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Fixed Point Results under New Contractive Conditions on Closed Balls

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Abstract: The goal of this manuscript is to present a new contractive mapping, namely a Ćirić-type rational $(\alpha_*, \eta_*, \Lambda, \Upsilon)$-multi-valued contraction mapping. In the framework of ordinary metric spaces, several fixed point results for semi α_*-admissible multi-valued contraction mappings with respect to η are also given. In addition, we have an example to back up our research. Finally, several fixed point results with a graph were discussed to improve the effectiveness of our contraction. In the same way, our findings expand, generalize and unify a large number of solid articles in the same direction.

Keywords: Fixed point technique, complete metric space, closed ball, semi α_*-admissible mapping.

1 Introduction

Fixed point (FP) theory acts a principle role in functional analysis, which is divided into two major areas: first area is the FP theory on contraction mappings on complete metric spaces and the second is the FP theory on continuous operators on compact and convex subsets of a normed space [1,2,3,4,5,6,7] . Recently, FP results have been proved under contractive mappings on a closed ball instead of a whole space. For further clarification, we advise the reader to read [8,9].

As another direction, Shoaib [9] discussed some new FP results for α_*-ψ-contractive type multivalued mappings in a closed ball of left (right) K-sequentially complete dislocated quasi metric space. Shoaib *et al.* [10] presented the concept of semi α_*-admissible mult-valued mappings and established FP consequences for semi α_*-admissible multivalued mappings satisfying a contractive condition of Reich type for elements in a sequence contained in a closed ball of a complete dislocated metric space. Rasham *et al.* [11] achieved FP theorems for a pair of semi α_*-dominated multivalued mappings fulfilling a generalized locally Ćirić type rational F-dominated multivalued contractive condition on a closed ball of complete dislocated $b-$metric space. Rasham and Shoaib [12] obtained common fixed point results for two families of multivalued mappings fulfilling generalized rational type A-dominated contractive conditions on a closed ball in complete dislocated $b-$metric spaces.

In 2008, Jachymski [13], proved a result on graphic contraction mappings on a metric space. Let $(\mho, \rho)$ be a metric space and Δ denotes the diagonal of the Cartesian product $\mho \times \mho$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with $\mho$, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. Assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. If λ and γ are vertices in a graph G, then a path in G from λ to γ of length $N (N \in \mathbb{N})$ is a sequence $\{\lambda_i\}_{i=0}^{N}$ of $N+1$ vertices such that $\lambda_0 = \lambda$, $\lambda_N = \gamma$ and $(\lambda_{n-1}, \lambda_n) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected, for more details, see for [13,14].

In 2012, the notions of α-ψ-contractive and α-admissible mappings are presented by Samet *et al.* [15]. They established under these concepts some FP theorems via various contraction mappings in complete metric spaces (CMSs). Over the years, altering distance functions there have been involved in a number of studies, for example, see, [16,17,18].

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Similar to previous works, in this manuscript, we discuss some new common FP results for Ćirić-type rational $(\alpha_*, \eta_*, \Lambda, \Upsilon)$-contraction multivalued mappings for a sequence contained in a closed ball on a CMS. Moreover, some new common FP theorems for ordered metric spaces endowed with a graph are derived.

2 Basic facts

We give some definitions and preliminaries in this section to aid understanding of our research.

Definition 1.[19] *Let $(\mho, \rho)$ be a metric space.*

(i)A sequence $\{\lambda_n\}$ in $(\mho, \rho)$ is called a Cauchy sequence if for all $\varepsilon > 0$, there is $n_0 \in N$ so that $\rho(\lambda_m, \lambda_n) < \varepsilon$ or $\lim_{n,m\to\infty} \rho(\lambda_n, \lambda_m) = 0$, $\forall n, m \geq n_0$.

(ii)A sequence $\{\lambda_n\}$ converges to a point λ in $\mho$ if $\lim_{n\to\infty} \rho(\lambda_n, \lambda) = 0$. In this case λ is called a limit of $\{\lambda_n\}$.

(iii)$(\mho, \rho)$ is complete if every Cauchy sequence in $\mho$ converges to a point $\lambda \in \mho$ such that $\rho(\lambda, \lambda) = 0$.

Note. For $\lambda \in \mho$ and $\varepsilon > 0$, $\overline{B(\lambda, \varepsilon)} = \{\gamma \in \mho : \rho(\lambda, \gamma) \leq \varepsilon\}$ is called a closed ball in the metric space $(\mho, \rho)$.

Definition 2.*Let K be a non-empty subset of a metric space $\mho$ and $\lambda \in \mho$. An element $\gamma_0 \in K$ is called a best approximation to λ in K if*

$$\rho(\lambda, \gamma_0) = \rho(\lambda, K) = \inf_{\gamma\in K} \rho(\lambda, \gamma).$$

If each $\lambda \in \mho$ has at least one best approximation in K, then K is called a proximal set.

Here, $\Xi\beta(\mho)$ represents the set of all proximal subsets of $\mho$.

Definition 3.[20] *The function $H_\rho : \Xi\beta(\mho) \times \Xi\beta(\mho) \to \mho$ defined by*

$$H_\rho(A, B) = \max\left\{\sup_{a\in A} \rho(a, B), \sup_{b\in B} \rho(A, b)\right\},$$

is a metric on $\Xi\beta(\mho)$, which is called Hausdorff metric induced by ρ. The pair $(\Xi\beta(\mho), H_\rho)$ is known as Hausdorff metric space.

Lemma 1.[21] *Let $A, B \in \Xi\beta(\mho)$, then for any $\lambda \in A$,*

$$D(\lambda, B) \leq H_\rho(A, B).$$

where

$$D(\lambda, B) = \inf\{d(\lambda, \gamma) : \gamma \in B\}.$$

In the context of a CMS, Nadler [22] presented that every multivalued contraction mapping has a FP as follows:

Definition 4.[23] *Let $\Gamma : \mho \to \Xi\beta(\mho)$ be a multivalued map. A point $\lambda \in \mho$ is called a FP of Γ if $\lambda \in \Gamma\lambda$.*

Let Ψ be a family of nondecreasing functions $\Upsilon : [0, \infty) \longrightarrow [0, \infty)$ so that $\sum_{n=1}^{\infty} \Upsilon^n(t) < +\infty$, $\forall t > 0$, where Υ^n symbolizes the $n-$th iterate of Υ.

The results below are useful in the sequel.

Lemma 2. *Let $\Upsilon \in \Psi$. Then the following postulates are true.*

(1)the sequence $\{\Upsilon^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, $\forall t \in (0, \infty)$;
(2)$\Upsilon(t) < t$, for each $t > 0$;
(3)$\Upsilon(t) = 0$ iff $t = 0$.

Definition 5.[23] *Let $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ be a mapping fulfilling*

$(\Phi_1)$$\Lambda$ is non-decreasing;
$(\Phi_2)$for each positive sequence $\{t_n\}$, we have

$$\lim_{n\to\infty} \Lambda(t_n) = 0 \text{ iff } \lim_{n\to\infty} t_n = 0;$$

$(\Phi_3)$$\Lambda$ is continuous.

Consider Φ represents the set of all functions $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ justifying the conditions $(\Phi_1) - (\Phi_3)$.

Mudhesh *et al.* [24] modified the Definition 5 by adding the following assumption:

$(\Phi_4)$for each $A_i \in (0, \infty)$, $i = 1, 2, ..., n$, we have $\Lambda\left(\sum_{n=1}^{\infty} A_i\right) \leq \sum_{n=1}^{\infty} \Lambda(A_i)$,

where Λ satisfies the conditions $(\Phi_1) - (\Phi_4)$.

Example 1.[25] The functions listed below are belong to Φ for all $t \in (0, \infty)$,

–$\Lambda(t) = at$, $a > 0$;
–$\Lambda(t) = |t|$.

The idea of semi α_*-admissible mapping on a set initiated in the work of [25] as follows:

Definition 6.[25] *Let $\Im : \mho \to \Xi\beta(\mho)$ be a multivalued mapping, $\alpha : \mho \times \mho \to [0, +\infty)$ be a function and A be a non-empty subset of $\mho$, we say that $\Im$ is semi α_*-admissible on A, whenever $\alpha(\lambda, \gamma) \geq 1$ implies that $\alpha_*(\Im\lambda, \Im\gamma) \geq 1$, for all $\lambda, \gamma \in A$, where*

$$\alpha_*(\Im\lambda, \Im\gamma) = \inf\{\alpha(a, b) : a \in \Im\lambda, b \in \Im\gamma\}.$$

It should be noted that if $A = \mho$, then we say that $\Im$ is an α_*-admissible on $\mho$.

Definition 6 extended to two mappings as follows:

Definition 7. *Let $\Im, \Gamma : \mho \to \Xi\beta(\mho)$ be two multivalued mappings, $\alpha : \mho \times \mho \to [0,+\infty)$ be a function and $A \subseteq \mho$. We say that $(\Im, \Gamma)$ is a pair of semi α_*-admissible on A, whenever $\alpha(\lambda,\gamma) \geq 1$ implies that $\alpha_*(\Im\lambda,\Gamma\gamma) \geq 1$ and $\alpha_*(\Gamma\lambda,\Im\gamma) \geq 1$, for all $\lambda,\gamma \in A$, where*

$$\alpha_*(\Im\lambda,\Gamma\gamma) = \inf\{\alpha(a,b) : a \in \Im\lambda, b \in \Gamma\gamma\}.$$

Also, if $A = \mho$, then we say that a pair $(\Im,\Gamma)$ is an α_*-admissible on $\mho$.

For two admissible functions Definition 6 and 7 can be written as:

Definition 8. *Let $\Im : \mho \to \Xi\beta(\mho)$ be multivalued mappings, $\alpha,\eta : \mho \times \mho \to [0,+\infty)$ and $A \subseteq \mho$. We say that $\Im$ is semi α_*-admissible with respect to η on A, whenever $\alpha(\lambda,\gamma) \geq \eta(\lambda,\gamma)$ implies that $\alpha_*(\Im\lambda,\Im\gamma) \geq \eta_*(\Im\lambda,\Im\gamma)$, for all $\lambda,\gamma \in A$, where*

$$\alpha_*(\Im\lambda,\Im\gamma) = \inf\{\alpha(a,b) : a \in \Im\lambda, b \in \Im\gamma\},$$

and

$$\eta_*(\Im\lambda,\Im\gamma) = \sup\{\eta(a,b) : a \in \Im\lambda, b \in \Im\gamma\}.$$

Moreover, $\Im$ is called α_*-admissible with respect to (wrt) η, if $A = \mho$.

Definition 9. *Let $\Im, \Gamma : \mho \to \Xi\beta(\mho)$ be two multivalued mappings, $\alpha,\eta : \mho \times \mho \to [0,+\infty)$ be functions and $A \subseteq \mho$. We say that a pair $(\Im,\Gamma)$ is semi α_*-admissible wrt η on A, whenever $\alpha(\lambda,\gamma) \geq \eta(\lambda,\gamma)$ implies that $\alpha_*(\Im\lambda,\Gamma\gamma) \geq \eta_*(\Im\lambda,\Gamma\gamma)$ and $\alpha_*(\Gamma\lambda,\Im\gamma) \geq \eta_*(\Gamma\lambda,\Im\gamma)$, for all $\lambda,\gamma \in A$, where*

$$\alpha_*(\Im\lambda,\Gamma\gamma) = \inf\{\alpha(a,b) : a \in \Im\lambda, b \in \Gamma\gamma\}$$

and

$$\eta_*(\Im\lambda,\Gamma\gamma) = \sup\{\eta(a,b) : a \in \Im\lambda, b \in \Gamma\gamma\}.$$

Again, if $A = \mho$, then the pair $(\Im,\Gamma)$ is called an α_*-admissible wrt η.

3 Main results

Let $(\mho,\rho)$ be a metric space, $\lambda_0 \in \mho$ and $\Im,\Gamma : \mho \to \Xi\beta(\mho)$ be multivalued mappings on $\mho$. Then there is $\lambda_1 \in \Im\lambda_0$ so that $\rho(\lambda_0,\Im\lambda_0) = \rho(\lambda_0,\lambda_1)$. Let $\lambda_2 \in \Gamma\lambda_1$ be such that $\rho(\lambda_1,\Gamma\lambda_1) = \rho(\lambda_1,\lambda_2)$. Continuing this process, we construct a sequence λ_n of points in $\mho$ so that

$$\lambda_{n+1} \in \Im\lambda_n \Rightarrow \rho(\lambda_n,\Im\lambda_n) = \rho(\lambda_n,\lambda_{n+1})$$

and

$$\lambda_{n+2} \in \Gamma\lambda_{n+1} \Rightarrow \rho(\lambda_{n+1},\Gamma\lambda_{n+1}) = \rho(\lambda_{n+1},\lambda_{n+2}).$$

In this part, $\{\Gamma\Im(\lambda_n)\}$ is called a sequence in $\mho$ generated by λ_0.

Now, we present our results by starting with the definition below.

Definition 10. *Let $(\mho,\rho)$ be a metric space, $\alpha,\eta : \mho \times \mho \to [0,+\infty)$ be two functions and $\Im,\Gamma : \mho \to \Xi\beta(\mho)$ be two multivalued mappings. The pair $(\Im,\Gamma)$ is called Ćirić-type rational $(\alpha_*,\eta_*,\Lambda,\Upsilon)$-contraction, if there exists $\Lambda \in \Phi$ and $\Upsilon \in \Psi$ such that $H_\rho(\Im\lambda,\Gamma\gamma) > 0$ implies*

$$\Lambda\left(\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\right) \leq \Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right], \quad (1)$$

for all $\lambda,\gamma \in \{\Gamma\Im(\lambda_n)\}$, where,

$$M_\rho(\lambda,\gamma) = \max\left\{\rho(\lambda,\gamma), D(\lambda,\Im\lambda), D(\gamma,\Gamma\gamma), \frac{D(\lambda,\Im\lambda).D(\gamma,\Gamma\gamma)}{1+\rho(\lambda,\gamma)}\right\}.$$

Theorem 1. *Let $(\mho,\rho)$ be a CMS, $\alpha,\eta : \mho \times \mho \to [0,+\infty)$ be given functions. Assume that $\Im,\Gamma : \mho \to \Xi\beta(\mho)$ are a pair of semi α_*-admissible multifunctions wrt η satisfying (1) on a closed ball $\overline{B_\rho(\lambda_0,r)}$, for $\lambda_0 \in B_\rho(\lambda_0,r)$ and $r > 0$. Suppose that $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\mho$ generated by λ_0, then $\{\Gamma\Im(\lambda_n)\} \to z \in \overline{B_\rho(\lambda_0,r)}$ and*

$$\Lambda(\rho(\lambda_0,\lambda_1)) \leq \sum_{i=0}^{\infty} \Upsilon^i\left[\Lambda(\rho(\lambda_0,\lambda_1))\right] \leq r \text{ where } r > 0. \quad (2)$$

Moreover, if for all $\lambda,\gamma \in \left(\overline{B_\rho(\lambda_0,r)} \cap \{\Gamma\Im(\lambda_n)\}\right) \cup \{z\}$, the contractive condition (1) holds. Then $\Im$ and Γ have a common FP in $\overline{B_\rho(\lambda_0,r)}$.

Proof. Since $\lambda_0 \in \overline{B_\rho(\lambda_0,r)}$, and $\Im,\Gamma : \mho \to \Xi\beta(\mho)$ are two multi-valued mappings on $\mho$, then there is $\lambda_1 \in \Im\lambda_0$ so that $D(\lambda_0,\Im\lambda_0) = \rho(\lambda_0,\lambda_1)$. If $\lambda_0 = \lambda_1$, then λ_0 is a FP in $\overline{B_\rho(\lambda_0,r)}$ of $\Im$. Let $\lambda_0 \neq \lambda_1$. From (2), we get

$$\Lambda(\rho(\lambda_0,\lambda_1)) \leq \sum_{i=0}^{\infty} \Upsilon^i\left[\Lambda(\rho(\lambda_0,\lambda_1))\right] \leq r, \ r > 0.$$

It follows that $\lambda_1 \in \overline{B_\rho(\lambda_0,r)}$. As $\alpha(\lambda_0,\lambda_1) \geq \eta(\lambda_0,\lambda_1)$ and $(\Im,\Gamma)$ is a pair of semi α_*-admissible multi-function with respect to η on $\overline{B_\rho(\lambda_0,r)}$, so $\alpha_*(\Im\lambda_0,\Gamma\lambda_1) \geq \eta_*(\Im\lambda_0,\Gamma\lambda_1)$. As $\alpha_*(\Im\lambda_0,\Gamma\lambda_1) \geq \eta_*(\Im\lambda_0,\Gamma\lambda_1)$, $\lambda_1 \in \Im\lambda_0$ and $\lambda_2 \in \Gamma\lambda_1$, so $\alpha(\lambda_1,\lambda_2) \geq \eta(\lambda_1,\lambda_2)$. Let $\lambda_2,\ldots,\lambda_i \in \overline{B_\rho(\lambda_0,r)}$ for some $i \in \mathbb{N}$. As $(\Im,\Gamma)$ is a pair of semi α_*-admissible multi-function on $\overline{B_\rho(\lambda_0,r)}$, thus, we have

$$\alpha_*(T\lambda_1,S\lambda_2) \geq \eta_*(T\lambda_1,S\lambda_2).$$

This implies that $\alpha(\lambda_2,\lambda_3) \geq \eta(\lambda_2,\lambda_3)$, which further implies

$$\alpha_*(\Im\lambda_2,\Gamma\lambda_3) \geq \eta_*(\Im\lambda_2,\Gamma\lambda_3).$$

Continuing this process and if $i = 2j+1$, $j = 1,2,\ldots\frac{i-1}{2}$, we have

$$\alpha_*(\Im\lambda_{2j},\Gamma\lambda_{2j+1}) \geq \eta_*(\Im\lambda_{2j},\Gamma\lambda_{2j+1}),$$

which this leads to

$$\alpha(\lambda_{2j+1},\lambda_{2j+2}) \geq \eta(\lambda_{2j+1},\lambda_{2j+2}).$$

Now, we can write

$$\begin{aligned}
&\Lambda\left(\rho(\lambda_{2j+1},\lambda_{2j+2})\right)\\
&\leq \Lambda\left(H_\rho(S\lambda_{2j},T\lambda_{2j+1})\right)\\
&\leq \Lambda\left(\alpha_*(S\lambda_{2j},T\lambda_{2j+1})H_d(S\lambda_{2j},T\lambda_{2j+1})\right)\\
&\leq \Upsilon\left[\Lambda\left(M_\rho(\lambda_{2j},\lambda_{2j+1})\right)\right]\\
&= \Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_{2j},\lambda_{2j+1}),D(\lambda_{2j},\Im\lambda_{2j}),\\ D(\lambda_{2j+1},\Gamma\lambda_{2j+1}),\\ \frac{D(\lambda_{2j},\Im\lambda_{2j}).D(\lambda_{2j+1},\Gamma\lambda_{2j+1})}{1+\rho(\lambda_{2j},\lambda_{2j+1})}\end{array}\right\}\right)\right]\\
&= \Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_{2j},\lambda_{2j+1}),\rho(\lambda_{2j},\lambda_{2j+1}),\\ \rho(\lambda_{2j+1},\lambda_{2j+2}),\\ \frac{\rho(\lambda_{2j},\lambda_{2j+1}).\rho(\lambda_{2j+1},\lambda_{2j+2})}{1+\rho(\lambda_{2j},\lambda_{2j+1})}\end{array}\right\}\right)\right]\\
&= \Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_{2j},\lambda_{2j+1}),\rho(\lambda_{2j+1},\lambda_{2j+2}),\\ \frac{\rho(\lambda_{2j},\lambda_{2j+1}).\rho(\lambda_{2j+1},\lambda_{2j+2})}{1+\rho(\lambda_{2j},\lambda_{2j+1})}\end{array}\right\}\right)\right].
\end{aligned}$$

If $M_\rho(\lambda_{2j},\lambda_{2j+1}) = \rho(\lambda_{2j+1},\lambda_{2j+2})$, then

$$\Lambda\left(\rho(\lambda_{2j+1},\lambda_{2j+2})\right) \leq \Upsilon\left[\Lambda\left(\rho(\lambda_{2j+1},\lambda_{2j+2})\right)\right].$$

Using (Φ_1) and properties of ψ, we get

$$\rho(\lambda_{2j+1},\lambda_{2j+2}) < \rho(\lambda_{2j+1},\lambda_{2j+2}),$$

which is a inconsistency as $\rho(\lambda_{2j+1},\lambda_{2j+2}) \geq 0$. Similarly, if

$$M_\rho(\lambda_{2j},\lambda_{2j+1}) = \frac{\rho(\lambda_{2j},\lambda_{2j+1}).\rho(\lambda_{2j+1},\lambda_{2j+2})}{1+\rho(\lambda_{2j},\lambda_{2j+1})},$$

we obtain a inconsistency, $M_\rho(\lambda_{2j},\lambda_{2j+1}) = \rho(\lambda_{2j},\lambda_{2j+1})$, which implies that

$$\begin{aligned}
&\Lambda\left(\rho(\lambda_{2j+1},\lambda_{2j+2})\right)\\
&\leq \Upsilon\left[\Lambda\left(\rho(\lambda_{2j},\lambda_{2j+1})\right)\right]\\
&\leq \Upsilon\left[\Lambda\left(\alpha_*(\Gamma\lambda_{2j-1},\Im\lambda_{2j})H_\rho(\Gamma\lambda_{2j-1},\Im\lambda_{2j})\right)\right]\\
&\leq \Upsilon^2\left[\Lambda\left(\rho(\lambda_{2j-1},\lambda_{2j})\right)\right]\\
&\quad\vdots\\
&\leq \Upsilon^{2j+1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right].
\end{aligned}$$

It follows that

$$\Lambda\left(\rho(\lambda_{2j+1},\lambda_{2j+2})\right) \leq \Upsilon^{2j+1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]. \quad (3)$$

Now, utilizing (ρ_3), (Φ_4), (2) and (3), we obtain

$$\begin{aligned}
&\Lambda\left(\rho(\lambda_0,\lambda_{2j+1})\right)\\
&\leq \Lambda\left(\rho(\lambda_0,\lambda_1)+\cdots+\rho(\lambda_{2j},\lambda_{2j+1})+\rho(\lambda_{2j+1},\lambda_{2j+2})\right)\\
&\leq \Lambda\left(\rho(\lambda_0,\lambda_1)\right)+\cdots\\
&\quad +\Lambda\left(\rho(\lambda_{2j},\lambda_{2j+1})\right)+\Lambda\left(\rho(\lambda_{2j+1},\lambda_{2j+2})\right)\\
&\leq \Lambda\left(\rho(\lambda_0,\lambda_1)\right)+\cdots\\
&\quad +\Upsilon^{2j}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]+\Upsilon^{2j+1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]\\
&\leq \sum_{i=0}^{2j+1}\Upsilon^i\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right] \leq r.
\end{aligned}$$

Thus, $\lambda_{2j+1} \in \overline{B_\rho(\lambda_0,r)}$. Therefore, by induction, $\lambda_n \in \overline{B_\rho(\lambda_0,r)}$ and $\alpha(\lambda_n,\lambda_{n+1}) \geq \eta(\lambda_n,\lambda_{n+1})$ for all $n \in \mathbb{N}$. Since $\Im$ and Γ are semi α_*-admissible multi-functions wrt η on $\overline{B_\rho(\lambda_0,r)}$, then $\alpha_*(\Im\lambda_n,\Gamma\lambda_{n+1}) \geq \eta_*(\Im\lambda_n,\Gamma\lambda_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$.

Now, inequality (3) can be written as

$$\Lambda\left(\rho(\lambda_{n+1},\lambda_{n+2})\right) \leq \Upsilon^{n+1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right], \text{ for all } n \in \mathbb{N}. \quad (4)$$

Passing $n \to \infty$ in (4), we get

$$0 \leq \lim_{n\to\infty}\Lambda\left(\rho(\lambda_{n+1},\lambda_{n+2})\right) \leq \lim_{n\to\infty}\Upsilon^{n+1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right] = 0,$$

hence

$$\lim_{n\to\infty}\Lambda\left(\rho(\lambda_{n+1},\lambda_{n+2})\right) = 0.$$

From (Φ_2), we get

$$\lim_{n\to\infty}\rho(\lambda_{n+1},\lambda_{n+2}) = 0. \quad (5)$$

This proved that $\{\lambda_n\}$ is a Cauchy sequence in $(\overline{B_\rho(\lambda_0,r)},d)$. Let $n,m \in \mathbb{N}$ with $m > n > p$. Then, we have

$$\begin{aligned}
&\Lambda\left(\rho(\lambda_n,\lambda_m)\right)\\
&\leq \Lambda\left(\rho(\lambda_n,\lambda_{n+1})+\rho(\lambda_{n+1},\lambda_{n+2})+\ldots+\rho(\lambda_{m-1},\lambda_m)\right)\\
&\leq \Lambda\left(\rho(\lambda_n,\lambda_{n+1})\right)+\Lambda\left(\rho(\lambda_{n+1},\lambda_{n+2})\right)+\ldots\\
&\quad +\Lambda\left(\rho(\lambda_{m-1},\lambda_m)\right)\\
&\leq \psi^n\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]+\psi^{n+1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]+\ldots \quad (6)\\
&\quad +\psi^{m-1}\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right].
\end{aligned}$$

Letting $n,m \to \infty$ in (6), one can write

$$\lim_{n,m\to\infty}\Lambda\left(\rho(\lambda_n,\lambda_m)\right) = 0.$$

Applying the condition (Φ_2), we have

$$\lim_{n,m\to\infty}\rho(\lambda_n,\lambda_m) = 0. \quad (7)$$

Since every closed ball in a CMS is also complete, so there is $\lambda^* \in \overline{B_\rho(\lambda_0,r)}$ so that $\lambda_n \to \lambda^*$ and

$$\lim_{n\to\infty}\rho(\lambda_n,\lambda^*) = 0. \quad (8)$$

Hence $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$ generated by λ_0 and $\{\Gamma\Im(\lambda_n)\} \to \lambda^* \in \overline{B_\rho(\lambda_0,r)}$. So, for $\lambda_n,\lambda_{n+1} \in \{\Gamma\Im(\lambda_n)\}$, one can write

$$\alpha(\lambda_n,\lambda_{n+1}) \geq \eta(\lambda_n,\lambda_{n+1}),\ \forall n \geq 0.$$

Because

$$\alpha_*(\Im\lambda_n,\Gamma\lambda_{n+1}) \geq \eta_*(\Im\lambda_n,\Gamma\lambda_{n+1})\ \forall n \geq 0,$$

then, we have

$$\alpha(\lambda_{n+1},\lambda_{n+2}) \geq \eta(\lambda_{n+1},\lambda_{n+2}).$$

From our assumption, we get

$$\alpha(\lambda_n,\lambda^*) \geq \eta(\lambda_n,\lambda^*),\ \forall n \geq 0.$$

Hence

$$\alpha_*(\Im\lambda_n,\Gamma\lambda^*) \geq \eta_*(\Im\lambda_n,\Gamma\lambda^*).$$

Now, to claim that $\lambda^* \in \Gamma\lambda^*$, assume that $d(\lambda^*,\Gamma\lambda^*) > 0$, then, we have

$$\begin{aligned}
&\Lambda\left(\rho(\lambda^*,\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})+\rho(\lambda_{n+1},\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)+\Lambda\left(\rho(\lambda_{n+1},\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)+\Lambda\left(\alpha_*(\Im\lambda_n,\Gamma\lambda^*)H_\rho(\Im\lambda_n,\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)+\Upsilon\left[\Lambda\left(M_\rho(\lambda_n,\lambda^*)\right)\right]\\
&= \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)\\
&\quad+\Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_n,\lambda^*),D(\lambda_n,\Im\lambda_n),\\ D(\lambda^*,\Gamma\lambda^*),\\ \frac{D(\lambda_n,\Im\lambda_n).D(\lambda^*,\Gamma\lambda^*)}{1+d(\lambda_n,\lambda^*)}\end{array}\right\}\right)\right]\\
&= \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)\\
&\quad+\Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_n,\lambda^*),\rho(\lambda_n,\lambda_{n+1}),\\ D(\lambda^*,\Gamma\lambda^*),\\ \frac{\rho(\lambda_n,\lambda_{n+1}).D(\lambda^*,\Gamma\lambda^*)}{1+\rho(\lambda_n,\lambda^*)}\end{array}\right\}\right)\right].
\end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (Φ_2), by properties of Υ and (8), we obtain that

$$\rho(\lambda^*,\Gamma\lambda^*) < \rho(\lambda^*,\Gamma\lambda^*),$$

a contradiction. Therefore $\rho(\lambda^*,\Gamma\lambda^*) = 0$ and $\lambda^* \in \Gamma\lambda^*$.

In the same scenario, one can write $\rho(\Im\lambda^*,\lambda^*) = 0$. Hence $\lambda^* \in \Im\lambda^*$. Therefore $\Im$ and Γ have a common FP in $\overline{B_\rho(\lambda_0,r)}$.

The following theorem illustrates that our results are valid in the context of partially ordered metric spaces (POMSs, for short).

Via this space, let $A,B \subseteq \mho$. If for each $a \in A$ there is $b \in B$ so that $a \preceq b$ and $a \preceq_r b$, then we say that $A \preceq B$ and $\Im A \preceq_r \Gamma B$, respectively.

Theorem 2. *Let $(\mho,\preceq,\rho)$ be a POMS, $\alpha,\eta : \mho\times\mho \to [0,\infty)$ be two functions and $\Im,\Gamma : \mho \to \Xi\beta(\mho)$ be two non-decreasing semi α_*-admissible multi-functions wrt η. Suppose also there is $\Lambda \in \Phi$ and $\Upsilon \in \Psi$ so that $H_\rho(\Im\lambda,\Gamma\gamma) > 0$ implies*

$$\Lambda\left(\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\right) \leq \Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right], \quad (9)$$

for all $\lambda,\gamma \in \overline{B_\rho(\lambda_0,r)} \cap \{\Gamma\Im(\lambda_n)\}$, $r > 0$, where

$$\begin{aligned}
&M_\rho(\lambda,\gamma)\\
&= \max\left\{\rho(\lambda,\gamma),D(\lambda,\Im\lambda),D(\gamma,\Gamma\gamma),\frac{D(\lambda,\Im\lambda).D(\gamma,\Gamma\gamma)}{1+\rho(\lambda,\gamma)}\right\},
\end{aligned}$$

with $\lambda \preceq \gamma$, $\Im\lambda \preceq_r \Gamma\gamma$ and $\sum_{i=0}^{n}\Upsilon^i[\Lambda(\rho(\lambda_0,\lambda_1))] \leq r$. Then $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $\lambda_n \preceq \lambda_{n+1}$ and $\{\Gamma\Im(\lambda_n)\} \to \lambda^* \in \overline{B_\rho(\lambda_0,r)}$. Moreover, if $\lambda^* \preceq \lambda_n$ or $\lambda_n \preceq \lambda^*$ and the inequality (9) holds for all $\lambda,\gamma \in \left(\overline{B_\rho(\lambda_0,r)} \cap \{\Gamma\Im(\lambda_n)\}\right) \cup \{\lambda^*\}$, then λ^* is a common FP of $\Im$ and Γ in $\overline{B_\rho(\lambda_0,r)}$.

Proof. Let $\lambda_0 \in \overline{B_\rho(\lambda_0,r)}$ so that $\lambda_0 \preceq \Im\lambda_0$. Define a sequence $\{\Gamma\Im(\lambda_n)\}_{n\in\mathbb{N}}$ by letting $\lambda_1 \in \Im\lambda_0$ so that $\lambda_0 \preceq \lambda_1$ and $\lambda_2 \in \Gamma\lambda_1$ so that $\lambda_1 \preceq \lambda_2$.

Since $\Im$ and Γ are non-decreasing, we have $\lambda_3 \in \Im\lambda_2$ so that $\lambda_2 \preceq \lambda_3$. Continuing in the same way, we obtain a sequence $\{\Gamma\Im(\lambda_n)\}_{n\in\mathbb{N}} \subseteq \overline{B_\rho(\lambda_0,r)}$ generated by λ_0 so that

$$\lambda_{2n+1} \in \Im\lambda_{2n} \text{ and } \lambda_{2n+2} \in \Gamma\lambda_{2n+1}$$

implies $\lambda_{2n} \preceq \lambda_{2n+1}$ and $\lambda_{2n+1} \preceq \lambda_{2n+2}$, $\forall n \geq 0$.

It follows that

$$\lambda_0 \preceq \lambda_1 \preceq \lambda_2 \preceq \cdots \preceq \lambda_n \preceq \lambda_{n+1} \preceq \cdots$$

Because the pair $(\Im,\Gamma)$ is semi α_*-admissible multi-functions with respect to η, we get

$$\alpha(\lambda_n,\lambda_{n+1}) \geq \eta(\lambda_n,\lambda_{n+1}),\ \forall n \geq 0.$$

Following the same technique used to prove Theorem 1, we conclude that

$$\lim_{n\to\infty}\rho(\lambda_n,\lambda^*) = 0. \quad (10)$$

Hence $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$ generated by λ_0 and $\{\Gamma\Im(\lambda_n)\} \to \lambda^* \in \overline{B_\rho(\lambda_0,r)}$. Also, for $\lambda_n,\lambda_{n+1} \in \{\Gamma\Im(\lambda_n)\}$ and for all $n \geq 0$, we get

$$\alpha(\lambda_n,\lambda_{n+1}) \geq \eta(\lambda_n,\lambda_{n+1}).$$

Since, for all $n \geq 0$, $\alpha_*(\Im\lambda_n,\Gamma\lambda_{n+1}) \geq \eta_*(\Im\lambda_n,\Gamma\lambda_{n+1})$, then we obtain

$$\alpha(\lambda_{n+1},\lambda_{n+2}) \geq \eta(\lambda_{n+1},\lambda_{n+2}).$$

It follows from our assumption that

$$\alpha(\lambda_n,\lambda^*) \geq \eta(\lambda_n,\lambda^*),\ \forall n \geq 0.$$

Thus

$$\alpha_*(\Im\lambda_n,\Gamma\lambda^*) \geq \eta_*(\Im\lambda_n,\Gamma\lambda^*).$$

Now, to prove $\lambda^* \in \Gamma\lambda^*$, let $\rho(\lambda^*,\Gamma\lambda^*) > 0$. Then, one gets

$$\begin{aligned}
&\Lambda\left(\rho(\lambda^*,\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})+\rho(\lambda_{n+1},\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)+\Lambda\left(\rho(\lambda_{n+1},\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)+\Lambda\left(\alpha_*(\Im\lambda_n,\Gamma\lambda^*)H_\rho(\Im\lambda_n,\Gamma\lambda^*)\right)\\
&\leq \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)+\Upsilon\left[\Lambda\left(M_\rho(\lambda_n,\lambda^*)\right)\right]\\
&= \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)\\
&\quad+\Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_n,\lambda^*),D(\lambda_n,\Im\lambda_n),\\ D(\lambda^*,\Gamma\lambda^*),\\ \frac{D(\lambda_n,\Im\lambda_n).D(\lambda^*,\Gamma\lambda^*)}{1+\rho(\lambda_n,\lambda^*)}\end{array}\right\}\right)\right]\\
&= \Lambda\left(\rho(\lambda^*,\lambda_{n+1})\right)\\
&\quad+\Upsilon\left[\Lambda\left(\max\left\{\begin{array}{c}\rho(\lambda_n,\lambda^*),\rho(\lambda_n,\lambda_{n+1}),\\ D(\lambda^*,\Gamma\lambda^*),\\ \frac{\rho(\lambda_n,\lambda_{n+1}).D(\lambda^*,\Gamma\lambda^*)}{1+\rho(\lambda_n,\lambda^*)}\end{array}\right\}\right)\right].
\end{aligned}$$

Passing $n \to \infty$ in the above inequality, using (Φ_2), by properties of Υ and (10), we have

$$\rho(\lambda^*,\Gamma\lambda^*) < \rho(\lambda^*,\Gamma\lambda^*),$$

a contradiction. Therefore $\rho(\lambda^*,\Gamma\lambda^*) = 0$ and $\lambda^* \in \Gamma\lambda^*$. Analogously, one can obtain that $\rho(\Im\lambda^*,\lambda^*) = 0$. Hence $\lambda^* \in \Im\lambda^*$. So $\Im$ and Γ have a common FP in $\overline{B_\rho(\lambda_0,r)}$.

If we put $\Im = \Gamma$ in Theorem 2, we have a result below:

Corollary 1. *Let $(\mho,\preceq,\rho)$ be a POMS, $\alpha,\eta : \mho\times\mho \to [0,\infty)$ be two functions and $\Im : \mho \to \Xi\beta(\mho)$ be non-decreasing semi α_*-admissible multi-functions wrt η. Also, suppose that there is $\Lambda \in \Theta$ and $\Upsilon \in \Psi$ so that $H_\rho(\Im\lambda,\Im\gamma) > 0$ implies*

$$\Lambda\left(\alpha_*(\Im\lambda,\Im\gamma)H_\rho(\Im\lambda,\Im\gamma)\right) \leq \Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right], \quad (11)$$

for all $\lambda,\gamma \in \overline{B_\rho(\lambda_0,r)} \cap \{\mho\Im(\lambda_n)\}$, $r > 0$, where

$$M_\rho(\lambda,\gamma) = \max\left\{\begin{array}{c}\rho(\lambda,\gamma),D(\lambda,\Im\lambda),\\ D(\gamma,\Gamma\gamma),\frac{D(\lambda,\Im\lambda).D(\gamma,\Gamma\gamma)}{1+\rho(\lambda,\gamma)}\end{array}\right\},$$

with $\lambda \preceq \gamma$, $\Im\lambda \preceq_r \Gamma\gamma$ and $\sum_{i=0}^{n}\Upsilon^i[\Lambda(\rho(\lambda_0,\lambda_1))] \leq r$. Then $\{\mho\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $\lambda_n \preceq \lambda_{n+1}$ and $\{\mho\Im(\lambda_n)\} \to \lambda^* \in \overline{B_\rho(\lambda_0,r)}$. Moreover, if $\lambda^* \preceq \lambda_n$ or $\lambda_n \preceq \lambda^*$ and the inequality (11) holds for all $\lambda,\gamma \in \left(\overline{B_\rho(\lambda_0,r)} \cap \{\Gamma\Im(\lambda_n)\}\right) \cup \{\lambda^*\}$ and $n \geq 0$, then λ^* is a FP of $\Im$ and Γ in $\overline{B_\rho(\lambda_0,r)}$.

Definition 11. *Assume that $f : \mho \longrightarrow \mho$ is a self-mapping and $\alpha,\eta : \mho\times\mho \to [0,+\infty)$ are given functions. We say that f is semi α-admissible wrt η, if*

$$\begin{aligned}
\alpha(\lambda,\gamma) &\geq \eta(\lambda,\gamma)\\
&\Longrightarrow \alpha(f\lambda,f\gamma) \geq \eta(f\lambda,f\gamma),
\end{aligned}$$

for some $\lambda,\gamma \in A \subseteq \mho$.

It should be noted that if $A = \mho$, then f is called α-admissible wrt η.

Based on the above definition, we state the following result:

Corollary 2. *Let $(\mho,\rho)$ be a CMS, $\Im : \mho \to \mho$ and λ_0 be an arbitrary point in $\overline{B_\rho(\lambda_0,r)}$, for $r > 0$. Let $\{\lambda_n\}$ be a Picard sequence in $\mho$ with initial guess λ_0 and $\alpha,\eta : \mho\times\mho \to [0,+\infty)$ be semi α-admissible mappings wrt η on $\overline{B_d(\lambda_0,r)}$ with $\alpha(\lambda_0,\lambda_1) \geq \eta(\lambda_0,\lambda_1)$. Assume that there are $\Lambda \in \Phi$ and $\Upsilon \in \Psi$ so that $\forall\lambda,\gamma \in \overline{B_\rho(\lambda_0,r)}$, $\alpha(\lambda,\gamma) \geq \eta(\lambda,\gamma)$ implies*

$$\Lambda\left(\rho(\Im\lambda,\Im\gamma)\right) \leq \Upsilon\left[\Lambda\left(E_\rho(\lambda,\gamma)\right)\right], \quad (12)$$

where

$$\begin{aligned}
&E_\rho(\lambda,\gamma)\\
&= \max\left\{\begin{array}{c}\rho(\lambda,\gamma),\rho(\lambda,\Im\lambda),\\ \rho(\gamma,\Im\gamma),\frac{\rho(\lambda,\Im\lambda).\rho(\gamma,\Im\gamma)}{1+\rho(\lambda,\gamma)}\end{array}\right\},
\end{aligned}$$

and $\sum_{i=0}^{n}\Upsilon^i(\Lambda(\rho(\lambda_0,\lambda_1))) \leq r$. Then $\{\lambda_n\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $\lambda_n \to \lambda^* \in \overline{B_\rho(\lambda_0,r)}$ and $\alpha(\lambda_n,\lambda_{n+1}) \geq \eta(\lambda_n,\lambda_{n+1})$ for all $n \geq 0$. Also, if

$$\alpha(\lambda_n,\lambda^*) \geq \eta(\lambda_n,\lambda^*),\ \forall n \geq 0,$$

and the inequality (12) holds for all $\lambda,\gamma \in \left(\overline{B_\rho(\lambda_0,r)} \cap \{\mho\Im(\lambda_n)\}\right) \cup \{\lambda^*\}$, then λ^* is a FP of $\Im$ in $\overline{B_\rho(\lambda_0,r)}$.

Corollary 3. *Let $(\mho,\rho)$ be a complete POMS and $\Im : \mho \to \mho$ be a nondecreasing mapping. Assume that λ_0 is an arbitrary point in $\overline{B_\rho(\lambda_0,r)}$, $\{\lambda_n\}$ is a Picard sequence in $\mho$ with initial guess λ_0 and $\lambda_0 \preceq \lambda_1$. Presume that there are $\Lambda \in \Phi$ and $\Upsilon \in \Psi$ so that*

$$\Lambda\left(\rho(S\lambda,S\gamma)\right) \leq \Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right], \quad (13)$$

where $M_\rho(\lambda,\gamma)$ is defined as in Corollary 2 for all λ,γ in $\overline{B_\rho(\lambda_0,r)} \cap \{\mho\Im(\lambda_n)\}$ with $\lambda \preceq \gamma$ and

$$\sum_{i=0}^{n}\Upsilon^i(\Lambda(\rho(\lambda_0,\lambda_1))) \leq r, \text{ where } r > 0.$$

Then $\{\lambda_n\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $\lambda_n \preceq \lambda_{n+1}$ and $\{\lambda_n\} \to \lambda^* \in \overline{B_\rho(\lambda_0,r)}$. Moreover, if $\lambda^* \preceq \lambda_n$ or $\lambda_n \preceq \lambda^*$ and the inequality (13) holds for each $\lambda,\gamma \in \left(\overline{B_\rho(\lambda_0,r)} \cap \{\mho\Im(\lambda_n)\}\right) \cup \{\lambda^*\}$, then λ^* is a FP of $\Im$ in $\overline{B_\rho(\lambda_0,r)}$.

To reinforce the theoretical results, we give the example below.

*Example 2.*Let $\mho=[0,\infty)$ with a metric $\rho(\lambda,\gamma)=|\lambda-\gamma|$. Then $(\mho,\rho)$ is a CMS. Define the multivalued mappings $\Im,\Gamma:\mho\to\Xi\beta(\mho)$ by

$$\Im\lambda=\begin{cases}[\frac{3\lambda}{e^3},\frac{\lambda}{e^3}], & \text{if } \lambda\in[0,1],\\ [1,\lambda+4], & \text{if } \lambda\in(1,\infty).\end{cases}$$

$$\text{and } \Gamma\lambda=\begin{cases}[\frac{3\lambda}{e^4},\frac{\lambda}{e^4}], & \text{if } \lambda\in[0,1],\\ [0,\lambda+5], & \text{if } \lambda\in(1,\infty).\end{cases}$$

Consider $\lambda_0=1$, $r=10$. Then, $\overline{B_\rho(\lambda_0,r)}=[0,11]$ and

$$\rho(\lambda_0,\Im\lambda_0)=\rho(1,\Im 1)=\rho(1,\frac{1}{e^3})=1-\frac{1}{e^3}.$$

Hence, we obtain a sequence $\{\Gamma\Im(\lambda_n)\}=\left\{1,\frac{1}{e^3},\frac{1}{e^7},\frac{1}{e^{10}},\frac{1}{e^{14}},\ldots\right\}$ in $\mho$ generated by λ_0. Let $\Lambda(t)=2t$ and $\psi(t)=\frac{2}{e}t$. Define the functions,

$$\alpha(\lambda,\gamma)=\begin{cases}2, & \text{if } \lambda,\gamma\in[0,1],\\ \frac{5}{4}, & \text{otherwise.}\end{cases}$$

$$\text{and } \eta(\lambda,\gamma)=\begin{cases}1, & \text{if } \lambda,\gamma\in[0,1],\\ \frac{1}{2}, & \text{otherwise.}\end{cases}$$

Now,

$$\begin{aligned}&\Lambda\left(\alpha_*(\Im 4,\Gamma 6)H_\rho(\Im 4,\Gamma 6)\right)\\ &=\frac{5}{4}\times 10>\frac{2}{e}\left(2\max\left\{2,4,5,\frac{20}{3},\frac{20}{9}\right\}\right)\\ &=\frac{80}{3e}=9.8.\end{aligned}$$

Hence the condition (1) does not hold on $\mho$ for all $\lambda,\gamma\in\mho$ and for all $\lambda,\gamma\in\overline{B_\rho(\lambda_0,r)}$. Now, for all $\lambda,\gamma\in\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}$, we get

$$\begin{aligned}&\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\\ &=2\max\left\{\sup_{a\in\Im\lambda}\rho(a,\Gamma\gamma),\ \sup_{b\in\Gamma\gamma}\rho(\Im\lambda,b)\right\}\\ &=2\max\left\{\begin{array}{c}\sup_{a\in\Im\lambda}\rho\left(a,[\frac{3\gamma}{e^4},\frac{\gamma}{e^4}]\right),\\ \sup_{b\in\Gamma\gamma}\rho\left([\frac{3\lambda}{e^3},\frac{\lambda}{e^3}],b\right)\end{array}\right\}\\ &=2\max\left\{\begin{array}{c}\rho\left(\frac{3\lambda}{e^3},[\frac{3\gamma}{e^4},\frac{\gamma}{e^4}]\right),\\ \rho\left([\frac{3\lambda}{e^3},\frac{\lambda}{e^3}],\frac{3\gamma}{e^4}\right)\end{array}\right\}\end{aligned}$$

which yields that

$$\begin{aligned}&\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\\ &=2\max\left\{\rho\left(\frac{3\lambda}{e^3},\frac{\gamma}{e^4}\right),\rho\left(\frac{\lambda}{e^3},\frac{3\gamma}{e^4}\right)\right\}\\ &=2\max\left\{\left|\frac{3\lambda}{e^3}-\frac{\gamma}{e^4}\right|,\left|\frac{\lambda}{e^3}-\frac{3\gamma}{e^4}\right|\right\}\\ &=\frac{2}{e}\max\left\{\left|\frac{3\lambda}{e^2}-\frac{\gamma}{e^3}\right|,\left|\frac{\lambda}{e^2}-\frac{3\gamma}{e^3}\right|\right\}\\ &\le\frac{1}{e}\times 2\max\left\{\begin{array}{c}|\lambda-\gamma|,\left|\lambda-\frac{\lambda}{e^3}\right|,\\ \left|\gamma-\frac{\gamma}{e^4}\right|,\frac{\left|\lambda-\frac{\lambda}{e^3}\right|\left|\gamma-\frac{\gamma}{e^4}\right|}{1+|\lambda-\gamma|}\end{array}\right\}.\end{aligned}$$

It follows that

$$2\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\le\frac{2}{e}\left[2M_\rho(\lambda,\gamma)\right],$$

which yields that

$$\Lambda\left(\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\right)\le\psi\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right].$$

Therefore the condition (1) holds on $\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}$. Also, for all $n\ge 0$, we obtain

$$\begin{aligned}\Lambda(\rho(\lambda_0,\lambda_1))&\le\sum_{i=0}^{n}\Upsilon^i[\Lambda(\rho(\lambda_0,\lambda_1))]\\ &=\sum_{i=0}^{n}\Upsilon^i\left[\Lambda\left(1-\frac{1}{e^3}\right)\right]\\ &=2\left(1-\frac{1}{e^3}\right)\sum_{i=0}^{n}\left(\frac{2}{e}\right)^i\\ &\le 10=r.\end{aligned}$$

Hence, all requirements of Theorem 1 are fulfilled. Moreover, $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $\alpha(\lambda_n,\lambda_{n+1})\ge\eta(\lambda_n,\lambda_{n+1})$ and $\{\Gamma\Im(\lambda_n)\}\longrightarrow 0\in\overline{B_\rho(\lambda_0,r)}$. Also, $\alpha(\lambda_n,0)\ge\eta(\lambda_n,0)$ or $\alpha(0,\lambda_n)\ge\eta(0,\lambda_n)$ for all $n\ge 0$. Further, the point 0 is a unique common FP of $\Im$ and Γ.

4 Fixed point results for graphic contractions

In this portion, we apply Theorem 1 in graph theory as an application.

Definition 12.[12] *Let $\mho$ be a non-empty set and $G=(V(G),E(G))$ be a graph so that $V(G)=\mho$ and let $\Gamma:\mho\to\Xi\beta(\mho)$. Γ is called edge preserving if the condition below hold:*

– for ecah $u\in\Gamma\lambda$ and $v\in\Gamma\gamma$, if $(\lambda,\gamma)\in E(G)$, then $(u,v)\in E(G)$.

Now, we introduce our main theorem in this part.

Theorem 3. *Let $(\mho,\rho)$ be a CMS endowed with a graph G, $\lambda_0 \in \overline{B_\rho(\lambda_0,r)}$, $r>0, \Im,\Gamma:\mho\to\Xi\beta(\mho)$ be two mappings, $\alpha,\eta:\mho\times\mho\to[0,\infty)$ be two functions and $\{\Gamma\Im(\lambda_n)\}$ be a sequence in $\mho$ generated by λ_0 with $(\lambda_0,\lambda_1)\in E(G)$. Suppose that the postulates below hold:*

($\heartsuit_1$)*the pair $(\Im,\Gamma)$ is edge preserving;*

($\heartsuit_2$)*for all $\lambda,\gamma\in\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}$ and $(\lambda,\gamma)\in E(G)$, there are $\Lambda\in\Phi$ and $\Upsilon\in\Psi$ so that $H_\rho(\Im\lambda,\Gamma\gamma)>0$ implies*

$$\Lambda\left(H_\rho(\Im\lambda,\Gamma\gamma)\right)\le\Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right], \tag{14}$$

where

$$M_\rho(\lambda,\gamma) = \max\left\{\begin{array}{c}\rho(\lambda,\gamma),D(\lambda,\Im\lambda),\\ D(\gamma,\Gamma\gamma),\frac{D(\lambda,\Im\lambda).D(\gamma,\Gamma\gamma)}{1+\rho(\lambda,\gamma)}\end{array}\right\}.$$

($\heartsuit_3$)*there is $\lambda_0\in\overline{B_\rho(\lambda_0,r)}$ so that*

$$\Lambda\left(\rho(\lambda_0,\Im\lambda_0)\right) \le\sum_{i=0}^{n}\Upsilon^i\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]\le r,\ for\ r>0,$$

Then $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $(\lambda_n,\lambda_{n+1})\in E(G)$ and $\{\Gamma\Im(\lambda_n)\}\to\lambda^*$. Also, if $(\lambda_n,\lambda^*)\in E(G)$ or $(\lambda^*,\lambda_n)\in E(G)$ for all $n\ge0$ and (14) holds for all $\lambda,\gamma\in\left(\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}\right)\cup\{\lambda^*\}$, then λ^* is a common FP of $\Im$ and Γ in $\overline{B_\rho(\lambda_0,r)}$.

Proof. Define the functions $\alpha,\eta:\mho\times\mho\to[0,+\infty)$ by

$$\alpha(\lambda,\gamma)=\eta(\lambda,\gamma)=\begin{cases}1, \text{ if } (\lambda,\gamma)\in E(G)\\ 0, \text{ otherwise.}\end{cases}$$

Since $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\mho$ generated by λ_0 with $(\lambda_0,\lambda_1)\in E(G)$, we have

$$\alpha(\lambda_0,\lambda_1)\ge\eta(\lambda_0,\lambda_1)\ge1.$$

Let $\alpha(\lambda,\gamma)\ge\eta(\lambda,\gamma)=1$. Then $(\lambda,\gamma)\in E(G)$. From ($\heartsuit_1$), we obtain $(u,v)\in E(G)$ for all $u\in\Im\lambda$ and $v\in\Gamma\gamma$. This implies that $\alpha(u,v)\ge\eta(u,v)=1$ for all $u\in\Im\lambda$ and $v\in\Gamma\gamma$. It follows that

$$\inf\{\alpha(u,v):u\in\Im\lambda,v\in\Gamma\gamma\} \ge\sup\{\eta(u,v):u\in\Im\lambda,v\in\Gamma\gamma\}=1.$$

Thus, $(\Im,\Gamma)$ is a pair of semi α_*-admissible multi-functions wrt η on $\overline{B_\rho(\lambda_0,r)}$. Moreover, if $(\lambda,\gamma)\in E(G)$, we have $\alpha(\lambda,\gamma)=\eta(\lambda,\gamma)=1$ and hence

$$\alpha_*(\Im\lambda,\Gamma\gamma)=\eta_*(\Im\lambda,\Gamma\gamma)=1.$$

Now, condition ($\heartsuit_2$) can be written as

$$\Lambda\left(\alpha_*(\Im\lambda,\Gamma\gamma)H_\rho(\Im\lambda,\Gamma\gamma)\right) =\Lambda\left(H_\rho(\Im\lambda,\Gamma\gamma)\right)\le\Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right],$$

for all $\lambda,\gamma\in\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}$. Condition ($\heartsuit_3$) leads to that all assumptions of Theorem 1. Now, we have $\{\Gamma\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $\alpha(\lambda_n,\lambda_{n+1})\ge\eta(\lambda_n,\lambda_{n+1})$, that is, $(\lambda_n,\lambda_{n+1})\in E(G)$ and $\{\Gamma\Im(\lambda_n)\}\to\lambda^*\in\overline{B_\rho(\lambda_0,r)}$. Further, if $(\lambda_n,\lambda^*)\in E(G)$ or $(\lambda^*,\lambda_n)\in E(G)$ for all $n\in\ge0$ and inequality (14) holds for all $\lambda,\gamma\in\left(\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}\right)\cup\{\lambda^*\}$, we can write

$$\alpha(\lambda_n,\lambda^*)\ge\eta(\lambda_n,\lambda^*) \text{ or }$$
$$\alpha(\lambda^*,\lambda_n)\ge\eta(\lambda^*,\lambda_n)\ \forall n\ge0.$$

Therefore, the existence of a FP λ^* in $\overline{B_\rho(\lambda_0,r)}$ of $\Im$ and Γ follows immediately by Theorem 1. This finished the proof.

Now, we preset some consequences that can be directly proven from Theorem 3. If we put $\Im=\Gamma$ in Theorem 3, we have the result below:

Corollary 4. *Let $(\mho,\rho)$ be a CMS endowed with a graph G, $\lambda_0\in\overline{B_\rho(\lambda_0,r)}$, $r>0$, $\Im:\mho\to\Xi\beta(\mho)$ be a given mapping, $\alpha,\eta:\mho\times\mho\to[0,\infty)$ be two functions and $\{\mho\Im(\lambda_n)\}$ be a sequence in $\mho$ generated by λ_0 with $(\lambda_0,\lambda_1)\in E(G)$. Suppose that the postulates below hold:*

–*the mapping $\Im$ is edge preserving;*

–*for all $\lambda,\gamma\in\overline{B_\rho(\lambda_0,r)}\cap\{\mho\Im(\lambda_n)\}$ and $(\lambda,\gamma)\in E(G)$, there are $\Lambda\in\Phi$ and $\Upsilon\in\Psi$ so that $H_\rho(\Im\lambda,\Im\gamma)>0$ implies*

$$\Lambda\left(H_\rho(\Im\lambda,\Im\gamma)\right)\le\Upsilon\left[\Lambda\left(M_\rho(\lambda,\gamma)\right)\right], \tag{15}$$

where

$$M_\rho(\lambda,\gamma) =\max\left\{\begin{array}{c}\rho(\lambda,\gamma),D(\lambda,\Im\lambda),\\ D(\gamma,\Im\gamma),\frac{D(\lambda,\Im\lambda).D(\gamma,\Im\gamma)}{1+\rho(\lambda,\gamma)}\end{array}\right\}.$$

–*there is $\lambda_0\in\overline{B_\rho(\lambda_0,r)}$ so that*

$$\Lambda\left(\rho(\lambda_0,\Im\lambda_0)\right) \le\sum_{i=0}^{n}\Upsilon^i\left[\Lambda\left(\rho(\lambda_0,\lambda_1)\right)\right]\le r,\ for\ r>0.$$

Then $\{\mho\Im(\lambda_n)\}$ is a sequence in $\overline{B_\rho(\lambda_0,r)}$, $(\lambda_n,\lambda_{n+1})\in E(G)$ and $\{\Gamma\Im(\lambda_n)\}\to\lambda^*$. Also, if $(\lambda_n,\lambda^*)\in E(G)$ or $(\lambda^*,\lambda_n)\in E(G)$ for all $n\ge0$ and (15) holds for all $\lambda,\gamma\in\left(\overline{B_\rho(\lambda_0,r)}\cap\{\Gamma\Im(\lambda_n)\}\right)\cup\{\lambda^*\}$, then λ^* is a FP of $\Im$ in $\overline{B_\rho(\lambda_0,r)}$.

Corollary 5. *Let $(\mho,\rho)$ be a CMS endowed with a graph G, $\lambda_0\in\overline{B_\rho(\lambda_0,r)}$, $r>0$, $\Im,\Gamma:\mho\to\Xi\beta(\mho)$ be two mappings, $\alpha,\eta:\mho\times\mho\to[0,\infty)$ be two functions and $\{\Gamma\Im(\lambda_n)\}$ be a sequence in $\mho$ generated by λ_0 with $(\lambda_0,\lambda_1)\in E(G)$. Suppose that the postulates below hold:*

–*the pair $(\Im,\Gamma)$ is edge preserving;*

–*for all* $\lambda, \gamma \in \overline{B_\rho(\lambda_0, r)} \cap \{\Gamma\Im(\lambda_n)\}$ *and* $(\lambda, \gamma) \in E(G)$, *there are* $\Lambda \in \Phi$ *and* $\Upsilon \in \Psi$ *so that* $H_\rho(\Im\lambda, \Gamma\gamma) > 0$ *implies*

$$\Lambda\left(H_\rho(\Im\lambda, \Gamma\gamma)\right) \le \Upsilon\left[\Lambda\left(\rho(\lambda, \gamma)\right)\right], \qquad (16)$$

–*there is* $\lambda_0 \in \overline{B_\rho(\lambda_0, r)}$ *so that*

$$\Lambda\left(\rho(\lambda_0, \Im\lambda_0)\right) \le \sum_{i=0}^{n} \Upsilon^i\left[\Lambda\left(\rho(\lambda_0, \lambda_1)\right)\right] \le r, \text{ for } r > 0,$$

Then $\{\Gamma\Im(\lambda_n)\}$ *is a sequence in* $\overline{B_\rho(\lambda_0, r)}$, $(\lambda_n, \lambda_{n+1}) \in E(G)$ *and* $\{\Gamma\Im(\lambda_n)\} \to \lambda^*$. *Also, if* $(\lambda_n, \lambda^*) \in E(G)$ *or* $(\lambda^*, \lambda_n) \in E(G)$ *for all* $n \ge 0$ *and (16) holds for all* $\lambda, \gamma \in \left(\overline{B_\rho(\lambda_0, r)} \cap \{\Gamma\Im(\lambda_n)\}\right) \cup \{\lambda^*\}$, *then* λ^* *is a common FP of* $\Im$ *and* Γ *in* $\overline{B_\rho(\lambda_0, r)}$.

*Proof.*In Theorem 3, take $M_\rho(\lambda, \gamma) = \rho(\lambda, \gamma)$ to obtain a common FP $\lambda^* \in \overline{B_d(\lambda_0, r)}$ so that $\lambda^* \in \Im\lambda^* \cap \Gamma\lambda^*$.

Conflict of Interest

The authors declare that they have no conflict of interest.

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