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On the Matrix Method Algorithm of Doubly Heavy Meson to Calculate Masses Spectra

H. I. Alrebdi[1], *Atef Ismail*[2], *Asmaa Sayed*[3] *and M. Allosh*[3,*]

[1]Department of Physics, College of Science, Princess Nourah bint Abdulrahman University, P.O.Box 84428, Riyadh 11671, Saudi Arabia
[2]Department of Physics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
[3]Physics Department, Faculty of Science, South Valley University, Qena 83523, Egypt

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Abstract: This work is devoted to propose a new solution to Schrödinger equation into a representation of the kinetic energy operator on a discrete lattice. The matrix method is illustrated by studying stability of some heavy charm mass spectra. Theoretical calculations are in good agreements with newly published experimental data.

Keywords: Matrix method, Schrödinger equation, kinetic energy, heavy charm.

1 Introduction

Quarkonia are mesons with hidden flavor. They consist of one quark and its anti–quarks. A charmonium is a 'charm–quarkonium'; a bound state consisting of a charm quark and its anti-quarks. Due to their huge masses, the quantum numbers and basic properties of most states in the charmonium family [1] can be described within a simple model of a non-relativistic quark–anti–quark pair ($c\overline{c}$). In this model, the states, characterized by the orbital angular momentum L, total spin S of the quark pair, and the total angular momentum J. The total angular momentum is reproduced easily by the vector sum of the orbital and the spin momenta as per below

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{1}$$

The total spin S is determined by the vector sum of the quark and anti–quark spins;

$$\mathbf{S} = \mathbf{S}_C + \mathbf{S}_{\overline{C}}, \tag{2}$$

Moreover, the total spin S should be even 0 or 1 only, thus splitting of the four possible spin states of the pair into singlet and triplet states. Moreover, the excitation of the radial motion of the $c\overline{c}$ pair [2,3,4] results in a spectrum of levels with the same L, S and J, and also differing by this spectrum. It is customary to insert the values of these quantum numbers for each charmonium state in well–known spectroscopic notation form $n^{2S+1}L_J$, where $n = n_r + 1$, n_r is the radial quantum number.

The time–independent Schrödinger Equation (SE) is one of the basic equations in quantum mechanics. Its solution is required in studying of atoms, molecules and their underlying structures in addition to their spectra. Many numerical methods (e.g., matrix method [5], Numerov method [6,7,8,9,10], eigenfunctions expansion method [8] and Newton method) have been used in solving SE. One of the most important and simplistic matrix schemes is extended to deduce the solution of time–independent Schrödinger equation in spherical symmetric $Q\overline{Q}$ potentials [10,11], and this scheme is referred to as Matrix method. In this work, the matrix method will be introduced effectively to solve SE numerically. The method is used to deduce the eigenvalues and eigenfunctions of SE. The obtained results are used to calculate spectra of some $c\overline{c}$ charmonium states. Computed masses will be compared to with experimental data. This paper is organized as follows: In Section 2, the potential model and the considered method used in solving Schrödinger equation are outlined. Section 3, is devoted to discuss the results of S, P and D charmonium states. And finally a short summary is given in Section 4.

* Corresponding author e-mail: m_allosh67@sci.svu.edu.eg

2 Numerical method

The potential model used in solving SE is [12,13]

$$V(r) = \frac{-4\alpha_s}{3r} + br + \frac{32\pi\alpha_s}{9m_c^2}(\frac{\sigma}{\sqrt{\pi}})^3 e^{-\sigma^2 r^2} S_C.S_{\overline{C}} + \frac{L(L+1)}{2\mu r^2} + \frac{1}{m_c^2}[(\frac{\alpha_s}{r^s} - \frac{b}{2r})\mathbf{L.S} + \frac{\alpha_s}{r^s}T], . \quad (3)$$

where $S_C.S_{\overline{C}} = \frac{S(S+1)}{2} - \frac{3}{4}$, The spin–spin contact hyperfine interaction term $S_C.S_{\overline{C}}$ is one of the spin–dependent terms predicted by OGE (One Gloun Exchange) forces [14,15]. The reduced mass of the quark–anti–quark is referred to as μ, m_c is the mass of the charm quark, and S is the total spin quantum number of the meson. For the considered mesons, the parameters α_s, b, σ, and m_c are taken to be 0.5461, 0.1425 GeV^2, 1.0946 GeV and 1.4796 GeV, respectively [16]. T represents the tensor operator, and the spin–orbit operator is diagonal in a $|J,L,S>$ basis, with the matrix elements

$$<\mathbf{L.S}>=[J(J+1)-L(L+1)-S(S+1)]/2\,, \quad (4)$$

Charmonium properties could be defined as a wave function of the bound quark–antiquark state that satisfies the SE using the potential model given in the Eq.(1). Radial Schrödinger equation with wave function $U(r) = rR(r)$ is written (in natural units) as

$$\nabla^2 U(r) + 2\mu(E - V(r))U(r) = 0\,, \quad (5)$$

where R(r) is the radial wave function, r is the interquark distance and E is the total energy of quark–anti–quark system. Eq.(3) could be rewritten as

$$\frac{-1}{2\mu}\frac{\delta^2}{dr^2}U(r) + [V(r) + \frac{l(l+1)}{2\mu r^2}]U(r) = EU(r)\,, \quad (6)$$

The second derivative form of U(r) function can be written as

$$\frac{d^2U(i)}{dr^2} = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + O(h^2)\,, \quad (7)$$

where h is the interval between two points

$$h = \frac{R_{Max} - R_{Min}}{N}\,, \quad (8)$$

where R_{max} and R_{min} are the extreme values of the distance between the quark–anti–quark, N is the matrix order.

An arbitrary value for r could be defined as per below

$$r_i = R_{min} + ih, \qquad i = 1,2,..,N-1\,, \quad (9)$$

Hence, we can rewrite Schrödinger equation for r_i as per blow

$$-\frac{U(r_i+h) - 2U(r_i) + U(r_i-h)}{2\mu h^2} + [V(r) + m_1 + m_2 + \frac{l(l+1)}{2\mu r^2}]U(r_i) = EU(r_i)\,, \quad (10)$$

$$-\frac{U_{i+1} - 2U_i + U_{i-1}}{2\mu h^2} + [V(r) + m_1 + m_2 + \frac{l(l+1)}{2\mu r^2}]U_i = EU_i\,, \quad (11)$$

where $U_i = U(r_i)$ and $U_{i\pm1} = U(r_i \pm h)$

$$-\frac{1}{2\mu h^2}U_{i+1} + [V(r) + m_1 + m_2 + \frac{l(l+1)}{2\mu r^2}]U_i - \frac{1}{2\mu h^2}Ur_i = U_i\,, \quad (12)$$

which could be written as

$$e_i U_{i+1} + d_i U_i + e_i U_{i-1} = EU_i\,, \quad (13)$$

$$d(i) = \frac{1}{\mu h^2} + V(r) + \frac{l(l+1)}{2\mu r^2}], e(i) = \frac{-1}{2\mu h^2}\,. \quad (14)$$

Eq.(11) could be transformed into a matrix form in which d (i) (e (i)) represents the diagonal (non–diagonal) elements, respectively. To solve this matrix to get the spectra of charmonium, N and R_{max} must be determined accurately.

3 Result and discussion

In this work, the reliability of the matrix method is studied by extracting the matrix method coefficients which are the matrix order N and the maximum distance between quark and anti–quark R_{max}. The matrix method coefficients is employed in studying the spectra of charmonium. The calculated spectra are in reasonable agreement with new published calculated data [16] and recent experimental data [17] by using χ^2 relation.

$$\chi^2 = \frac{1}{n}\sum_{k=1}^{n}(M_k^{exp.} - M_k^{cal.})^2\,. \quad (15)$$

where n is the number of existing experimental data, $M_k^{exp.}$ is the experimental mass and $M_k^{cal.}$ is calculated mass. To solve Schrödinger equation by using matrix method, an accurate values of N and R_{max} must give a stable value for the charmonium mass spectra by setting $R_{max} = 20\ fm$ while changing N to the best fitting value. The relation between N and the reverse of the calculated spectra masses in GeV^{-1} of charmonium is studied by changing the values of N from 50 to 300 with interval 5. A FORTRAN program is constructed to accomplish this task.

The stability of the reverse of mass is found around $N \geq 45,75,110$ and 142 for $1S,2S,3S$ and $4S$ states, respectively, at $R_{max} = 20\ fm$. Similarly, we get the best value for N by fitting the relationship between reverse masses of P,D states and N at $N = 61,86$ and 110 for $1P,1D$ and $2D$, respectively. Figures 1,2 and 3, represent the relation between calculated spectrum and the order of the matrix N at $R_{max} = 20\ fm$ for S,P and D charmonium states, respectively. Furthermore, studying the behavior of the reversed calculated masses by changing the values of R_{max} from 2 fm to 50 fm at $N = 200$ is done perfectly.

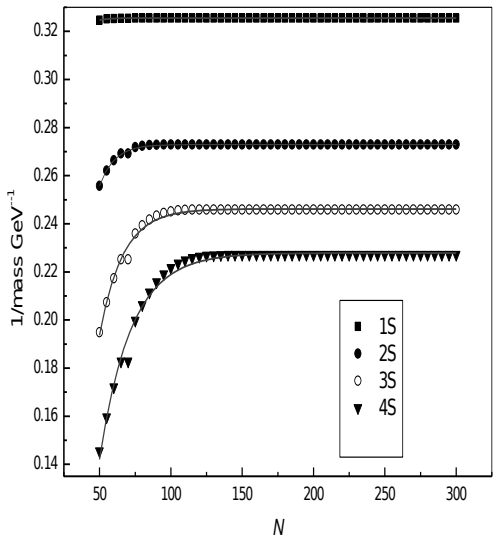

Fig. 1: The relation between theoretical spectrum and the order of the matrix N at $R_{max} = 20fm$ of $S-$ charmonium states.

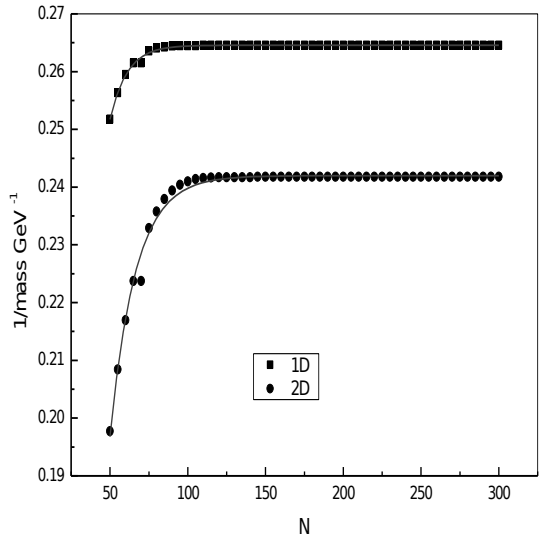

Fig. 3: : the relation between theoretical spectrum and the order of the matrix N at $R_{max} = 20fm$ of $D-$ charmonium states.

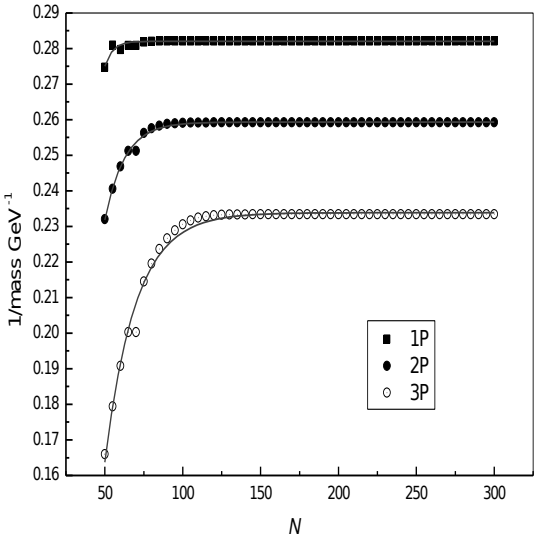

Fig. 2: The relation between theoretical spectrum and the order of the matrix N at $R_{max} = 20fm$ of $P-$ charmonium states.

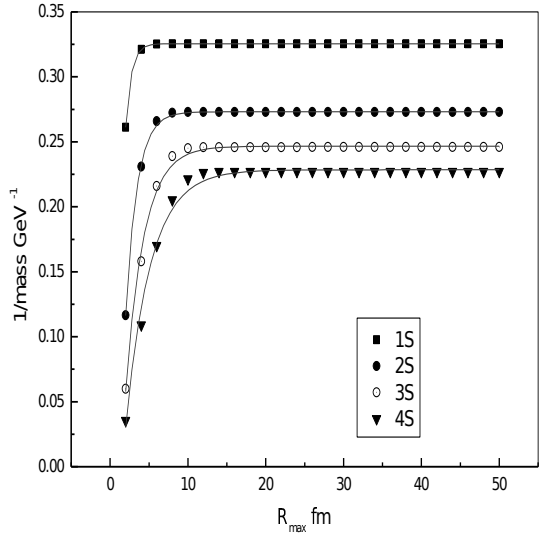

Fig. 4: the relation between theoretical spectrum and the distance between the quark and anti-quark R_{max} for $S-$ states.

The values which make stability of spectra are found at $R_{max} \geq 4,9,12$ and 17 for $1S, 2S, 3S$ and $4S$ states, respectively, at $N = 200$, and $14, 10, 12$ for $1P, 1D, 2D$, respectively.

The calculated values of the masses spectra as well as the experimental values for the considered states and best fits χ^2 are listed in table 1. From this table, one can easily notice that the computed values of the spectra by using the matrix method algorithm are in good agreements with the measured data. This conclusion comes closer to the smaller values of the fitting parameter χ^2, ranges from 0.0002 to 0.0003, imply the success of describing the experimental data for the above algorithm.

Figures (1 - 3) represent the relation between the calculated spectrum and the order of matrix N at $R_{max} = 2\ (fm)$ for S, P and D charmonium states, respectively. Figures (4 - 6) represent the relation between calculated spectrum and the distance between the quark and anti–quark R_{max} at N=200 for S, P and D charmonium states, respectively.

From the previous figures, the best value of N could be set to be larger than or equal to 200. This value could reflect perfectly the calculated masses as well as the

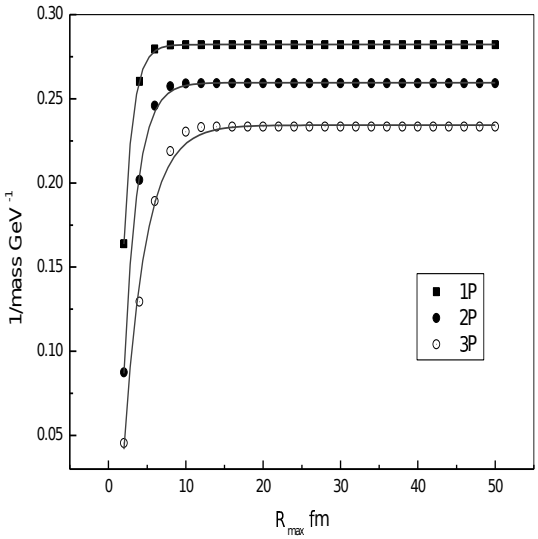

Fig. 5: The relation between theoretical spectrum and the distance between the quark and anti-quark R_{max} for $P-$ states.

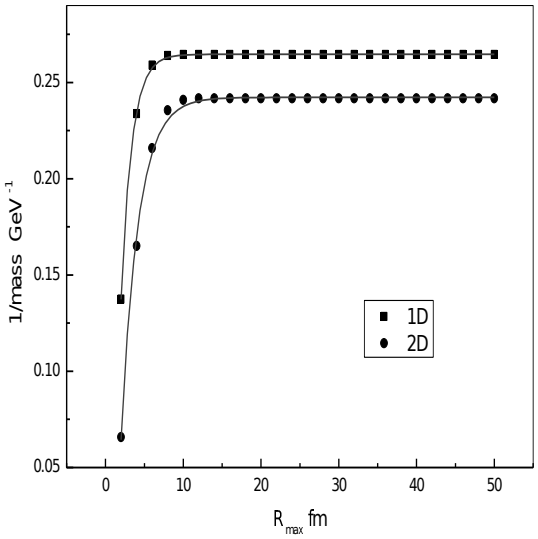

Fig. 6: the relation between theoretical spectrum and the distance between the quark and anti-quark R_{max} of $D-$ states.

experimental data. $R_{max} = 20fm$ could be used safely to obtain theoretical spectra in good accordance with the recently published available experimental data [17]. The small value of χ^2 shows that there is a good agreement between our calculations and the available experimental data as shown in table 1.

4 Conclusion

In this work, Schrödinger equation is solved numerically as an eigenvalue problem within the matrix method. From the current study we noticed that the corresponding obtained calculated spectra reflect reasonable description to the new published experimental data [17] and recent theoretical data [16]. The obtained results revealed that predicted masses of S, P and D waves are very close to the existing experimental values and other hypothetical assessments. We have newly predicted states for the S, P and D waves state of heavy mesons. It is found that the method is simple for calculation and plotting of accurate eigenvalues. The prediction of the model agrees well with the experimental data. New extended studies by using the matrix method to obtain the series of mesons spectra such as $b\overline{b}$, $c\overline{n}$, $c\overline{s}$, and $c\overline{b}$ are recommended.

Table 1: Experimental and calculated mass spectra of $(c\overline{c})$ states in GeV. Summation of errors squares between Calculated and measured masses χ^2 have been outlined.

Name	State	Experimental Masses	Calculated Masses	Ref.[16]
		MeV[17]	MeV	
$j/\psi(1S)$	1^3S_1	3096.87± 0.04	3072	3090
$\eta_C(1S)$	1^0S_1	2979.2 ± 1.3	3047	2982
$\psi(2S)$	2^3S_1	3685.96 ± 0.09	3664	3672
$\eta_C(2S)$	2^0S_1	3637.7 ± 4.4	3653	3630
$\psi(3S)$	3^3S_1	4040 ± 10	4066	4072
$\eta_C(3S)$	3^0S_1	-	4059	4043
$\psi(4S)$	4^3S_1	4415 ± 6	4401	4406
$\eta_C(4S)$	4^0S_1	-	4395	4384
$\chi_2(1P)$	1^3P_2	3556.18 ± 0.13	3545	3556
$\chi_1(1P)$	1^3P_1	3510.51 ± 0.12	3507	3505
$\chi_0(1P)$	1^3P_0	3415.3 ± 0.4	3401	3424
$h_c(1P)$	1^1P_1	-	3521	3516
$\chi_2(2P)$	2^3P_2	-	3959	3972
$\chi_1(2P)$	2^3P_1	-	3929	3925
$\chi_0(2P)$	2^3P_0	-	3857	3852
$h_c(2P)$	2^1P_1	-	3939	3934
$\chi_2(3P)$	3^3P_2	-	4303	4317
$\chi_1(3P)$	3^3P_1	-	4276	4271
$\chi_0(3P)$	3^3P_0	-	4219	4202
$h_c(3P)$	3^1P_1	-	4284	4279
$\psi_3(1D)$	1^3D_3	-	3805	3806
$\psi_2(1D)$	1^3D_2	-	3800	3800
$\psi_1(1D)$	1^3D_1	3769.9 ± 2.5	3780	3785
$\psi_{c2}(1D)$	1^1D_2	-	3800	3799
$\psi_3(2D)$	2^3D_3	-	4165	4167
$\psi_2(2D)$	2^3D_2	-	4158	4158
$\psi_1(2D)$	2^3D_1	4159 ± 20	4136	4142
$\psi_{c2}(2D)$	2^1D_2		4158	4158
χ^2	-	-	0.0003	0.0002

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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H. I. Alrebdi is working as Assistant Professor of Theoretical Physics at Princess Nourah Bint Abdulrahamn University, Saudi Arabia. Dr. Haifa has received her PhD in 2016 from King Saud University in the area of Theoretical Physics. Her research interests focused on High energy physics, particle physics, nuclear physics, Standard model and beyond, cosmology and computational physics.

Atef Ismail is working as Assistant Professor of Theoretical Physics at Al Azhar University, Egypt. Dr. Atef has received his PhD in 2016 from Al Azhar University in the area of nuclear reactions and radiotherapy. His research interests focused on cancer therapeutics by using the Heavy-Ions (HI) techniques, particle physics, Standard model and beyond, cosmology and computational physics.

Asmaa Sayed is affiliated with the Department of physics Department, Faculty of Science, South valley University, Egypt. Dr. Asmaa has received her PhD in 2021 from South Valley University in the area of particle physics. Her research activities are mainly focused on: search theory, applied Mathematics, High energy physics, particle physics, nuclear physics, Standard model and beyond, cosmology and computational physics.

M. Allosh is affiliated with the Department of physics Department, Faculty of Science, South valley University, Egypt. Dr. Mohamed has received his PhD in 2016 from South Valley University in the area of particle physics. He has published several research papers in leading and well reputed international journals of physical sciences. His research activities are mainly focused on: search theory, applied Mathematics, High energy physics, particle physics, nuclear physics, Standard model and beyond, cosmology and computational physics.