Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160404

A Study on an Extended Total Fuzzy Graph

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Received: 2 Mar. 2022, Revised: 12 Apr. 2022, Accepted: 6 May 2022
Published online: 1 Jul. 2022

Abstract: Total fuzzy graph is defined and its properties as well as its total coloring have been discussed and studied by some scholars. This manuscript defines an extended total fuzzy graph and elaborates it with illustrations. In addition some related properties for extended total fuzzy graphs are stated and proved. Finally, we extant the definition of total coloring of an extended total fuzzy graph and determine the bound for its total chromatic number.

Keywords: Total fuzzy graph, Extended total fuzzy graph, Total coloring

1 . Introduction

Kauffman [1] was the first person to introduce the notion of fuzzy graph (F-graph). But, the base for the development of F-graph theory is due to Rosenfeld [2] as well as Yeh and Band [3]. Rosenfeld described the connectivity concept of F-graphs and Yeh et.al presented its parameter. Most of the theoretical development of F-graph theory is based on Rosenfeld's initial work. Isomorphism in generalized F-graph has been introduced by Samanta [4] to capture the similarity of uncertainties in different networks.
Homomorphism, weak isomorphism, co-weak isomorphism and nearly isomorphism are defined and an application of image visualization was described. A new method to explain the homeomorphic between some fuzzy topological graphs which will be applied in smart cities has been presented by Atefa etal [5]. The idea of an isomorphic picture F−graph as well as its application on a social network have been described by Zuo [6]. The notations of $\mu-$complement, homomorphism, isomorphism, weak isomorphism and co weak isomorphism of regular picture F-graph and mathematical model of communication network and transportation network by using picture fuzzy multi-graph is also studies and its application on transportation network/communication network is also presented by Xiao [7]. S. Kavitha and S. Lavanya defined the total F-graph and studied total chromatic number of total graphs of F-graphs [8].
The main objectives of this study are to define and draw the graph of the new concept called an extended total F-graph, to show its properties and discus the total coloring of such graphs. To show these the manuscript is organized as follows: The first section deals with introduction of the concept under study,while the second section is devoted for preliminary concepts about F-graphs. The key idea of the paper, which is an ETFG is discussed in section thrree. The next two sections, section four and five concentrate on the discussion of results from an ETFG and total colouring of an ETFG. Finally, the paper is concluded in Section six.

2 . Preliminaries

This section presented basic concepts of fuzzy graphs, total fuzzy graphs and total coloring of fuzzy graphs. Unless otherwise mentioned all the concepts are from [9] to [19].

Definition 1. *An ordered triple $G=(V,\sigma,\mu)$, where V is the node set $\{x_1,x_2,\cdots,x_n\},\sigma$ is a fuzzy subset of V, such that $\sigma:V\to[0,1]$ and μ be a fuzzy relation on σ with $\mu:V\to[0,1]$ and that $\sigma:V\times V\to[0,1]$ such that $\mu(x,y)\leq\sigma(x)\wedge\sigma(y)\,\forall x,y\in V$ is called F-graph.*

Definition 2. *$G^*=(V,E)$, where $E\subseteq V\times V$ is called the underlying crisp graph of a F-graph $G=(V,\sigma,\mu)$.*

Definition 3. *Let $G=(V,\sigma,\mu)$ be a F-graph with the underlying set V. Then, the order of G denoted by*

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Order(G) is defined as:

$$Order(G) = \sum_{x \in V} \sigma(x)$$

and size of G denoted by size(G) defined as:

$$Size(G) = \sum_{x,y \in V} \mu(x,y)$$

Definition 4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The degree of a vertex $x \in V$ is defined as*

$$d_G(G) = \sum_{y \neq x, y \in V} \mu(x,y)$$

Definition 5. *Let $G = (V, \sigma, \mu)$ be a F-graph. The busy value of the vertex x in G is $D(x) = \sum_i \sigma(x) \wedge \sigma(x_i)$ where x_i are neighbors of x and the busy value of G is $D(x) = \sum_i \sigma(x_i)$ where x_i's are the vertexes of G.*

Definition 6. *If $\mu(x,y) > 0$, then x and y are said to be adjacent to each other and lie on the edge, $e = (x,y)$. A path P in a F-graph $G = (V, \sigma, \mu)$ is a sequence of distinct nodes $x_0, x_1, x_2, \cdots, x_n$ such that $\mu(x_{i-1}, x_i) > 0, 1 \leq i \leq n$. Here n is called the length of the path.*

Definition 7. *If x,y are nodes in G and if they are connected by means of a path, then the strength of that path is defined as $\Lambda_{i=1}^n = \mu(x_{i-1}, x_i)$. If x,y are connected by means of paths of length k, then $\mu^k(y,x) = sup\{\mu(y,x_1) \wedge \mu(x_1,x_2) \wedge \mu(x_2,x_3) \wedge \cdots \wedge \mu(x_{k-1},x) : y, x_1, x_2, \cdots, x_{k-1}, x \in V\}$. If $y,x \in V$, then the strength of connectedness between y and x is,*

$$\mu^\infty(y,x) = sup\{\mu^k(y,x) : k = 1,2,\cdots\}$$

Definition 8. *Let $G = (V, \sigma, \mu)$ be a F-graph. Then, G is said to be connected if $\mu^\infty(y,x) > 0$ for all $y,x \in \sigma^*$. An arc (y,x) is said to be a strong arc if $\mu(y,x) \geq x\mu^\infty(y,x)$ and a node y, is said to be an isolated node, if $\mu(y,x) = 0$ for all $y \neq x$.*

Definition 9. *$G = (V, \sigma, \mu)$ is a fuzzy cycle if and only if (σ^*, μ^*) is cycle and there does not exist a unique $(x,y) \in \mu^*$ such that $\mu(x,y) = \wedge\{\mu(x,y) : (x,y) \in \mu^*\}$.*

Definition 10. *A family $\Gamma = \gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_k$ of fuzzy sets on $V \cup E$ is called a k-fuzzy total coloring of $G = (V, \sigma, \mu)$, if:*

a) $\max\{\gamma_i(x)\} = \sigma(x)$ *for all $x \in V$ and* $\max\{\gamma_i(x,y)\} = \mu(x,y)$ *for all edges $(x,y) \in E$*
b) $\gamma_i \wedge \gamma_j = 0$
c) For every adjacent vertices x,y of $G, \min\{\gamma_i(x), \gamma_i(y)\} = 0$.

The least value of k for which there exists a k-fuzzy coloring is called the total chromatic number of F-graph G and is denoted by $\chi_T^f(G)$.

3 . An Extended Total Fuzzy Graph (ETFG)

In this section, we introduce the definition of an extended total F-graph and draw its graph.

Definition 11. *Let $G = (V, \sigma, \mu)$ be a F-graph with its underlying set V and crisp graph $G^* = (\sigma^*, \mu^*)$. The pair $ETFG(G) = (\sigma_{ETFG}, \mu_{ETFG}$ of the F-graph G is defined as follows:*
Let the node set of ETFG(G) be $V \cup E$, where V is the vertex set and E is the edge set of the underlying crisp graph. The fuzzy subset σ_{ETFG} is defined on $V \cup E$ as:

$$\sigma_{ETFG}(x) = \sigma(x), \ if \ x \in V$$

$$\sigma_{ETFG}(e) = \mu(e), \ if \ e \in E$$

The fuzzy relation μ_{ETFG} is defined on $(V \cup E) \times (V \cup E)$, called edges of $ETFG(G)$ as:

$$\mu_{ETFG}(x,y) = \mu(x,y), \ if \ x,y \in V$$

$$\mu_{ETFG}(e_i, e_j) = \mu(e_i) \wedge \mu(e_j), \text{ if } e_i \text{ and } e_j \ \ are \ adjacent$$

$$= 0, \text{ Otherwise.}$$

By definition; $\mu_{ETFG}(x,y) \leq \sigma_{ETFG}(x) \wedge \sigma_{ETFG}(y)$ for all $x,y \in V \cup E$. Hence, μ_{ETFG} is a fuzzy relation on the fuzzy subset σ_{ETFG}. Thus, the pair $ETF(\sigma_{ETFG}, \mu_{ETFG})$ is a F-graph, and it is termed as an extended total F-graph of G.

Example 1. Consider the F-fuzzy graph $G = (V, \sigma, \mu)$ where its underlying crisp graph $G^* = (V, E)$ has vertex set $V = \{x_1, x_2, x_3\}$ and edge set $E = \{x_1x_2, x_2x_3, x_3x_1\}$. Let the fuzzy vertex set defined on V be as follows:

$$\sigma(x_1) = 0.4, \sigma(x_2) = 0.5, \sigma(x_3) = 0.7.$$

Let the fuzzy relation defined on the fuzzy edge set be as follows:

$$\mu(x_1,x_2) = 0.2, \mu(x_2,x_3) = 0.4, \mu(x_3,x_1) = 0.4.$$

Since, $\mu(x_i,x_j) \leq \sigma(x_i) \wedge \sigma(x_j)$ for all $x_i, x_j \in V$, the graph $G = (V, \sigma, \mu)$ is a F-graph and its graph is as shown in the figure 4 .

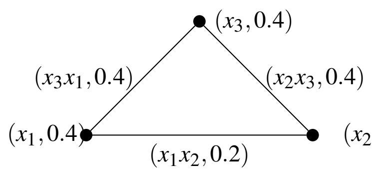

Fig. 1: Fuzzy graph

Now, let us construct the ETFG of the F-graph in the example 1: That is, $ETFG(\sigma_{ETF}, \mu_{ETF})$ of the F-graph G, where the node set of ETFG is $V \cup E$, which is the set $\{x_1, x_2, x_3, x_1x_2, x_2x_3, x_3x_1\}$. We define the fuzzy subset μ_{ETFG} as follows:

$$\mu_{ETFG}(x) = \sigma(x), if u \in V$$
$$\sigma_{ETFG}(e) = \mu(e), if e \in E.$$

Hence, we have the following fuzzy subsets σ_{ETFG}:

$$\sigma_{ETFG}(x_1) = \sigma_(x_1) = 0.4$$
$$\sigma_{ETFG}(x_2) = \sigma(x_2) = 0.5$$
$$\sigma_{ETFG}(x_3) = \sigma(x_3) = 0.7$$
$$\sigma_{ETFG}(x_1x_2) = \mu(x_1, x_2) = 0.2$$
$$\sigma_{ETFG}(x_2x_3) = \mu(x_2, x) = 0.4$$
$$\sigma_{ETFG}(x_3x_1) = \mu(x_3, x_1) = 0.4$$

The fuzzy relations μ_{ETFG} will be as follows:

$$\mu_{ETFG}(x, y) = \mu(x, y), if x, y \in V$$

$\mu_{ETFG}(e_i, e_j) = \mu(e_i) \wedge \mu(e_j)$, if e_i and e_j have a node in common between them.

$= 0$, Otherwise.

Hence,

$$\mu_{ETFG}(x_1, x_2) = \mu(x_1, x_2) = 0.2,$$
$$\mu_{ETFG}(x_2, x_3) = \mu(x_2, x_3) = 0.4,$$
$$\mu_{ETFG}(x_3, x_1) = \mu(x_3, x_1) = 0.4,$$
$$\mu_{ETFG}(x_1x_2, x_2x_3)) = \mu(x_1x_2) \wedge \mu(x_2x_3) = 0.2 \wedge 0.4 = 0.2$$
$$\mu_{ETFG}(x_1x_2, x_3x_1) = \mu(x_1x_2) \wedge \mu(x_3x_1) = 0.2 \wedge 0.4 = 0.2$$
$$\mu_{ETFG}(x_2x_3, x_3x_1) = \mu(x_2x_3) \wedge \mu(x_3x_1) = 0.4 \wedge 0.4 = 0.4$$

Since, $\mu_{ETFG}(x_i, x_j) \leq \sigma_{ETFG}(x_i) \wedge \sigma_{ETFG}(x_j)$ for all $x_i, x_j \in V \cup E$, the graph $ETFG(\sigma_{ETFG}, \mu_{ETFG})$ is a F-graph and from the above node sets $V \cup E$, fuzzy subsets σ_{ETFG} and fuzzy relations μ_{ETFG}, the graph of ETFG of G is as shown in figure 2.

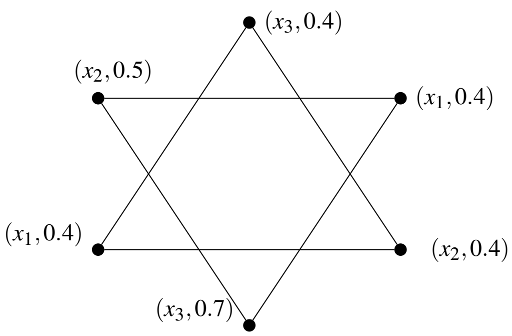

Fig. 2: ETFG of a Fuzzy Graph G

Example 2. Consider the fuzzy graph $G = (V, \sigma, \mu)$ with the fuzzy vertex set;

$$\sigma(x_1) = 0.3, \sigma(x_2) = 0.4, \sigma(x_3) = 0.6, \sigma(x_4) = 0.8$$

and fuzzy edge set:

$$\mu(x_1, x_2) = 0.2, \mu(x_2, x_3) = 0.3,$$
$$\mu(x_3, x_4) = 0.5, \mu(x_4, x_1) = 0.1$$

Since, $\mu(x_i, x_j) \leq \sigma(x_i) \wedge \mu(x_j)$ for all $x_i, x_j \in V$, the graph $G = (V, \sigma, \mu)$ is a F-graph and its graph is as shown in the figure 3 below.

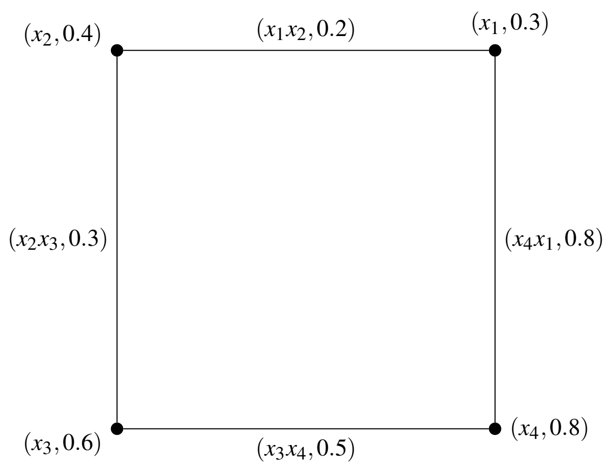

Fig. 3: Fuzzy Graph

Now, from the F-graph in example 2 above the ETFG of G, $ETFG(\sigma_{ETFG}, \mu_{ETFG})$ will be defined as follows:

i.Fuzzy vertex set σ_{ETFG} is as follows:

$$\sigma_{ETFG}(x) = \mu(x)\ , if\ x \in V$$
$$\sigma_{ETFG}(e) = \mu(e),\ if\ e \in E.$$

Hence;

$$\sigma_{ETFG}(x_1) = \sigma(x_1) = 0.3,$$
$$\sigma_{ETFG}(x_2) = \sigma(x_2) = 0.4,$$
$$\sigma_{ETFG}(x_3) = \sigma(x_3) = 0.6,$$
$$\sigma_E TF(x_4) = \sigma(x_4) = 0.8$$
$$\sigma_{ETFG}(x_1x_2) = \mu(x_1, x_2) = 0.2,$$
$$\sigma_{ETFG}(x_2x_3) = \mu(x_2, x_3) = 0.3,$$
$$\sigma_{ETFG}(x_3x_4) = \mu(x_3, x_4) = 0.5$$
$$\sigma_{ETFG}(x_4x_1) = \mu(x_4, x_1) = 0.1$$

ii. The fuzzy edge set μ_{ETFG} is as follows:

$$\mu_{ETFG}(x,y) = \mu(x,y), if x,y \in V$$

$$\mu_{ETFG}(e_i,e_j) = \mu(e_i) \wedge \mu(e_j), \text{ if } e_i \text{ and } e_j \text{ are adjacent.}$$

$$= 0, \text{ Otherwise.}$$

Hence,

$$\begin{aligned}
&\mu_{ETFG}(x_1,x_2) = \mu(x_1,x_2) = 0.2,\\
&\mu_{ETFG}(x_2,x_3) = \mu(x_2,x_3) = 0.3,\\
&\mu_{ETFG}(x_3,x_4) = \mu(x_3,x_4) = 0.5,\\
&\mu_{ETFG}(x_4,x_1) = \mu(x_4,x_1) = 0.1\\
&\mu_{ETFG}(x_1x_2,x_2x_3) = \mu(x_1x_2) \wedge \mu(x_2x_3) = 0.2\\
&\mu_{ETFG}(x_1x_2,x_3x_4,) = \mu(x_1x_2) \wedge \mu(x_3x_4) = 0\\
&\mu_{ETFG}(x_1x_2,x_4x_1) = \mu(x_1x_2) \wedge \mu(x_4x_1) = 0.1\\
&\mu_{ETFG}(x_2x_3,x_3x_4) = \mu(x_2x_3) \wedge \mu(x_3x_4) = 0.3,\\
&\mu_{ETFG}(x_2x_3,x_4x_1) = 0\\
&\mu_{ETFG}(x_3x_4,x_4x_1) = \mu(x_3x_4) \wedge \mu(x_4x_1) = 0.1
\end{aligned}$$

Clearly, $\mu_{ETFG}(x_i,x_j) \leq \sigma_{ETFG}(x_i) \wedge \sigma_{ETFG}(x_j)$ for all $x_i,x_j \in V$ and hence the graph $ETFG(\sigma_{ETFG},\mu_{ETFG})$ is a F-graph.

The graph $ETFG(G) : (\sigma_{ETFG},\mu_{ETFG})$ is an ETFG and its graph is as shown in figure 4:

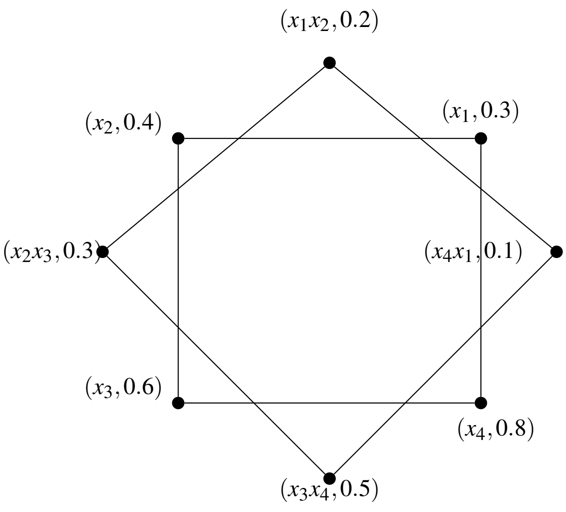

Fig. 4: ETFG of a F-graph G

4 . Some Results from an Extended Total F-graph

Theorem 1. *Let $G = (V,\sigma,\mu)$ be a F-graph, then $Order(ETFG(G)) = Order(G) + Size(G)$.*

Proof. By the definition of an ETFG, the node set of ETFG(G) is $V \cup E$and the fuzzy subset $\sigma_ETFG(x) = \sigma(x)$,if $x \in V$ and $\sigma(e) = \mu(e)$,if $e \in E$. Now,

$$\begin{aligned}
Order(ETFG(G)) &= \sum_{x \in V \cup E} \sigma_{ETFG}(x), \ by\ definition\\
&= \sum_{x \in V} \sigma_{ETFG}(x) + \sum_{x \in E} \sigma_{ETFG}(x)\\
&= \sum_{x \in V} \sigma(x) + \sum_{x \in E} \sigma(x)\\
&= Order(G) + Size(G).
\end{aligned}$$

Theorem 2. *Let $G = (V,\sigma,\mu)$ be a F-graph, then $Size(ETFG(G)) = Size(G) + \sum_{e_i,e_j \in E} \mu(e_i) \wedge \mu(e_j)$*

Proof. By definition of size of a F-graph, we have;

$$\begin{aligned}
size(ETFG(G)) &= \sum_{x,y \in V \cup E} \mu_{ETFG}(x,y), \ by\ definition\\
&= \sum_{x,y \in V} \mu_{ETFG}(x) + \sum_{x \in V, e \in E} \mu_{ETFG}(x,e)\\
&+ \sum_{e_i \in, e_j \in E} \mu_{ETFG}(e_i,e_j)
\end{aligned}$$

(The second summation is zero, since there is no fuzzy relation between $x \in V$ and $e \in E$ in the ETFG)

$$= \sum_{x,y \in V} \mu(x,y) + \sum_{e_i,e_j \in E} \mu(e_i,e_j)$$

$$= size(G) + \sum_{e_i,e_j \in E} \mu(e_i) \wedge \mu(e_j)$$

Theorem 3. *Let $G = (V,\sigma,\mu)$ be a F-graph, then*

$$\begin{aligned}
d(ETFG(G)) &= d_G(x) \ \ if\ x \in V\\
&= \text{busy value of } e_i \text{ in } ETFG(G), \text{if } x \in E
\end{aligned}$$

Proof. By definition of degree of a vertex of a F-graph (definition 3), we have the following two cases to prove the theorem.

Case 1: Let $x \in V$. Then,

$$d(ETFG(x)) = \sum_{x,y \in V} \mu_{ETFG}(x,y) + \sum_{x \in V, e \in E} \mu_{ETFG}(x,e)$$

(where x lies on the edge of e in the second summation).

$$= \sum_{x,y \in V} \mu_{ETFG}(x,y) + 0$$

(The second summation is zero, since there is no fuzzy relation between $x \in V$ and $e \in E$ in the ETFG)

$$= \sum_{x,y \in V} \mu_{ETFG}(x,y) = d_G(x)$$

Case 2: Let $e_i \in E$, then

$$d(ETFG(e_i)) = \sum_{x\in V} \mu_{ETFG}(e_i,x) + \sum_{e_j\in E} \mu_{ETFG}(e_i,e_j)$$

$$= 0 + \sum_{e_j\in E} \mu_{ETFG}(e_i) \wedge \mu_{ETFG}(e_j)$$

(The first summation is zero, since there is no fuzzy relation between $x \in V$ and $e \in E$ in the ETFG)

$$= bussy\ value\ of\ e_i\ in\ ETFG(G).$$

Theorem 4. *An ETFG of any F-graph is disconnected graph.*

Proof. Let $G = (V,\sigma,\mu)$ be a F-graph. The fuzzy vertex set of ETFG(G) consists of $V \cup E$ of G and the fuzzy vertex relation is defined only between $x,y \in V$ and $e_i,e_j \in E$. Since there is no fuzzy relation between $x \in V$ and$e \in E$ of elements in the vertex set of ETFG(G), then there is no path that connects x and e in ETFG(G) and $\mu^\infty(x,y) = 0$. Hence, ETFG(G) is disconnected graph.

5 . Total Coloring of an Extended Total F-graph

In this section we introduce the concept of total coloring of an ETFG and discuss some of its properties.

Definition 12. *A family $\gamma = \{\gamma_1,\gamma_2,\cdots,\gamma_k\}$, of a fuzzy set on $V \cup E$ is called an ETFG k-fuzzy total coloring of F-graph,$G = (V,\sigma,\mu)$, if the following three conditions met.*

i. $Max\gamma_i(x) = \sigma(x)$ for all $x \in V$ and $Max\gamma_i(x,y) = \mu(x,y)$ for all edges $(x,y) \in E$.
ii. $\gamma_i \wedge \gamma_j = 0$
iii. For every adjacent vertices x,y of ETFG(G), $Min\gamma_i(x),\gamma_i(y) = 0$.

The least number of colors possible is called the total chromatic number of ETFG(G) and it is denoted by $X^f_{ETFG}(G)$.

Theorem 5. *Let $G = (V,\sigma,\mu)$ be a F-graph. The total chromatic number of an ETFG is bounded above by its total chromatic number.*

Proof.
Case 1: Let $G = (V,\sigma,\mu)$ be complete graph.
Suppose $G = (V,\sigma,\mu))$ be a complete F-graph. An ETFG of G is two disconnected complete graphs G and G', where G' is a F-graph whose fuzzy vertex is $e \in E$ for all $e \in E$ and fuzzy relation (e_i,e_j), where e_i and e_j have nodes in common in G.

Thus, $\chi^f_{ETFG}(G) = \chi^f_T(G)$. On the other hand, since G' is a fuzzy graph which is isomorphic to G, then $\chi^f_{ETFG}(G') = \chi^f_{TG}(G)$. Therefore, $\chi^f_{ETFG}(G) = \chi^f_{TG}(G)$.
Case 2: Let ,$G = (V,\sigma,\mu)$ be a fuzzy cyclic graph.
Suppose $G = (V,\sigma,\mu)$ be a fuzzy cyclic graph. The ETFG of G is two disconnected fuzzy cyclic graphs G and G', where G' is a F-graph whose fuzzy vertex is $e \in E$ for all $e \in E$and fuzzy relation (e_i,e_j), where e_i and e_j have nodes in common in G.

Thus, $\chi^f_{ETFG}(G) = \chi^f_{TG}(G)$. But, since G' is a F-graph which is isomorphic to G, then $\chi^f_{ETFG}(G') = \chi^f_{TG}(G)$. Therefore, $\chi^f_{ETFG}(G) = \chi^f_{TG}(G)$.
Case 3: Let, $G = (V,\sigma,\mu)$ be disconnected F-graph. Suppose $G = (V,\sigma,\mu)$ be a disconnected F-graph. The ETFG of G is two disconnected complete graphs G and G', where G' is a disconnected F-graph whose fuzzy vertex is $e \in E$ for all $e \in E$ and fuzzy relation (e_i,e_j), where e_i and e_j have nodes in common in G with one edge less than the edges of G. Thus, $\chi^f_{ETFG}(G) = \chi^f(G)$. Since G' is a F-graph which is disconnected and less edges than the edges of G, then then $\chi^f_{ETFG}(G') < \chi^f_{TG}(G)$.

Therefore, $\chi^f_{ETFG}(G) \leq \chi^f_{TG}(G)$.

*Example 3.*Consider a fuzzy graph $G = (V,\sigma,\mu)$ with vertex set $V = \{x_1,x_2,x_3,x_4,x_5,x_6\}$ and edge set $E = \{x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_6,x_6x_1\}$, whose membership functions are defined as follows:

$$\sigma(x_i) = \begin{cases} 0.2, & \text{if } i = 1 \\ 0.7, & \text{for } i = 2 \\ 0.5, & \text{for } i = 3 \\ 0.4, & \text{for } i = 4 \\ 0.6, & \text{for } i = 5 \\ 0.3, & \text{for } i = 6 \end{cases}$$

$$\mu(x_i,x_j) = \begin{cases} 0.2, & \text{for } (i,j) = (1,2) \\ 0.5, & \text{for } (i,j) = (2,3) \\ 0.1, & \text{for } (i,j) = (3,4) \\ 0.4, & \text{for } (i,j) = (4,5) \\ 0.3, & \text{for } (i,j) = (5,6) \\ 0.1, & \text{for } (i,j) = (6,1) \end{cases}$$

Let us define a family of fuzzy sets $\Gamma = \gamma_1,\gamma_2$ on $V \cup E$ as follows:

$$\gamma_1(x_i) = \begin{cases} 0.2, & \text{for } i = 1 \\ 0.5, & \text{for } i = 3 \\ 0.6, & \text{for } i = 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1(x_i,x_j) = \begin{cases} 0.2, & \text{for } (i,j) = (1,2) \\ 0.1, & \text{for } (i,j) = (3,4) \\ 0.3, & \text{for } (i,j) = (5,6) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} 0.7, & \text{for } i = 2 \\ 0.4, & \text{for } i = 4 \\ 0.3, & \text{for } i = 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_2(x_i, x_j) = \begin{cases} 0.5, & \text{for } (i,j) = (2,3) \\ 0.4, & \text{for } (i,j) = (3,4) \\ 0.1, & \text{for } (i,j) = (6,1) \\ 0, & \text{otherwise} \end{cases}$$

Clearly, a family of fuzzy sets $\Gamma = \{\gamma_1, \gamma_2\}$ defined as above satisfies the definition of total coloring of F-graphs and hence, $\chi^f_{TG}(G) = 2$.

When we come to our point of concern, we need to determine the total chromatic number of an ETFG of the F-graph in the example 3.
Now, to construct an ETFG, $ETFG(G) = (V \cup E, \sigma_{ETFG}, \mu_{ETFG}$, where

$V \cup E =$
$\{x_1, x_2, x_3, x_4, x_5, x_6, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1\}$.

The fuzzy subset of ETFG(G) will be as follows;

$$\sigma_{ETFG}(x_i) = \begin{cases} 0.2, & \text{if } i = 1 \\ 0.7, & \text{for } i = 2 \\ 0.5, & \text{for } i = 3 \\ 0.4, & \text{for } i = 4 \\ 0.6, & \text{for } i = 5 \\ 0.3, & \text{for } i = 6 \end{cases}$$

$$\sigma_{ETFG}(x_i, x_j) = \begin{cases} 0.2, & \text{for } (i,j) = (1,2) \\ 0.5, & \text{for } (i,j) = (2,3) \\ 0.1, & \text{for } (i,j) = (3,4) \\ 0.4, & \text{for } (i,j) = (4,5) \\ 0.3, & \text{for } (i,j) = (5,6) \\ 0.1, & \text{for } (i,j) = (6,1) \end{cases}$$

The fuzzy relation will be:

$$\sigma_{ETFG}(x_i, x_j) = \begin{cases} 0.2, & \text{for } (i,j) = (1,2) \\ 0.5, & \text{for } (i,j) = (2,3) \\ 0.1, & \text{for } (i,j) = (3,4) \\ 0.4, & \text{for } (i,j) = (4,5) \\ 0.3, & \text{for } (i,j) = (5,6) \\ 0.1, & \text{for } (i,j) = (6,1) \end{cases}$$

$$\mu_{ETFG}(x_ix_j, x_jx_k) = \begin{cases} 0.2, & \text{if } (ij, jk) = (12,23) \\ 0.1, & \text{for } (ij, jk) = (23,34) \\ 0.1, & \text{for } (ij, jk) = (34,45) \\ 0.3 & \text{for } (ij, jk) = (45,56) \\ 0.1, & \text{for } (ij, jk) = (56,61) \\ 0.1, & \text{for } (ij, jk) = (61,12) \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be a family of fuzzy subset defined on $V \cup E$ as follows:

i. For the vertex set;

$$\gamma_1(x_i) = \begin{cases} 0.2, & \text{for } i = 1 \\ 0.5, & \text{for } i = 3 \\ 0.6, & \text{for } i = 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1 2(x_i) = \begin{cases} 0.7, & \text{for } i = 2 \\ 0.4, & \text{for } i = 4 \\ 0.3, & \text{for } i = 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1(x_ix_j) = \begin{cases} 0.2, & \text{for } ij = 12 \\ 0.1, & \text{for } ij = 34 \\ 0.3, & \text{for } ij = 56 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_2(x_ix_j) = \begin{cases} 0.5, & \text{for } (i,j) = (2,3) \\ 0.4, & \text{for } (i,j) = (4,5) \\ 0.1, & \text{for } (i,j) = (6,1) \\ 0, & \text{otherwise} \end{cases}$$

ii.For the edge set;

$$\gamma_1(x_i, x_j) = \begin{cases} 0.2, & \text{for } (i,j) = (1,2) \\ 0.1, & \text{for } (i,j) = (3,4) \\ 0.3, & \text{for } (i,j) = (5,6) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1(x_ix_j, x_jx_k) = \begin{cases} 0.2, & \text{for } (ij, jk) = (23,34) \\ 0.1, & \text{for } (ij, jk) = (34,45) \\ 0.1, & \text{for } (ij, jk) = (56,61) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_2(x_ix_j) = \begin{cases} 0.5, & \text{for } (i,j) = (2,3) \\ 0.4, & \text{for } (i,j) = (4,5) \\ 0.1, & \text{for } (i,j) = (6,1) \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_2(x_ix_j, x_jx_k) = \begin{cases} 0.1, & \text{for } (ij, jk) = (23,34) \\ 0.3, & \text{for } (ij, jk) = (45,56) \\ 0.1, & \text{for } (ij, jk) = (61,12) \\ 0, & \text{otherwise} \end{cases}$$

Using table 1 below we can check whether Γ satisfies the definition of total coloring of ETFG of G.

Table 1: Total Coloring of an ETFG

V and E	γ_1	γ_2	Maximum	$\gamma_1 \wedge \gamma_2$	Min.
x_1	0.2	0	0.2	0	0
x_2	0	0.7	0.7	0	0
x_3	0.5	0	0.5	0	0
x_4	0	0.4	0.4	0	0
x_5	0.6	0	0.6	0	0
x_6	0	0.3	0.3	0	0
x_1x_2	0.2	0	0.2	0	0
x_2x_3	0	0.5	0.5	0	0
x_3x_4	0.1	0	0.1	0	0
x_4x_5	0	0.4	0.4	0	0
x_5x_6	0.3	0	0.3	0	0
x_6x_1	0	0.1	0.1	0	0
(x_1,x_2)	0.2	0	0.2	0	
(x_2,x_3)	0	0.5	0.5	0	
(x_3,x_4)	0.1	0	0.1	0	
(x_4,x_5)	0	0.4	0.4	0	
(x_5,x_6)	0.3	0	0.3	0	
(x_6,x_5)	0.1	0	0.1	0	
(x_1x_2,x_2x_3)	0.2	0	0.2	0	
(x_2x_3,x_3x_4)	0	0.1	0.1	0	
(x_3x_4,x_4x_5)	0.1	0	0.1	0	
(x_4x_5,x_5x_6)	0	0.3	0.3	0	
(x_3x_4,x_4x_5)	0.1	0	0.1	0	
(x_6x_1,x_1x_2)	0	0.1	0.1	0	

As shown in the table 1 above $\Gamma = \{\gamma_1, \gamma_2\}$ satisfies the definition of total coloring of an ETFG of fuzzy graph G.
Therefore, $\chi^f_{ETFG}(G) = 2$.

Hence, $\chi^f_{ETFG}(G) \leq \chi^f_T(G)$.

6 . Conclusions

This article defined an extension of total F-graph $(ETFG)$ for a given F-graph and illustrated with examples. Some properties of $ETFG$ were proposed and proved. Furthermore, we compare the results with the total F-graphs and justified. Also total coloring of $ETFG$ of given F-graph was defined and resulting its upper bound. Theories and properties related with TFGs is well presented and discussed by different researchers. The article paves the way to deal the isomorphic properties of an ETFG and their application in real life situations like social networking, communication networking and transportation problems.

Conflicts of Interests

The authors declare that they have no conflicts of interests.

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