{\eta-1}u_{k,l-1,\mu,v} - b(k-1,\mu) + \sum_{v=0}^{\eta-1}u_{k-1,l,\mu,v} + b(k,\mu) - \sum_{v=0}^{\eta-1}u_{k,l,\mu,v}]$$
$$= \sum_{\mu=\rho}^{k-1}\sum_{v=\eta}^{l-1}(-u_{k-1,l-1,\mu,v} + u_{k,l-1,\mu,v} + u_{k-1,l,\mu,v} - u_{k,l,\mu,v})$$

$$= \sum_{v=0}^{\eta-1}\sum_{\mu=\rho}^{k-1}(-u_{k-1,l-1,\mu,v}+u_{k,l-1,\mu,v}+u_{k-1,l,\mu,v}-u_{k,l,\mu,v})$$
$$= \sum_{v=0}^{\eta-1}[-u(k-1,v)+\sum_{\mu=0}^{\eta-1}u_{k-1,l-1,\mu,v}+u(k,v)$$
$$-\sum_{\mu=0}^{\rho-1}u_{k,l-1,\mu,v}+u(k-1,v)-\sum_{\mu=0}^{\rho}u_{k-1,l,\mu,v}-u(k,v)$$
$$+\sum_{\mu=0}^{\rho}u_{k,l,\mu,v}]$$
$$= \sum_{\mu=0}^{\rho-1}\sum_{v=0}^{\eta-1}\Delta_{11}u_{k-1,l-1,\mu,v}\geq 0. \tag{37}$$

Next, using (15) and (37),

$$\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}$$
$$= \left(\sum_{\mu=0}^{\rho-1}\sum_{v=0}^{\eta-1}-\sum_{\mu=0}^{\rho}\sum_{v=0}^{\eta-1}-\sum_{\mu=0}^{\rho-1}\sum_{v=0}^{\eta}+\sum_{\mu=0}^{\rho}\sum_{v=0}^{\eta}\right)\Delta_{11}u_{k-1,l-1,\mu,v}$$
$$= \Delta_{11}u_{k-1,l-1,\rho,\eta}.$$

Again, from the condition (24),

$$\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}$$
$$= \sum_{\rho=0}^{k-1}(b(k-1,\rho)-b(k,\rho)-b(k-1,\rho)+u_{k-1,l,\rho,l}$$
$$+b(k,\rho)-u_{kl\rho l})$$
$$= \sum_{\rho=0}^{k-1}(u_{k-1,l,\rho,l}-u_{kl\rho l})$$
$$= u(l,l)-u(l,l)+u_{klkl}.$$

Now, using the condition (25), we get

$$J_1 = \mathscr{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(klu_{klkl})^{\gamma(q-1)}(kl)^{\gamma\delta q}$$
$$\times\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}||\lambda_{\rho\eta}|^q\chi_{\rho\eta}^q$$
$$= \mathscr{O}(1)\sum_{k=1}^{K}\sum_{l=1}^{L}(|\lambda_{\rho\eta}|\chi_{\rho\eta})^q\sum_{k=\rho+1}^{K+1}\sum_{l=\eta+1}^{L+1}(kl)^{\gamma\delta q}|\Delta_{\rho\eta}\hat{u}_{kl\rho\eta}|.$$

Moreover, using the condition (28) and (29), we have

$$J_1 = \mathscr{O}(1)\sum_{k=1}^{K}\sum_{l=1}^{L}(\rho\eta)^{\gamma\delta q}u_{ijij}(|\lambda_{\rho\eta}|\chi_{\rho\eta})^q$$
$$= \mathscr{O}(1).$$

Next, for $r=2$, we have

$$J_2 = \mathscr{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}$$
$$\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\Delta_{0\eta}\hat{u}_{k,l,\rho+1,\eta}||\Delta_{\rho 0}\lambda_{\rho\eta}|\chi_{\rho\eta}\right]$$
$$\times\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\Delta_{0\eta}\hat{u}_{k,l,\rho+1,\eta}||\Delta_{\rho 0}\lambda_{\rho\eta}|\chi_{\rho\eta}\right]^{q-1}.$$

Using (37) and (24), we have

$$0 \leq \hat{u}_{k,l,\rho+1,\eta} \tag{38}$$
$$= \sum_{\mu=0}^{\rho}\sum_{v=0}^{\eta-1}\Delta_{11}u_{k-1,l-1,\mu,v}$$
$$\leq \sum_{\mu=0}^{k-1}\sum_{v=0}^{l-1}(u_{k-1,l-1,\mu,v}-u_{k,l-1,\mu,v}-u_{k-1,l,\mu,v}+u_{k,l,\mu,v})$$
$$= \sum_{\mu=0}^{k-1}(b(k-1,\mu)-b(k,\mu)-b(k-1,\mu)+u_{k-1,l,\mu,l}$$
$$+b(k,\mu)-u_{kl\mu v})$$
$$= \sum_{\mu=0}^{k-1}(u_{k-1,l,\mu,l}-u_{kl\mu v})$$
$$= u(l,l)-u(l,l)+u_{klkl}. \tag{39}$$

Again, since

$$|\Delta_{0\eta}\hat{u}_{k,l,\rho+1,\eta}| \leq \hat{u}_{k,l,\rho+1,\eta}+\hat{u}_{k,l,\rho+1,\eta+1},$$

so by using properties (25), (29) and (32), we get

$$J_2 = \mathscr{O}(1)\sum_{k=1}^{K}\sum_{l=1}^{L}|\Delta_{\rho 0}\lambda_{\rho\eta}|\chi_{\rho\eta}$$
$$\sum_{k=\rho+1}^{K+1}\sum_{l=\eta+1}^{L+1}(\rho\eta)^{\gamma\delta q}|\Delta_{0\eta}\hat{u}_{k,l,\rho+1,\eta}|$$
$$= \mathscr{O}(1)\sum_{k=1}^{K}\sum_{l=1}^{L}|\Delta_{\rho 0}\lambda_{\rho\eta}|\chi_{\rho\eta}$$
$$\sum_{k=\rho+1}^{K+1}\sum_{l=\eta+1}^{L+1}(\rho\eta)^{\gamma\delta q}(\hat{u}_{k,l,\rho+1,\eta}+\hat{u}_{k,l,\rho+1,\eta+1})$$
$$= \mathscr{O}(1).$$

In the similar way, it can be proved that

$$J_3 = \mathscr{O}(1).$$

Next, for $r=4$,

$$J_4 = \mathscr{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}$$
$$\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\hat{u}_{k,l,\rho+1,\eta+1}||\Delta_{\rho\eta}\lambda_{\rho\eta}|\chi_{\rho\eta}\right]$$
$$\times\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\hat{u}_{k,l,\rho+1,\eta+1}||\Delta_{\rho\eta}\lambda_{\rho\eta}|\chi_{\rho\eta}\right]^{q-1}.$$

Using (24), (37) and follow the concept used in (38), we have

$$0 \leq \hat{u}_{k,l,\rho+1,\eta+1} = \sum_{\mu=0}^{\rho}\sum_{v=0}^{\eta-1} \Delta_{11} u_{k-1,l-1,\mu,v}$$
$$= u(l,l) - u(l,l) + u_{klkl}.$$

So, by using properties (25), (29) and (31),

$$J_4 = \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(klu_{klkl})^{\gamma(q-1)}(kl)^{\gamma\delta q}$$
$$\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\hat{u}_{k,l,\rho+1,\eta+1}||\Delta_{\rho\eta}\lambda_{\rho\eta}|\chi_{\rho\eta}\right]$$
$$\times\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\Delta_{\rho\eta}\lambda_{\rho\eta}|\chi_{\rho\eta}\right]^{q-1}$$
$$= \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma\delta q}\left[\sum_{\rho=0}^{k-1}\sum_{\eta=0}^{l-1}|\hat{u}_{k,l,\rho+1,\eta+1}||\Delta_{ij}\lambda_{ij}|\chi_{ij}\right]$$
$$= \mathcal{O}(1)\sum_{\rho=0}^{K}\sum_{\eta=0}^{L}(kl)^{\gamma\delta q}|\Delta_{\rho\eta}\lambda_{\rho\eta}|\chi_{\rho\eta}$$
$$= \mathcal{O}(1).$$

Now, for $r = 5$, we have

$$J_5 = \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}\left(\sum_{\rho=0}^{k-1}\lambda_{\rho l}|\Delta_{\rho 0}\hat{u}_{kl\rho l}|\chi_{\rho l}\right)^{q}$$
$$= \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}\left[\sum_{\rho=0}^{k-1}|\Delta_{\rho 0}\hat{u}_{kl\rho l}|(|\lambda_{\rho l}|\chi_{\rho l})^{q}\right]$$
$$\times\left[\sum_{\rho=0}^{k-1}|\Delta_{\rho 0}\hat{u}_{kl\rho l}|\right]^{q-1}.$$

Also, from (19),

$$\Delta_{\rho 0}\hat{u}_{kl\rho l} = \Delta_{\rho 0}(\Delta_{11}\overline{u}_{k-1,l-1,\rho,l})$$
$$= \Delta_{\rho 0}\left(-\sum_{\mu=\rho}^{k-1}u_{k-1,l,v,l} + \sum_{\mu=\rho}^{k}u_{kl\mu l}\right)$$
$$= u_{k-1,l,\rho,l} + u_{kl\rho l} \leq 0.$$

Again, by the property (25), (26) and (29),

$$\sum_{\rho=0}^{k-1}|\Delta_{\rho 0}\hat{u}_{kl\rho l}| = \sum_{\rho=0}^{k-1}(u_{k-1,l-1,\rho,l} - u_{kl\rho l})$$
$$= u(l,l) - u(l,l) + u_{klkl}.$$

Thus, by using property (25), (26) and (30),

$$J_5 = \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(klu_{klkl})^{\gamma(q-1)}(kl)^{\gamma\delta q}$$
$$\left[\sum_{\rho=0}^{k-1}|\Delta_{\rho 0}\hat{u}_{kl\rho l}|(|\lambda_{\rho l}|\chi_{\rho l})^{q}\right]$$
$$= \mathcal{O}(1)\sum_{l=1}^{L+1}\sum_{\rho=0}^{K}(|\lambda_{\rho l}|\chi_{\rho l})^{q}\left(\sum_{\rho=0}^{k-1}(kl)^{\gamma\delta q}|\Delta_{\rho 0}\hat{u}_{kl\rho l}|\right)$$
$$= \mathcal{O}(1).$$

Further, for r=6

$$J_6 = \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}\left[\sum_{\rho=0}^{k-1}|\hat{u}_{k,l,\rho+1,l}||(\Delta_{\rho 0}\lambda_{\rho l})|\chi_{\rho l}\right]$$
$$\times\left[\sum_{\rho=0}^{k-1}|\hat{u}_{k,l,\rho+1,l}||(\Delta_{\rho 0}\lambda_{\rho l})|\chi_{\rho l}\right]^{q-1}.$$

We have, from (19), (24), and using the concept in (38)

$$\hat{u}_{k,l,\rho+1,l} = u(l,l) - u(l,l) + u_{klkl}.$$

Clearly, using conditions (25), (29) and (32), we get

$$J_6 = \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(klu_{klkl})^{\gamma(q-1)}(kl)^{\gamma\delta q}$$
$$\left[\sum_{\rho=0}^{k-1}|\hat{u}_{k,l,\rho+1,l}||(\Delta_{\rho 0}\lambda_{\rho l})|\chi_{\rho l}\right]$$
$$\times\left[\sum_{\rho=0}^{k-1}|(\Delta_{\rho 0}\lambda_{\rho l})|\chi_{\rho l}\right]^{q-1}$$
$$= \mathcal{O}(1)\sum_{k=1}^{K}\sum_{l=1}^{L+1}(kl)^{\gamma\delta q}|\Delta_{\rho 0}\lambda_{\rho l}|\chi_{\rho l}$$
$$= \mathcal{O}(1).$$

Furthermore, for $r = 7$

$$J_7 = \mathcal{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}$$
$$\left[\sum_{\eta=0}^{l-1}|\Delta_{0\eta}\hat{u}_{klk\eta}|(|\lambda_{k\eta}|\chi_{k\eta})^{q}\right]$$
$$\times\left[\sum_{\eta=0}^{l-1}|\Delta_{0\eta}\hat{u}_{klk\eta}|\right]^{q-1}.$$

Also, from (15),

$$\hat{u}_{klk\eta} = -\sum_{v=\eta}^{l-1}u_{k,l-1,k,\eta} + \sum_{v=\eta}^{l}u_{k,l,k,\eta}.$$

Again, since

$$\Delta_{0\eta}\hat{u}_{klk\eta} = -u_{k,l-1,k,\eta} + u_{k,k,k,\eta},$$

so, properties (26) and (24), yields

$$\sum_{\eta=0}^{l-1}|\Delta_{0\eta}\hat{u}_{klk\eta}| = b(k,k) - b(k,k) + u_{klkl}.$$

Clearly, using (25), (28) and (31), we get

$$J_7 = \mathscr{O}(1).$$

Next, for $r = 8$

$$J_8 = \mathscr{O}(1)\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\gamma(\delta q+q-1)}\left[\sum_{\eta=0}^{l-1}\hat{u}_{k,l,k,\eta+1}(\Delta_{0\eta}\lambda_{k\eta})\chi_{k\eta}\right] \times\left[\sum_{\eta=0}^{l-1}\hat{u}_{k,l,k,\eta+1}(\Delta_{0\eta}\lambda_{k\eta})\chi_{k\eta}\right]^{q-1}.$$

Now in the similar lines as in the proof of J_6 and by using properties (25), (29) and (31), we get

$$J_8 = \mathscr{O}(1).$$

Finally, for r=9 and from properties (24), (25), (27) and (34), together with (20) and under the consideration of $\hat{u}_{klkl} = u_{klkl}$, we get

$$J_9 = \sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(klu_{klkl})^{\gamma(q-1)}(kl)^{\gamma\delta q}u_{klkl}(|\lambda_{kl}|\chi_{kl})^q = \mathscr{O}(1).$$

The proof of Theorem 1 has been completed.

3 Concluding Remarks and Observations

In this concluding section of our investigation, we present here various remarks and observations concerning the criterion for double triangular matrix $(\overline{N},p,q)$ [5] and accordingly establish a factorable double weighted mean matrix $(\overline{N},p,q,\delta)$ with entries,

$$u_{kl\rho\eta} = \frac{p_\rho q_\eta}{P_k Q_l},$$

where (p_k), (q_l) are non-negative sequnces with $p_0, q_0 > 0$, and

$$P_k = \sum_{\rho=0}^{k} p_\rho \to \infty;\quad Q_l = \sum_{\eta=0}^{l} q_\eta \to \infty.$$

*Remark.*Suppose that $(\overline{N},p,q,\delta)$ satisfies

$$\frac{kl p_k q_l}{P_k Q_l} = O(1); \tag{40}$$

$$\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\delta q}\left|\frac{p_k q_l}{P_k Q_l P_{k-1} Q_{l-1}}\right| = O\left(\frac{(\rho\eta)^{\delta q}}{P_{\rho\eta} Q_{\rho\eta}}\right), \tag{41}$$

let (χ_{kl}) be a given double sequence of positive numbers and suppose that $(s_{kl}) = O(\chi_{kl})$ $(k,l \to \infty)$. If $(\lambda_{kl}) \in R$ satisfying

$$\sum_{k=1}^{K+1}\sum_{l=1}^{L+1}(kl)^{\delta q}\frac{p_k q_l}{P_k Q_l P_k Q_l}(|\lambda_{kl}|\chi_{kl})^q < \infty, \tag{42}$$

and condition (31) to (34) of Theorem 1, then $\sum\sum b_{kl}\lambda_{kl}$ is summable $|\overline{N},p,q,\delta|_q$ $(q \geq 0)$.

*Remark.*Let $s_{kl} = \sum_{\rho=0}^{k}\sum_{\eta=0}^{l} b_{\rho\eta}$, define

$$U_q = \left\{s_{kl} : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(kl)^{\delta q+q-1}|b_{kl}|^q \leq \infty\right\}.$$

A double infinite matrix $(U,\delta) \in B(U_q)$, if every sequence in U_q is summable $|U,\delta|_q$.

*Remark.*Let $\gamma = 1$ and U satisfy conditions (23) to (29) of Theorem 1. Then $(U,\delta) \in B(U_q)$.

*Remark.*In the result of this paper by taking $\delta = 0$, the double absolute $|U|_q$-summability can be obtained from Theorem 1.

*Remark.*If we take $\gamma = 1$ in the Theorem 1, then we get a result of Jena *et. al* [3] on double absolute indexed matrix summability.

*Remark.*Motivated by the recently-published results of Das *et al.* [11] and Pradhan *et al.* (see [12] and [13]) the interested reader's attention is drawn toward the possibility of investigating the basic idea of summability of infinite series towards application areas of science like a rectification of signals in FIR filter and IIR filter to speed of the rate of convergence. Using these techniques, the output of the waves can be made more balanced because of the behaviour of the input.

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Conflict of Interest The authors declare that they have no conflict of interest

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