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Bayesian Inference based on Pooled Sample from Two Independent Samples of Record Values

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Abstract: Instead of discussing the problem of estimating the unknown parameters for every underlying distribution separately, this paper develops a general procedure for estimating the unknown parameters based on an ordered pooled sample from two independent sequences of record values using a general exponential form for the underlying distributions. Maximum likelihood and Bayesian methods are used to estimate the unknown parameters. Bayesian estimation is discussed using three different loss functions. The problem of predicting record values from a future sample is also discussed. In addition, the results of the exponential and Pareto distributions are shown as examples. Furthermore, a Monte Carlo simulation study is carried out to compare the maximum likelihood and Bayesian estimates, as well as to examine the performance of point and interval predictions. Finally, a numerical example is provided to demonstrate all of the inferential procedures discussed here.

Keywords: Bayesian estimation, Bayesian prediction, Exponential distribution, Maximum likelihood estimation, Ordered pooled sample, Pareto distribution, Record values

1 Introduction

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent and identically distributed (iid) random variables. Then, an observation X_j is called an upper record value if its value exceeds that of all previous observations, i.e., if $X_j > X_i$ for every $i < j$. Scientists and engineers place a higher value on record values and associated statistics. The record values have been studied extensively and can be found in a variety of real-life situations. For example, climatologists and hydrologists are interested in predicting a river's flood level that is higher than previous flood levels. Seismologists are also interested in predicting the magnitude of an earthquake in a given region that is larger than previous ones.

Furthermore, record values can be used to analyze data from a minimal-repair system; see [1]. The system is put back into operation in a minimal repair experiment after a failure that demanded a minimal repair of the system. Surprisingly, the observed repair times have the same joint distribution as the upper record values in this case. Chandler [2] was the first that introduced the theory of record values, and many authors have since studied record values and the associated statistics; see, for example, [3,4,5,6,7,8,9].

In a random sample of size n, the expected number of observed record values is approximately $\log n + \gamma$, where γ is the Eulers constant 0.5772. As a result, we should expect to find only 7 records in a sequence of 1000 observations. Thus, the statistical inference developed based on this data will have a low degree of precision. In this situation, if taking an additional independent sample of record values is possible and convenient, the ordered pooled sample from these two samples could be used to increase the statistical inference's precision.

Beutner and Cramer [10] looked at a situation where data from two different minimal-repair systems (two independent sequences of record values) are pooled. They derived non-parametric prediction intervals for the future repair times of a minimal-repair system with the same structure. Amini and Balakrishnan [11] looked into a general problem for pooling from k independent samples of record values. They produced exact distribution-free confidence intervals for population quantiles and exact prediction intervals for future record values.

For deriving a general procedure for estimating the unknown parameters of the underlying distribution, we consider here the general exponential form for the underlying distribution, proposed by AL-Hussaini [12],

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with a cumulative distribution function (CDF) which can be written in the form

$$F(x|\theta) = 1 - \exp[-\lambda(x;\theta)], \tag{1}$$

where $\lambda(x;\theta) = -\ln(1 - F(x|\theta))$. Of course, several requirements must be met in order for $F(x|\theta)$ is a proper CDF. These conditions are: $\lambda(x;\theta)$ is continuous, monotone increasing and differentiable function, with $\lambda(x;\theta) \to 0$ as $x \to -\infty$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. The probability density function (PDF) is given by

$$f(x|\theta) = \lambda'(x;\theta)\exp[-\lambda(x;\theta)]. \tag{2}$$

By selecting an appropriate choice of $\lambda(x;\theta)$, we may derive various distributions as special examples from the general exponential form (1), including; exponential, Pareto, Weibull, and Burr Type-XII distributions. Because this exponential form provides some flexibility in developing general results, several authors have developed a general procedure of statistical inference based on different forms of observed data using the general exponential form (1) , see for example; [13,14, 15,16,17,18,19].

In this paper, the general exponential form (1) is used to derive a general procedure for estimating the unknown parameters of the underlying distribution as well as predicting record values from a future sample using an ordered pooled sample of two independent sequences of record values. The rest of this paper is structured as follows. In Section 2, the model of the ordered pooled sample from two independent sequences of record values is described and then the ML method for estimating unknown parameters is discussed. In Section 3, three different loss functions are used to calculate Bayesian estimators for the unknown parameters. The problem of predicting record values from a future sample is discussed in Section 4. Section 5 provides illustrative examples of the exponential and Perato distributions. Finally, the results of a simulation study and a numerical example are presented in Section 6 to illustrate all of the inferential procedures developed in this paper.

2 The Model Description and ML Estimation

Let $X_{(1)},\ldots,X_{(m)}$ and $Y_{(1)},\ldots,Y_{(n)}$ be two independent sequences of record values from the same population. In the following, the ordered pooled sample from $\{X_{(1)},\ldots,X_{(m)};Y_{(1)},\ldots,Y_{(n)}\}$ is denoted by $\mathbf{W} = (W_1,\ldots,W_{m+n})$. The joint density function of the pooled sample $\mathbf{W} = (W_1,\ldots,W_{m+n})$ was developed by Beutner and Cramer [10] as a mixture of the joint distributions of particular generalized order statistics from the same population as follows:

$$f(\mathbf{w}) = \sum_{i=0}^{m-1} K_i f^{\mathbf{U}^{(n+i)}}(\mathbf{w}) + \sum_{j=0}^{n-1} K_j^* f^{\mathbf{V}^{(m+j)}}(\mathbf{w}), \tag{3}$$

where $\mathbf{w} = (w_1,\ldots,w_{m+n})$ is a vector of realizations, $\mathbf{U}^{(n+i)} = (U_{(*1)}^{(n+i)},\ldots,U_{(*m+n)}^{(n+i)})$ for $i = 0,\ldots,m-1$, and $\mathbf{V}^{(m+j)} = (V_{(*1)}^{(m+j)},\ldots,V_{(*m+n)}^{(m+j)})$ for $j = 0,\ldots,n-1$, are generalized order statistics from the same population based on parameters

$$\gamma_\ell^{(n+i)} = 1 + 1_{[1,\ldots,n+i]}(\ell),\ 0 \le i \le m-1,$$
$$\eta_\ell^{(m+j)} = 1 + 1_{[1,\ldots,m+j]}(\ell),\ 0 \le j \le n-1,\ 1 \le \ell \le m+n,$$

respectively with $1_A(\cdot)$ denotes the indicator function on A, and the mixture probabilities are given by

$$K_i = \binom{n+i-1}{n-1} 2^{-(n+i)},\ \ 0 \le i \le m-1,$$
$$K_j^* = \binom{m+j-1}{m-1} 2^{-(m+j)},\ \ 0 \le j \le n-1.$$

By using the joint density function of the generalized order statistics given by Kamps [20], the joint density function (3) of the ordered pooled sample $\mathbf{W} = (W_1,\ldots,W_{m+n})$ is derived by Mohie El-Din et al. [7] as

$$f(\mathbf{w}) = \sum_{i=0}^{m-1} \beta_i \left(\prod_{\substack{k=1 \\ k \ne n+i}}^{m+n-1} \frac{f(w_k)}{1 - F(w_k)} \right) f(w_{n+i}) f(w_{m+n}) + \sum_{j=0}^{n-1} \beta_j^* \left(\prod_{\substack{k=1 \\ k \ne m+j}}^{m+n-1} \frac{f(w_k)}{1 - F(w_k)} \right) f(w_{m+j}) f(w_{m+n}), \tag{4}$$

where

$$\beta_i = \binom{n+i-1}{n-1},\ \ 0 \le i \le m-1,$$

and

$$\beta_j^* = \binom{m+j-1}{m-1},\ \ 0 \le j \le n-1.$$

Upon using (1) and (2) in (4), we obtain the likelihood function as

$$L(\theta;\mathbf{w}) = \sum_{i=0}^{m-1} \beta_i C(\theta;\mathbf{w}) \exp\left[-D_i(\theta;\mathbf{w})\right] + \sum_{j=0}^{n-1} \beta_j^* C(\theta;\mathbf{w}) \exp\left[-D_j^*(\theta;\mathbf{w})\right], \tag{5}$$

where $C(\theta;\mathbf{w}) = \prod_{k=1}^{m+n} \lambda'(w_k;\theta)$,

$$D_i(\theta;\mathbf{w}) = \lambda(w_{n+i};\theta) + \lambda(w_{m+n};\theta), \quad \text{for } i = 0,1,\ldots,m-1,$$

and

$$D_j^*(\theta;\mathbf{w}) = \lambda(w_{m+j};\theta) + \lambda(w_{m+n};\theta), \quad \text{for } j = 0,1,\ldots,n-1.$$

From (5), the log-likelihood function of θ is given by

$$\log L(\theta;\mathbf{z}) = \log\left(\sum_{i=0}^{m-1}\beta_i C(\theta;\mathbf{w})\exp[-D_i(\theta;\mathbf{w})] + \sum_{j=0}^{n-1}\beta_j^* C(\theta;\mathbf{w})\exp\left[-D_j^*(\theta;\mathbf{w})\right]\right).$$

Suppose $\theta = (\theta_1,...,\theta_N)$ is the vector of parameters of the exponential form in (1). Then, the ML estimator $\widehat{\theta}_{qML}$ of θ_q, for $q = 1,...,N$, can be obtained by solving the following system of equations

$$\frac{\partial \log L(\theta;\mathbf{z})}{\partial \theta_q} = 0. \quad q = 1,...,N. \tag{6}$$

Clearly, the system of equations in (6) do not have explicit solution but it is not difficult to carry out a numerical method for this propose.

3 The Bayesian Estimation

In this section, we use the Bayesian method to estimate the unknown parameters of the underlying distribution using a general conjugate prior proposed by AL-Hussaini [12], which is written as

$$\pi(\theta;\delta) \propto A(\theta;\delta)\exp[-B(\theta;\delta)], \tag{7}$$

where δ is the vector of prior parameters. Several priors used in the literature are included in the prior family (7).

Upon combining (5) and (7), the posterior density function of θ, given $\mathbf{W} = \mathbf{w}$, is then given by

$$\pi^*(\theta|\mathbf{w}) = I^{-1}\left\{\sum_{i=0}^{m-1}\beta_i\phi(\theta;\mathbf{w})\exp[-\psi_i(\theta;\mathbf{w})] + \sum_{j=0}^{n-1}\beta_j^*\phi(\theta;\mathbf{w})\exp\left[-\psi_j^*(\theta;\mathbf{w})\right]\right\}, \tag{8}$$

where $\phi(\theta;\mathbf{w}) = A(\theta;\delta)C(\theta;\mathbf{z})$,

$$\psi_i(\theta;\mathbf{w}) = D_i(\theta;\mathbf{w}) + B(\theta;\delta), \text{ for } i = 0,1,\ldots,m-1,$$

$$\psi_j^*(\theta;\mathbf{w}) = D_j^*(\theta;\mathbf{w}) + B(\theta;\delta), \text{ for } j = 0,1,\ldots,n-1,$$

and

$$I = \sum_{i=0}^{m-1}\beta_i\int_{\theta\in\Theta}\phi(\theta;\mathbf{w})\exp[-\psi_i(\theta;\mathbf{w})]\,d\theta + \sum_{j=0}^{n-1}\beta_j^*\int_{\theta\in\Theta}\phi(\theta;\mathbf{w})\exp\left[-\psi_j^*(\theta;\mathbf{w})\right]d\theta.$$

The squared error (SE), linear-exponential (LINEX), and generalized entropy (GE) loss functions are all used to develop the Bayesian estimation in this paper. These loss functions have been considered by many authors; see, for example, [21,22,23,24].

For $q = 1,...,N$, the Bayesian estimator of θ_q under the SE loss function is

$$\widehat{\theta}_{qBS} = E[\theta_q] = I^{-1}\left\{\sum_{i=0}^{m-1}\beta_i\int_{\theta\in\Theta}\theta_q\phi(\theta;\mathbf{w})\exp[-\psi_i(\theta;\mathbf{w})]\,d\theta + \sum_{j=0}^{n-1}\beta_j^*\int_{\theta\in\Theta}\theta_q\phi(\theta;\mathbf{w})\exp\left[-\psi_j^*(\theta;\mathbf{w})\right]d\theta\right\}. \tag{9}$$

For $q = 1,...,N$, the Bayesian estimator of θ_q under the LINEX loss function is

$$\widehat{\theta}_{qBL} = \frac{-1}{\upsilon}\log\left(E\left[e^{-\upsilon\theta_q}\right]\right) = \frac{-1}{\upsilon}\log\left(I^{-1}\left\{\sum_{i=0}^{m-1}\beta_i\int_{\theta\in\Theta}\phi(\theta;\mathbf{w}) \times\exp[-\{\psi_i(\theta;\mathbf{w})+\upsilon\theta_q\}]\,d\theta + \sum_{j=0}^{n-1}\beta_j^*\int_{\theta\in\Theta}\phi(\theta;\mathbf{w}) \times\exp\left[-\{\psi_j^*(\theta;\mathbf{w})+\upsilon\theta_q\}\right]d\theta\right\}\right). \tag{10}$$

For $q = 1,...,N$, the Bayesian estimator of θ_q under the GE loss function is

$$\widehat{\theta}_{qBE} = \left(E[\theta_q^{-\mu}]\right)^{-\frac{1}{\mu}} = \left(I^{-1}\left\{\sum_{i=0}^{m-1}\beta_i\int_{\theta\in\Theta}\theta_q^{-\mu}\phi(\theta;\mathbf{w})\exp[-\psi_i(\theta;\mathbf{w})]\,d\theta + \sum_{j=0}^{n-1}\beta_j^*\int_{\theta\in\Theta}\theta_q^{-\mu}\phi(\theta;\mathbf{w})\exp\left[-\psi_j^*(\theta;\mathbf{w})\right]d\theta\right\}\right)^{-\frac{1}{\mu}}. \tag{11}$$

4 Bayesian prediction

Let $Z_{(1)}, Z_{(2)}, Z_{(3)}, \ldots$ be the record values from a future independent sample from the same population. The Bayesian prediction of $Z_{(r)}$, for $r = 1,2,3...$, is discussed here based on the observed pooled sample $\mathbf{W} = (W_1,...,W_{m+n})$. The Bayesian predictive distribution for $Z_{(r)}$ is derived and the Bayesian point predictor and prediction interval for $Z_{(r)}$ are then calculated.

The marginal density function of $Z_{(r)}$ is well known to be given by; see [25]

$$f_{Z_{(r)}}(z|\theta) = \frac{1}{\Gamma(r)}[-\log\overline{F}(z;\theta)]^{r-1}f(z;\theta), \tag{12}$$

where $\Gamma(\cdot)$ is the complete gamma function.

The marginal density function of $Z_{(r)}$ is obtained by substituting (1) and (2) in (12) as follows:

$$f_{Z_{(r)}}(z|\theta) = \frac{1}{\Gamma(r)}\lambda'(z;\theta)[\lambda(z;\theta)]^{r-1}\exp[-\lambda(z;\theta)], \tag{13}$$

and the Bayesian predictive density function of $Z_{(r)}$, given $\mathbf{W}=\mathbf{w}$, is then obtained by combining (8) and (13) as follows:

$$f^*_{Z_{(r)}}(z|\mathbf{w}) = \int_{\theta\in\Theta} f_{Z_{(r)}}(z|\theta)\pi^*(\theta|\mathbf{w})d\theta$$

$$= \frac{I^{-1}}{\Gamma(r)}\left\{\sum_{i=0}^{m-1}\beta_i \int_{\theta\in\Theta}\lambda'(z;\theta)[\lambda(z;\theta)]^{r-1}\phi(\theta;\mathbf{w}) \exp\left[-\{\psi_i(\theta;\mathbf{w})+\lambda(z;\theta)\}\right]d\theta + \sum_{j=0}^{n-1}\beta^*_j \int_{\theta\in\Theta}\lambda'(z;\theta)[\lambda(z;\theta)]^{r-1}\phi(\theta;\mathbf{w}) \exp\left[-\{\psi^*_j(\theta;\mathbf{w})+\lambda(z;\theta)\}\right]d\theta\right\}. \quad (14)$$

Given $\mathbf{W}=\mathbf{w}$, we can easily get the predictive survival function of $Z_{(r)}$ from (14) as follows:

$$\overline{F}^*_{Z_{(r)}}(t|\mathbf{z}) = \int_{v}^{\infty} f^*_{Z_{(r)}}(z|\mathbf{z})dz$$

$$= \frac{I^{-1}}{\Gamma(k)}\left\{\sum_{i=0}^{m-1}\beta_i \int_{t}^{\infty}\int_{\theta\in\Theta}\lambda'(z;\theta)[\lambda(z;\theta)]^{r-1}\phi(\theta;\mathbf{w}) \exp\left[-\{\psi_i(\theta;\mathbf{w})+\lambda(z;\theta)\}\right]d\theta dz + \sum_{j=0}^{n-1}\beta^*_j \int_{t}^{\infty}\int_{\theta\in\Theta}\lambda'(z;\theta)[\lambda(z;\theta)]^{r-1}\phi(\theta;\mathbf{w}) \exp\left[-\{\psi^*_j(\theta;\mathbf{w})+\lambda(z;\theta)\}\right]d\theta dz\right\}. \quad (15)$$

Under the SE loss function, the Bayesian point predictor of $Z_{(r)}$ is calculated as the mean of the predictive density given by (14).

By solving the following two equations, we can obtain the Bayesian predictive bounds of the two-sided equi-tailed $100(1-\gamma)\%$ interval for $Z_{(r)}$:

$$\bar{F}^*_{Z_{(r)}}(L|\mathbf{w}) = 1-\frac{\gamma}{2}, \quad \text{and} \quad \bar{F}^*_{Z_{(r)}}(U|\mathbf{w}) = \frac{\gamma}{2}, \quad (16)$$

where L and U denote the lower and upper bounds, respectively.

By solving the following two equations, we can obtain the Bayesian predictive bounds of the highest posterior density (HPD) $100(1-\gamma)\%$ interval for $Z_{(r)}$:

$$\bar{F}^*_{Z_{(r)}}(L_{HPD}|\mathbf{w}) - \bar{F}^*_{Z_{(r)}}(U_{HPD}|\mathbf{w}) = 1-\gamma \quad \text{and} \quad f^*_{Z_{(r)}}(L_{HPD}|\mathbf{w}) - f^*_{Z_{(r)}}(U_{HPD}|\mathbf{w}) = 0, \quad (17)$$

where L_{HPD} and U_{HPD} denote the HPD lower and upper bounds, respectively.

5 Illustrative examples

The results of the exponential and Pareto distributions are presented in this section as illustrative examples from the general exponential form (1).

5.1 Exponential(θ) distribution

The CDF in this case is

$$F(x|\theta) = 1-\exp[-\theta x], \quad x>0, \quad (18)$$

where $\theta>0$, and so we have

$$\lambda(x;\theta) = \theta x \quad \text{and} \quad \lambda'(x;\theta) = \theta.$$

Therefore, the likelihood function is given by (5), where $C(\theta;\mathbf{z}) = \theta^{m+n}$,

$$D_i(\theta;\mathbf{w}) = \theta\{w_{n+i}+w_{m+n}\}, \text{ for } i=0,1,\ldots,m-1,$$

and

$$D^*_j(\theta;\mathbf{w}) = \theta\{w_{m+j}+w_{m+n}\}, \text{ for } j=0,1,\ldots,n-1.$$

By solving (6) numerically, we can obtain the ML estimator of θ.

For the Bayesian estimation and prediction, we use the conjugate gamma prior of θ which is given by

$$\pi(\theta;a,b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp[-b\theta], \quad \theta>0, \quad (19)$$

where a and b are positive hyperparameters that could be chosen based on prior knowledge of the mean and variance of θ. Furthermore, by substituting $a=b=0$, we can obtain Jeffrey's prior as a special case of (19). Thus, we have

$$A(\theta;\delta) = \theta^{a-1} \quad \text{and} \quad B(\theta;\delta) = \theta b, \quad (20)$$

where $\delta=(a,b)$.

The posterior density function is then given by (8), where $\phi(\theta;\mathbf{w}) = \theta^{m+n+a-1}$,

$$\psi_i(\theta;\mathbf{w}) = \theta\{w_{n+i}+w_{m+n}+b\}, \text{ for } i=0,1,\ldots,m-1,$$

$$\psi^*_j(\theta;\mathbf{w}) = \theta\{w_{m+j}+w_{m+n}+b\}, \text{ for } j=0,1,\ldots,n-1,$$

and

$$I = \Gamma(m+n+a)\left\{\sum_{i=0}^{m-1}\beta_i[w_{n+i}+w_{m+n}+b]^{-(m+n+a)} + \sum_{j=0}^{n-1}\beta^*_j[w_{m+j}+w_{m+n}+b]^{-(m+n+a)}\right\}.$$

As a result, using the SE, LINEX, and GE loss functions, the Bayesian estimators of θ are given, respectively, by

$$\widehat{\theta}_{BS} = \frac{(m+n+a)}{I_1}\left(\sum_{i=0}^{m-1}\beta_i[w_{n+i}+w_{m+n}+b]^{-(m+n+a+1)} + \sum_{j=0}^{n-1}\beta^*_j[w_{m+j}+w_{m+n}+b]^{-(m+n+a+1)}\right), \quad (21)$$

where

$$I_1 = \left(\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b\right]^{-(m+n+a)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b\right]^{-(m+n+a)}\right).$$

$$\widehat{\theta}_{BL} = \frac{1}{\upsilon}\log I_2 - \frac{1}{\upsilon}\log\left(\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b + \upsilon\right]^{-(m+n+a)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b + \upsilon\right]^{-(m+n+a)}\right), \quad (22)$$

where

$$I_2 = \left(\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b\right]^{-(m+n+a)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b\right]^{-(m+n+a)}\right).$$

$$\widehat{\theta}_{BE} = \left(\frac{\Gamma(m+n+a-c)}{I_3}\right)^{-\frac{1}{c}} \times \left(\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b\right]^{-(m+n+a-c)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b\right]^{-(m+n+a-c)}\right)^{-\frac{1}{c}}, \quad (23)$$

where

$$I_3 = \left(\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b\right]^{-(m+n+a)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b\right]^{-(m+n+a)}\right).$$

In this case, the Bayesian predictive density function of $Z_{(r)}$, given $\mathbf{W} = \mathbf{w}$, is then given by

$$f^*_{Z_{(r)}}(z|\mathbf{w}) = \frac{I^{-1}\Gamma(m+n+r+a)}{\Gamma(r)} \left\{\sum_{i=0}^{m-1} \beta_i z^{r-1} \left[w_{n+i} + w_{m+n} + z + b\right]^{-(m+n+r+a)} + \sum_{j=0}^{n-1} \beta_j^* z^{r-1} \left[w_{m+j} + w_{m+n} + z + b\right]^{-(m+n+r+a)}\right\}, \quad (24)$$

and the predictive survival function of $Z_{(r)}$, given $\mathbf{W} = \mathbf{w}$, is given by

$$\bar{F}^*_{Z_{(r)}}(t|\mathbf{w}) = I^{-1}\left\{\sum_{i=0}^{m-1}\sum_{k=0}^{r-1} \frac{\beta_i \Gamma(m+n+k+a)}{k!} t^k \left[w_{n+i} + w_{m+n} + t + b\right]^{-(m+n+k+a)} + \sum_{j=0}^{n-1}\sum_{k=0}^{r-1} \frac{\beta_j^* \Gamma(m+n+k+a)}{k!} t^k \left[w_{m+j} + w_{m+n} + t + b\right]^{-(m+n+k+a)}\right\}. \quad (25)$$

Under the SE loss function, the Bayesian point predictor of $Z_{(r)}$ is given by

$$\widehat{Z}_{(r)} = \frac{k}{I_4(m+n+a-1)} \times \left\{\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b\right]^{-(m+n+a-1)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b\right]^{-(m+n+a-1)}\right\}, \quad (26)$$

where

$$I_4 = \left\{\sum_{i=0}^{m-1} \beta_i \left[w_{n+i} + w_{m+n} + b\right]^{-(m+n+a)} + \sum_{j=0}^{n-1} \beta_j^* \left[w_{m+j} + w_{m+n} + b\right]^{-(m+n+a)}\right\}.$$

Using $f^*_{Z_{(r)}}(z|\mathbf{w})$ and $\bar{F}^*_{Z_{(r)}}(t|\mathbf{w})$ given in (24) and (25), respectively, the Bayesian predictive bounds of the two-sided equi-tailed $100(1-\gamma)\%$ interval for $Z(r)$ can be obtained by solving the two equations in (16). Also, the Bayesian predictive bounds of the HPD $100(1-\gamma)\%$ interval for $Z(r)$ can be obtained by solving the two equations in (17).

5.2 *Pareto(α, σ) distribution*

The CDF in this case is

$$F(x|\alpha,\sigma) = 1 - \left(\frac{\sigma}{x}\right)^{\alpha}, \quad x \geq \sigma, \quad (27)$$

where $\alpha > 0$ and $\sigma > 0$, and so we have

$$\lambda(x;\alpha,\sigma) = \alpha \ln\left(\frac{x}{\sigma}\right) \quad \text{and} \quad \lambda'(x;\alpha,\sigma) = \frac{\alpha}{x}.$$

Therefore, the likelihood function is given by (5), where

$$C(\alpha,\sigma;\mathbf{w}) = \alpha^{r+s} \prod_{i=1}^{m+n} \frac{1}{w_i},$$

$$D_i(\alpha,\sigma;\mathbf{w}) = \alpha\left(\ln w_{n+i} + \ln w_{m+n} - 2\ln\sigma\right), \quad \text{for } i = 0,1,\ldots,m-1,$$

$$D_j^*(\alpha,\sigma;\mathbf{w}) = \alpha\left(\ln w_{m+j} + \ln w_{m+n} - 2\ln\sigma\right) \quad \text{for } j = 0,1,\dots,n-1.$$

The likelihood function (5) is obviously a monotone increasing function in σ, so its maximum value will be when $\sigma = w_1$. As a result, the ML estimator of α can be found by solving (6) numerically with respect to α.

Under the assumption that both parameters α and σ are unknown, for Bayesian estimation and prediction, we may use the joint power-gamma prior of α and σ proposed by Arnold and Press [25] as follows;

$$\pi_2(\alpha,\sigma) \propto \alpha^a\sigma^{-1}\exp[-\alpha(\ln c - b\ln\sigma)], \quad \alpha>0,\ 0<\sigma<d, \tag{28}$$

where a, b, c, d are positive constants and $d^b < c$. Thus, we have

$$A(\alpha,\sigma;\delta) = \alpha^a\sigma^{-1} \text{ and } B(\alpha,\sigma;\delta) = \alpha(\ln c - b\ln\sigma), \tag{29}$$

where $\delta = (a,b,c,d)$.

The posterior density function is then given by (8), with

$\phi(\alpha,\sigma;\mathbf{w}) = \alpha^{m+n+a}\sigma^{-1}$,

$$\psi_i(\alpha,\sigma;\mathbf{w}) = \alpha\left(\ln w_{n+i} + \ln w_{m+n} - (b+2)\ln\sigma + \ln c\right), \quad \text{for } i = 0,1,\dots,m-1,$$

$$\psi_j^*(\alpha,\sigma;\mathbf{w}) = \alpha\left(\ln w_{m+j} + \ln w_{m+n} - (b+2)\ln\sigma + \ln c\right), \quad \text{for } j = 0,1,\dots,n-1,$$

$$I = \frac{\Gamma(m+n+a)}{b+2}\left\{\sum_{i=0}^{m-1}\beta_i\left[\rho_i(\mathbf{w},M)\right]^{-(m+n+a)} + \sum_{j=0}^{n-1}\beta_j^*\left[\rho_j^*(\mathbf{w},M)\right]^{-(m+n+a)}\right\},$$

where

$$\rho_i(\mathbf{w},y) = \ln w_{n+i} + \ln w_{m+n} - (b+2)\ln y + \ln c, \quad \text{for } i = 0,1,\dots,m-1,$$

$$\rho_j^*(\mathbf{w},y) = \ln w_{m+j} + \ln w_{m+n} - (b+2)\ln y + \ln c, \quad \text{for } j = 0,1,\dots,n-1,$$

and $M = \min(w_1,d)$.

The Bayesian estimators of α and σ under the SE, LINEX and GE loss functions can obtained from equations (21), (22) and (23), respectively.

The Bayesian predictive density function of $Z_{(r)}$, given $\mathbf{W}=\mathbf{w}$, in this case is then given by

$$f_{Z_{(r)}}^*(z|\mathbf{w}) = \begin{cases} f_{1Z_{(r)}}^*(z|\mathbf{w}), & 0<z\le M,\\ f_{2Z_{(r)}}^*(z|\mathbf{w}), & z>M,\end{cases} \tag{30}$$

where

$$f_{1Z_{(r)}}^*(z|\mathbf{w}) = \int_0^z\int_0^\infty f_{Z_{(r)}}(z|\alpha,\sigma)\pi^*(\alpha,\sigma|\mathbf{w})d\alpha d\sigma = \frac{(-1)^{r-1}\Gamma(m+n+a+1)I^{-1}}{(b+3)^r}\left\{\sum_{i=0}^{m-1}\beta_i z^{-1}[\rho_i(\mathbf{w},z)]^{-(m+n+a+1)} + \sum_{j=0}^{n-1}\beta_j^* z^{-1}[\rho_j^*(\mathbf{w},z)]^{-(m+n+a+1)}\right\}$$

and

$$f_{2Z_{(r)}}^*(z|\mathbf{w}) = \int_0^M\int_0^\infty f_{Z_{(r)}}(z|\alpha,\sigma)\pi^*(\alpha,\sigma|\mathbf{w})d\alpha d\sigma$$

$$= I^{-1}\left\{\sum_{i=0}^{m-1}\sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}\Gamma(m+n+k+a+1)}{k!(b+3)^{r-k}}\beta_i z^{-1}[\ln z - \ln M]^k [\rho_i(\mathbf{w},M)+\ln z - \ln M]^{-(m+n+k+a+1)} + \sum_{j=0}^{n-1}\sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}\Gamma(m+n+k+a+1)}{k!(b+3)^{r-k}}\beta_j^* z^{-1}[\ln z - \ln M]^k \left[\rho_j^*(\mathbf{w},M)+\ln z - \ln\right]^{-(m+n+k+a+1)}\right\}.$$

From (30), we simply obtain the predictive survival function of $Z_{(r)}$, given $\mathbf{W}=\mathbf{w}$, as

$$\bar{F}_{Z_{(r)}}^*(t|\mathbf{w}) = \begin{cases}\bar{F}_{1Z_{(r)}}^*(t|\mathbf{w}), & 0<t<M,\\ \bar{F}_{2Z_{(r)}}^*(t|\mathbf{w}), & t\ge M,\end{cases} \tag{31}$$

where

$$\bar{F}_{1Z_{(r)}}^*(t|\mathbf{w}) = \int_t^M f_{1Z_{(r)}}^*(z|\mathbf{w})dz + \int_M^\infty f_{2Z_{(r)}}^*(z|\mathbf{w})dz$$

$$= \frac{(-1)^r(m+n+a)I^{-1}}{(b+3)^r(b+2)}\left\{\sum_{i=0}^{m-1}\beta_i\left([\rho_i(\mathbf{w},M)]^{-(m+n+a)} - [\rho_i(\mathbf{w},t)]^{-(m+n+a)}\right) + \sum_{j=0}^{n-1}\beta_j^*\left([\rho_j^*(\mathbf{w},M)]^{-(m+n+a)} - [\rho_j^*(\mathbf{w},t)]^{-(m+n+a)}\right)\right\}$$

$$+ I^{-1}\Gamma(m+n+a)\left\{\sum_{i=0}^{m-1}\sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}}{(b+3)^{r-k}}\beta_i[\rho_i(\mathbf{w},M)]^{-(m+n+a)} + \sum_{j=0}^{n-1}\sum_{k=0}^{r-1}\frac{(-1)^{r-k-1}}{(b+3)^{r-k}}\beta_j^*[\rho_j^*(\mathbf{w},M)]^{-(m+n+a)}\right\}$$

and

$$\bar{F}^{*}_{2Z_{(r)}}(t|\mathbf{w}) = \int_{t}^{\infty} f^{*}_{2Z_{(r)}}(z|\mathbf{w})dz$$

$$= I^{-1}\left\{\sum_{i=0}^{m-1}\sum_{k=0}^{r-1}\sum_{q=0}^{k}\frac{(-1)^{r-k-1}\Gamma(m+n+q+a)}{q!(b+3)^{r-k}}\beta_i \times[\ln t-\ln M]^{q}[\rho_i(\mathbf{w},M)+\ln t-\ln M]^{-(m+n+q+a)} + \sum_{j=0}^{n-1}\sum_{k=0}^{r-1}\sum_{q=0}^{k}\frac{(-1)^{r-k-1}\Gamma(m+n+q+a)}{q!(b+3)^{r-k}}\beta^{*}_{j} \times[\ln t-\ln M]^{q}[\rho^{*}_{j}(\mathbf{w},M)+\ln t-\ln M]^{-(m+n+q+a)}\right\}.$$

Under the SE loss function, the Bayesian point predictor of $Z_{(r)}$ is given by

$$\widehat{Z}_{(r)} = \int_{0}^{M} z f^{*}_{1Z_{(r)}}(z|\mathbf{w})dz + \int_{M}^{\infty} z f^{*}_{2Z_{(r)}}(z|\mathbf{w})dz,$$

Using $f^{*}_{Z_{(r)}}(z|\mathbf{w})$ and $\bar{F}^{*}_{Z_{(r)}}(t|\mathbf{w})$ given in (30) and (31), respectively, the Bayesian predictive bounds of the two-sided equi-tailed $100(1-\gamma)\%$ interval for $Z(r)$ can be obtained by solving the two equations in (16). Also, the Bayesian predictive bounds of the HPD $100(1-\gamma)\%$ interval for $Z(r)$ can be obtained by solving the two equations in (17).

6 Results and discussion

Some computational findings for the exponential distribution are provided in this section. A Monte Carlo simulation study is conducted to compare the ML and Bayesian estimates, as well as to examine the performance of point and interval prediction. Finally, numerical findings from real data are shown to demonstrate all of the inferential outcomes.

6.1 Monte Carlo simulation

In this simulation study, the performance of the ML and Bayesian estimates, as well as the point and interval prediction, is tested. With different values for m and n, the parameter θ was chosen to be $0.1, 1, 5$. We calculated the ML and Bayesian estimates of θ under the SE, LINEX, and GE loss functions using informative gamma prior (GP) with $(a,b) = (0.1,1)$ (when $\theta = 0.1$), (1,1) (when $\theta = 1$), and (5,1) (when $\theta = 5$), and Jeffreys' prior (JP) with $(a,b) = (0,0)$. Also, when $\theta = 1$, we computed the point predictor as well as the 95% equi-tailed and the HPD prediction intervals for record values $Z_{(r)}$, for $r = 1, 5, 10, 15$, from a future sample from the same population.

We repeated this process 10000 times and then computed the estimated bias (EB) and used the root mean square error to calculate the estimated risk (ER) for each estimate. Table 1 provides the EB and ER of all the estimates of θ. In addition, we calculated the estimated average value $(\widehat{Z}_{(r)})$ and mean squared prediction error (MSPE) of the point predictor, as well as the estimated average value of the lower limit (L), upper bound (U), and width for each prediction interval, which are all shown in Table 2.

Table 1: The estimated bias and risk of the ML and Bayesian estimates of θ for different choices of θ, m and n.

				$\hat{\theta}_{ML}$		$\hat{\theta}_{BS}$		$\hat{\theta}_{BL}$		$\hat{\theta}_{BE}$	
θ	m	n		*EB*	*ER*	*EB*	*ER*	*EB*	*ER*	*EB*	*ER*
0.1	4	4	GP	0.0300	0.1118	0.0180	0.0534	0.0175	0.0527	0.0064	0.0457
			JP	–	–	0.0187	0.0552	0.0181	0.0544	0.0068	0.0472
	6	4	GP	0.0221	0.0661	0.0150	0.0447	0.0146	0.0443	0.0055	0.0391
			JP	–	–	0.0154	0.0458	0.0150	0.0453	0.0058	0.0399
	8	6	GP	0.0108	0.0367	0.0100	0.0338	0.0097	0.0336	0.0035	0.0306
			JP	–	–	0.0102	0.0343	0.0099	0.0341	0.0036	0.0310
	10	8	GP	0.0097	0.0319	0.0075	0.0297	0.0074	0.0296	0.0026	0.0276
			JP	–	–	0.0077	0.0300	0.0075	0.0299	0.0027	0.0278
1	4	4	GP	0.1885	0.6029	0.1362	0.4212	0.0956	0.3807	0.0460	0.3658
			JP	–	–	0.1868	0.5515	0.1341	0.4850	0.0684	0.4717
	6	4	GP	0.1711	0.4784	0.1202	0.3749	0.0869	0.3445	0.0365	0.3311
			JP	–	–	0.1542	0.4579	0.1136	0.4137	0.0581	0.3992
	8	6	GP	0.1084	0.3668	0.0858	0.3042	0.0630	0.2865	0.0259	0.2773
			JP	–	–	0.1017	0.3429	0.0760	0.3207	0.0360	0.3099
	10	8	GP	0.0733	0.3016	0.0669	0.2737	0.0496	0.2611	0.0204	0.2547
			JP	–	–	0.0766	0.3004	0.0577	0.2853	0.0267	0.2781
5	4	4	GP	0.7924	3.0144	0.3031	1.1788	0.2069	0.9599	0.2350	1.0743
			JP	–	–	0.9338	2.7575	0.5036	1.7064	0.6418	2.3583
	6	4	GP	0.6553	2.3920	0.3022	1.1508	0.1578	0.9470	0.1971	1.0554
			JP	–	–	0.7711	2.2897	0.3639	1.5496	0.4906	1.9960
	8	6	GP	0.5420	1.8339	0.2526	1.0756	0.1152	0.9169	0.1293	1.0040
			JP	–	–	0.5087	1.7144	0.2547	1.3150	0.3800	1.5496
	10	8	GP	0.4868	1.5953	0.2146	1.0289	0.0907	0.8975	0.1031	0.9728
			JP	–	–	0.3831	1.5020	0.1444	1.2189	0.2336	1.3905

6.2 Illustrative example

Jarrett [26] corrects and expands the British coal-mining disasters accident data of Maguire et al. [27], which covers the period from 15 March 1851 to 22 March 1962 and includes 191 explosions with 10 or more men killed. The values represent the number of days between successive coal-mining disasters. The time intervals between two successive disasters are assumed to be independent and exponentially distributed with $\theta = 0.0039$. We divided the time period from 15 March 1851 to 22 March 1962 into two sub-periods and saved only the upper record values for each sub-period.

Table 3 shows the record values generated from the data in each sub-period. We used the GP with

Table 2: Bayesian prediction of $Z_{(r)}$ for $r = 1, 5, 10, 15$, and for different choices of m and n when $\theta = 1$

				Point Predictor		*Equi − Tailed Interval*			*HPD Interval*		
r	m	n		$\widehat{Z}_{(r)}$	MSPE	L	U	*Width*	L	U	*Width*
1	4	4	GP	1.1106	0.1285	0.0248	4.5247	4.4998	0.0000	3.5201	3.5201
			JP	1.1265	0.1708	0.0247	4.6493	4.6246	0.0000	3.5956	3.5956
	6	4	GP	1.0863	0.1065	0.0248	4.3598	4.3340	0.0000	3.4154	3.4154
			JP	1.0959	0.1347	0.0247	4.4389	4.4142	0.0000	3.4633	3.4633
	8	6	GP	1.0648	0.0787	0.0250	4.1795	4.1545	0.0000	3.3064	3.3064
			JP	1.0697	0.0927	0.0249	4.2192	4.1943	0.0000	3.3308	3.3308
	10	8	GP	1.0527	0.0602	0.0251	4.0784	4.0533	0.0000	3.2446	3.2446
			JP	1.0557	0.0683	0.0251	4.1026	4.0775	0.0000	3.2597	3.2597
5	4	4	GP	5.5532	3.5848	1.4086	14.2902	12.8816	0.7889	12.1852	11.3963
			JP	5.6325	4.2696	1.3838	14.8953	13.5115	0.7485	12.5839	13.3324
	6	4	GP	5.4317	2.6624	1.4274	13.5248	12.0974	0.8330	11.6663	10.8333
			JP	5.4798	3.3665	1.4074	13.9133	12.5059	0.8010	11.9234	11.1224
	8	6	GP	5.3240	1.9676	1.4776	12.6247	11.1471	0.9170	11.0705	10.1535
			JP	5.3485	2.1355	1.4663	12.8195	11.3529	0.8977	11.2014	10.3037
	10	8	GP	5.2633	1.5043	1.5089	12.1200	10.6111	0.9714	10.7325	9.7611
			JP	5.2786	1.7085	1.5019	12.2380	10.7361	0.9584	10.8127	9.8543
10	4	4	GP	11.1063	12.8515	3.8752	25.5609	21.6857	2.7608	22.2802	19.5197
			JP	11.2650	17.0775	3.7831	26.7968	23.0137	2.6203	23.1144	20.4941
	6	4	GP	10.8633	10.6504	3.9548	24.0043	20.0495	2.9126	21.2039	18.2913
			JP	10.9595	13.4652	3.8797	24.8034	20.9237	2.7999	21.7489	18.9490
	8	6	GP	10.6480	7.8697	4.1470	22.1320	17.9850	3.2100	19.9252	16.7152
			JP	10.6970	9.2708	4.1025	22.5362	18.4337	3.1409	20.2063	17.0654
	10	8	GP	10.5265	6.0167	4.2715	21.0739	16.8024	3.4059	19.1909	15.7850
			JP	10.5573	6.8335	4.2427	21.3201	17.0774	3.3599	19.3646	16.0047
15	4	4	GP	16.6595	28.9153	6.4980	36.6563	30.0065	4.9275	32.2032	27.2757
			JP	16.8975	38.4251	6.3244	38.5356	32.2112	4.6715	33.4868	28.8153
	6	4	GP	16.2950	23.9630	6.6546	34.2892	27.6345	5.1092	30.1428	25.0336
			JP	16.4393	30.2973	6.3850	34.9863	28.60127	4.8923	30.9248	26.0325
	8	6	GP	15.9520	17.7069	7.0041	31.2686	24.2645	5.7381	28.4279	22.6898
			JP	16.0454	20.8595	6.9169	31.8793	24.9624	5.6098	28.8592	23.2494
	10	8	GP	15.7898	13.5378	7.1418	29.2068	22.0650	6.0134	26.8783	20.8649
			JP	15.8359	15.3759	7.0765	29.5441	22.4676	5.9211	27.1143	21.1932

Table 3: Record values from British coal-mining disasters accident data

	Record values						
Sub-period 1	157	216	232	826			
Sub-period 2	176	315	345	388	1205	1643	2366

Table 4: The ML and Bayes estimates of θ.

	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{BS}$	$\widehat{\theta}_{BL}$	$\widehat{\theta}_{BE}$
GP	0.0038	0.0038	0.0038	0.0035
JP	–	0.0038	0.0038	0.0035

$(a,b) = (0.004, 1)$ and JP with $(a,b) = (0,0)$ to compute the ML and Bayesian estimates of θ under the SE, LINEX (with $v = 0.5$), and GE (with $c = 0.5$) loss functions as shown in Table 4. We also calculated the point predictor, the bounds of the 95% equi-tailed and the HPD prediction intervals for record values $Z_{(r)}$, for $r = 1, 2, \ldots, 10$, from a future sample from the same population, and these obtained results are presented in Table 5.

Table 5: Bayesian prediction of $Z_{(r)}$ for $r = 1, \ldots, 10$

	Point predictor		Equi-tailed interval		HPD interval	
r	GP	JP	GP	JP	GP	JP
1	292.8	292.8	(6.7, 1178.3)	(6.7, 1178.6)	(0.000, 921.7)	(0.000, 921.9)
2	585.6	585.7	(61.3, 1858.4)	(61.3, 1858.8)	(6.6, 1517.3)	(6.6, 1517.6)
3	878.3	878.5	(152.2, 2482.6)	(152.2, 2483.2)	(53.7, 2079.4)	(53.7, 2079.9)
4	1171.1	1171.3	(262.1, 3083.4)	(262.2, 3084.2)	(129.9, 2624.3)	(129.9, 2624.9)
5	1463.9	1464.2	(383.2, 3671.3)	(383.2, 3672.3)	(222.0, 3158.0)	(221.9, 3158.7)
6	1756.7	1757.0	(511.4, 4251.2)	(511.4, 4252.4)	(323.5, 3684.1)	(323.5, 3685.0)
7	2049.4	2049.8	(644.5, 4825.7)	(644.5, 4827.1)	(431.5, 4205.0)	(431.5, 4206.1)
8	2342.2	2342.7	(781.1, 5396.4)	(781.2, 5397.9)	(543.8, 4722.1)	(543.8, 4723.3)
9	2635.0	2635.5	(920.4, 5964.2)	(920.5, 5965.9)	(659.5, 5236.5)	(659.4, 5237.8)
10	2927.8	2928.3	(1061.7, 6529.8)	(1061.8, 6531.8)	(777.6, 5748.6)	(777.6, 5750.1)

The above results can be used in different fields [28]-[32].

7 Conclusions

In this paper, a general procedure for estimation and prediction based on ordered pooled sample from two independent sequences of record values is developed using the general exponential form of the underlying distributions and the general form of the prior distributions. The ML estimator for unknown parameter is obtained. The SE, LINEX, and GE loss functions are used to calculate the Bayesian estimator. The point and interval predictions for record values a future sample values from the same population are computed. As an illustration, the results of the exponential distribution are shown.

From Tables 1-5, we observe that:

- For all different choices of θ, m and n, the EB and ER of the Bayesian estimate are smaller than those of the ML estimate.
- The EB and ER of Bayesian estimates using the LINEX and GE loss functions are smaller than those using the SE loss function for all different values of θ, m, and n.
- The EB and ER of all estimates decrease with increasing m and n.
- In all of the cases considered, the HPD prediction interval is more precise than the corresponding equi-tailed interval.
- The width of the equi-tailed and HPD intervals and the corresponding mean squared prediction error decrease with increasing m and n.
- The width of the equi-tailed and HPD intervals and the corresponding mean squared prediction error increase with increasing r.
- When the results of the informative priors are compared to those of Jeffreys' non-informative priors, it is clear that the former provides more precise results.

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Conflicts of Interests

The authors declare that they have no conflicts of interests

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