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On Categorical Property of Fuzzy Soft Topological Spaces

S. Saleh[1,2,*], *Radwan Abu-Gdairi*[3], *Tareq M. Al-shami*[4] *and Mohammed S. Abdo*[2]

[1]Computer Science Department, Cihan University-Erbil, Kurdistan region, Iraq
[2]Mathematics Department, Hodeidah University, Hodeidah, Yemen
[3]Department of Mathematics, Faculty of Science, Zarqa University, P.O. Box 13110 Zarqa, Jordan
[4]Department of Mathematics, Sana'a University, Sana'a, Yemen

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Abstract: The main aim of this work is to provide a categorical framework for fuzzy soft topological spaces ***FST***. We define some of their subcategories such as ***Top***, ***STop***, δ_τ-***FST***, $\widetilde{\chi}$-***FST***, and $\widetilde{\alpha}$-***FST***. In general, we initiate some functors between them and discuss some results and relations of fuzzy soft maps. The relationships between FST and their subcategories are studied, for instance, we show that *Top*, ***STop*** are bireflective full subcategories of ***FST*** and $\widetilde{\alpha}$-***FST***is a bicoreflective full subcategory of ***FST***. Moreover, we demonstrate that ***Top***, δ_τ-***FST***, and $\widetilde{\chi}$-***FST*** are isomorphic.

Keywords: Soft set, fuzzy soft set, fuzzy soft mapping, soft topology, fuzzy soft topology, category theory.

1 Introduction and preliminaries

It is known that there are some mathematical tools for dealing with uncertainties such as fuzzy sets (F-sets), introduced by Zadeh [1], soft setss (S-sets) introduced by Molodtsov [2], and fuzzy softs (FS-sets), developed by Maji et al.[3]. To expand the grade spaces of membership and nonmembership functions, the concepts of SR-fuzzy sets [4] and (3,2)-fuzzy sets [6] were defined and discussed. In 1986, Chang [7] studied topological ideas and notions via fuzzy settings. At this year, Ameen et al. [8] explored the space of infra-fuzzy topology as a new generalization of fuzzy topology.

In 2011, Shabir-Naz [9] applied soft sets to define the concept of soft topology. They discussed the notions of soft interior and closure operators and soft separation axioms. Then, many authors redefined the classical topological concepts via soft topological structure like [10,11,12,13,14]. Al-shami successfully applied some theoretical soft topological concepts to to information science [15] and decision-making problems [16].

Recently, Al-shami [17] has extended this concept to infra-soft topologies. He has investigated the main topological notions via the structure of infra-soft topology such as compactness and fixed point theorem [18], Homeomorphism and quotient mappings [19], and separation axioms [20,21,5]. Tanay et al. [22] presented the concepts of topological structures with respect to FS-sets and gave some of their properties. Also, Roy and Samanta [23] gave the definition of FS-topology over the initial universe set. Then FS-sets have been studied and applied in different fields in recent time, e.g. [24,25,26, 27,28,29,30].

This paper organized as follows, after the introductory section, we give some basic definitions which are needed throughout this paper. In section 2, we give some results and properties. The relations between the continuity in fuzzy soft topologies (FSTs) and some induced soft topologies (STs) are studied. In section 3, in order to provide a categorical framework for fuzzy soft topological spaces ***FST***, we define some subcategories of ***FST***, and construct some functors between them. The relationship between ***FST*** and some of their subcategories are studied, for instance, we illustrate that ***Top***, ***STop*** are bireflective full subcategories of ***FST*** and $\widetilde{\alpha}$-***FST*** are a bicoreflective full subcategory of ***FST***. Moreover, we elucidate that the categories ***Top***, δ_τ-***FST***, and $\widetilde{\chi}$-***FST*** are isomorphic.

* Corresponding author e-mail: salem.saleh@cihanuniversity.edu.iq

Through this manuscript, X is an universe set, E is the set of parameters for X, $P(X)$ is the power set of X and I^X is the set of all F-subsets of X, where $I=[0,1]$. For background of category theory we refer to [31].

A category is a quadruple $\mathscr{C}=(\partial, hom, id, \circ)$ consisting of:
1. A class ∂, whose members are called $\mathscr{C}$-objects
2. For each pair (A,B) of $\mathscr{C}$-objects, a set $hom(A,B)$, whose members are called $\mathscr{C}$-morphisms from A to B,
3. For each $\mathscr{C}$-object A, a morphism $idA : A \to A$, called the $\mathscr{C}$-identity on A
4. A composition law associating with each $\mathscr{C}$-morphism $f : A \to B$ and each $\mathscr{C}$-morphism $g : B \to C$ an $\mathscr{C}$-morphism $g \circ f : A \to C$, called the composite of f and g, subject to the following conditions:
(a) composition is associative
(b) $\mathscr{C}$-identities act as identities with respect to composition
(c) the sets $hom(A,B)$ are pairwise disjoint.

For $A \subset X$. The characteristic function χ_A of A is a function $\chi_A : X \to \{0,1\}$ given by $\chi_A(x)=1$ if $x \in A$ and $\chi_A(x)=0$ if $x \notin A$ [32].

A Fuzzy set(F-set) A of X is a set characterized by the membership function $A : X \to I$ and A can be written as $A=\{(x,A(x)) : x \in X, A(x) \in I\}$. The support of $A \in I^X$ is the crisp set $S(A)=\{x \in X : A(x)>0\}$. For $A,B \in I^X$ and $A_i \in I^X, i \in J$ we have $S(A \cap B)=S(A) \cap S(B)$, $S(\cup_{i \in J} A_i)=\cup_{i \in J} S(A_i)$, $S(\underline{0})=\emptyset, S(\underline{1})=X$, and $S(\chi_A)=A$. For the basic operations on F-sets see [1].

Definition 1.([34, 2]) *A soft set (S-set) $F_E=(F,E)$ on X is a mapping $F : E \to P(X)$ where, the value $F(e)$ is a subset of X for all $e \in E$. Thus a S-set F_E on X can be represented by the set of ordered pairs $F_E=\{(e,F(e)) : e \in E, F(e) \in P(X)\}$. The family of all S-sets on X is denoted by $SS(X,E)$.*

The ST-space ([35, 33]) (X,τ^*,E) where X is an universe set, E is a fixed set of parameters, and τ^* is the collection of soft sets on X which are closed under arbitrary union, finite intersection, and $\emptyset_E, X_E \in \tau^*$. The members of τ^* are called soft open sets in X and denoted by $SO(X)$ and S-set F_E on X is said to be soft closed in X iff $F_E^c \in \tau^*$, denoted by $SC(X)$.

Definition 2.([36, 37]) *A fuzzy soft set (FS-set) $f_E=(f,E)$ on X is a mapping $f : E \longrightarrow I^X$, where the value $f(e)$ is a F-set for all $e \in E$. Thus f_E can be written as the set of ordered pairs $f_E=\{(e,f(e)) : e \in E,\ f(e) \in I^X\}$. $FSS(X,E)$ refer to the family of all FS-sets on X.*

Let X be a universe set, E is a fixed set of parameters, and δ is the family of FS-sets on X which are closed under arbitrary union, finite intersection, and $\widetilde{0}_E$, $\widetilde{1}_E$ belong to δ. In this case (X,δ,E) is called a FST-space [26, 23].

Definition 3.([39]) *Let $SS(X,E)$ and $SS(Y,K)$ are two families of all S-sets on X,Y respectively and let $f : X \to Y$, $u : E \to K$ are two maps, then the map $\varphi_{uf} : SS(X,E) \to SS(Y,K)$ is said to be a soft map (S-map) for which:*

1) If $U_E \in SS(X,E)$ then the image $\varphi_{uf}(U_E)$ of $U_E$is the S-set on Y given by $\varphi_{uf}(U_E)(k)=\cup\{f(U(e)) : e \in u^{-1}(k)\}$ if $u^{-1}(k) \neq \emptyset$ and $\varphi_{uf}(U_E)(k)=\emptyset_E$ otherwise for all $k \in K$.
2) If $V_K \in SS(Y,K)$ then the preimage of V_K denoted by $\varphi_{uf}^{-1}(V_K)$ is the S-set on X given by $\varphi_{uf}^{-1}(e)=f^{-1}(V(u(e)))$ for all $e \in E$. The S-map φ_{uf} is called one-one, if f and u are one-one and φ_{uf} is called onto, if f and u are onto.

Definition 4.([38]) *Let $FSS(X,E)$ and $FSS(Y,K)$ are two families of all FS-sets on X,Y respectively and let $f : X \longrightarrow Y, u : E \to K$ are two maps, then the map $\widetilde{\varphi}_{uf} : FSS(X,E) \to FSS(Y,K)$ is called a fuzzy soft map (FS-map)for which:*

1) If $h_E \in FSS(X,E)$ then the image $\widetilde{\varphi}_{uf}$ of $h_E$is the FS-set on Y given by $\widetilde{\varphi}_{uf}(h_E)(k)=sup\{f(h(e)) : e \in u^{-1}(k)\}$ if $u^{-1}(k) \neq \emptyset$ and $\widetilde{\varphi}_{uf}(h_E)(k)=\widetilde{\emptyset}_Y$ otherwise for all $k \in K$.
2) If $g_K \in FSS(Y,K)$ then the preimage $\widetilde{\varphi}_{uf}^{-1}(g_K)$ of g_K is the FS-set on X defined by $\widetilde{\varphi}_{uf}^{-1}(e)=f^{-1}(g(u(e)))$ for all $e \in E$. The FS-map $\widetilde{\varphi}_{uf}$ is called one-one, if f and u are one-one and $\widetilde{\varphi}_{uf}$ is called onto, if f and u are onto.

For more details about the properties of image and preimage of S-sets see [39].

Theorem 1.([38]) *Let $h_{iE} \in FSS(X,E), g_{iK} \in FSS(Y,K)$ $\forall i \in J$, then for a FS-map $\widetilde{\varphi}_{uf} : FSS(X,E) \to FSS(Y,K)$, we have:*

1) if $h_{1E} \sqsubseteq h_{2E}$, then $\widetilde{\varphi}_{uf}(h_{1E}) \sqsubseteq \widetilde{\varphi}_{uf}(h_2E)$,
2) if $g_{1K} \sqsubseteq g_{2K}$, then $\widetilde{\varphi}_{uf}^{-1}(g_{1K}) \sqsubseteq \widetilde{\varphi}_{uf}^{-1}(g_{2K})$,
3) $\widetilde{\varphi}_{uf}(\sqcup_{i \in J} h_{iE})=\sqcup_{i \in J} \widetilde{\varphi}_{uf}(h_{iE})$,
4) $\widetilde{\varphi}_{uf}(\sqcap_{i \in J} h_{iE}) \sqsubseteq \sqcap_{i \in J} \widetilde{\varphi}_{uf}(h_{iE})$,
5) $\widetilde{\varphi}_{uf}^{-1}(\sqcup_{i \in J} g_{iK})=\sqcup_{i \in J} \widetilde{\varphi}_{uf}^{-1}(g_{iK})$,
6) $\widetilde{\varphi}_{uf}^{-1}(\sqcap_{i \in J} g_{iK})=\sqcap_{i \in J} \widetilde{\varphi}_{uf}^{-1}(g_{iK})$,
7) $\widetilde{\varphi}_{uf}^{-1}(\widetilde{1}_K)=\widetilde{1}_E$ and $\widetilde{\varphi}_{uf}^{-1}(\widetilde{0}_K)=\widetilde{0}_E$,
8) $\widetilde{\varphi}_{uf}(\widetilde{0}_E)=\widetilde{0}_K$.

Definition 5.([38]) *Let $\widetilde{\varphi}_{uf} : FSS(X,\delta,E) \to FSS(Y,\eta,K)$ be a FS-map, then $\widetilde{\varphi}_{uf}$ is said to be:*

1) fuzzy soft continuous (FSC) if $\widetilde{\varphi}_{uf}^{-1}(g_K) \in \delta$ $\forall g_K \in \eta$,
2) fuzzy soft open (FS-open)if $\widetilde{\varphi}_{uf}(h_E) \in \eta$ $\forall h_E \in \delta$,
3) fuzzy soft closed (FS-closed)if $\widetilde{\varphi}_{uf}(h_E) \in \eta^c$ $\forall h_E \in \delta^c$.

Definition 6.([29]) *Let $A \subset X$ and $F_E \in SS(X,E)$, then:*
i) A soft characteristic function $\widetilde{\chi}_A$ of A is a FS-set defined by $\widetilde{\chi}_A=\{(e,\widetilde{\chi}_A(e)) : \widetilde{\chi}_A(e)=\chi_A\ \forall e \in E\}$.

ii) A soft characteristic function $\widetilde{\chi}_{F_E}$ of F_E is a FS-set defined by $\widetilde{\chi}_{F_E}=\{(e,\widetilde{\chi}_{F_E}(e)) : \widetilde{\chi}_{F_E}(e)=\chi_{F(e)}\ \forall e \in E\}$.

Theorem 2.([29]) *(1) For any topological space (X,τ), the collection $\delta_\tau = \{\widetilde{\chi}_A : A \in \tau\}$ defines a FS-topology on X generated by τ. In this case (X,δ_τ,E) call a g-FST space.*

(2) Every FST-space (X,δ,E) defines a topology on X in the form $\tau_\delta = \{A \subseteq X : \widetilde{\chi}_A \in \tau\}$ which is generated by δ.

Theorem 3.([29]) *(1) For a STS (X,τ^*,E), the collection $\delta_\Delta = \{\widetilde{\chi}_{F_E} : F_E \in \tau^*\}$ defines a FS-topology on X generated by τ^*.*

(2) For a FSTS (X,δ,E) the collection $\tau^*_\Delta = \{F_E : \widetilde{\chi}_{F_E} \in \tau^*\}$ defines a S-topology on X which is generated by δ.

2 Some properties and relations

In the next, we give some results, properties, and the relations between the continuity in FSTs and that of some induced STs are studied.

Definition 7.*Let $\alpha \in I$. The FS-constant set $\widetilde{\alpha}_E$ on X is a FS-set given by the form $\widetilde{\alpha}_E = \{(e,\widetilde{\alpha}_E(e)) : \widetilde{\alpha}_E(e) = \underline{\alpha}(e)\ \forall e \in E, \underline{\alpha}(e)(x) = \alpha\ \forall x \in X\}$ where $\underline{\alpha}(e)$ is the constant fuzzy set.*
A FSTS (X,δ,E) is said to be fully stratified iff $\widetilde{\alpha}_E$ for all $\alpha \in I$.

Definition 8.*Let $f_E \in FSS(X,E)$, then the soft support $\widetilde{S}(f_E)$ of f_E is a S-set given by $\widetilde{S}(f_E) = \{(e,\widetilde{S}(f_E)(e)) : \widetilde{S}(f_E)(e) = S(f(e))\ \forall e \in E, f(e) \in I^X\}$ where $S(f(e))$ is the support of $f(e)$ given by $S(f(e)) = \{x \in X : f(e)(x) > 0\}$.*

Lemma 1.*Let f_E and $g_E \in FSS(X,E)$ and $\{f_{iE} : i \in J\} \subseteq FSS(X,E)$, we have:*

i) $\widetilde{S}(\widetilde{1}_E) = X_E$ and $\widetilde{S}(\widetilde{0}_E) = \widetilde{0}_E$,
ii) $f_E \sqsubseteq g_E \Longrightarrow \widetilde{S}(f_E) \sqsubseteq \widetilde{S}(g_E)$,
iii) $(\widetilde{\cup}_{i\in J}\widetilde{S}(f_{iE}) = \widetilde{S}(\sqcup_{i\in J}(f_{iE})$,
iv) $\widetilde{S}(f_E)\widetilde{\cap}\widetilde{S}(g_E) = \widetilde{S}(f_E \sqcap g_E)$.

Proof. It fellows directly from the above Definition.

By using the above lemma, one can check the following Proposition that show how construct FST from ST and vice versa.

Proposition 1.*(i) For a STS (X,τ^*,E), the collection: $\delta_{\tau^*} = \{f_E : \widetilde{S}(f_E) \in \tau^*\}$ defines a FST on X which is generated by τ^*.*
(ii)For a FSTS (X,δ,E) the collection $\tau^*_\delta = \{\widetilde{S}(f_E) : f_E \in \delta\}$ defines a ST on X which is generated by δ.

Lemma 2.*Let $f : X \to Y$ and $u : E \to K$ are two maps, $\widetilde{\varphi}_{uf} : FSS(X,E) \to FSS(Y,K)$ be a FS-map, $\varphi_{uf} : SS(X,E) \to SS(Y,K)$ be a S-map, and $B \subset Y, G_K \in SS(Y,K)$, then:*
1) $f^{-1}(\chi_B) = \chi_{f^{-1}(B)}$,
2)$\widetilde{\varphi}^{-1}_{uf}(\widetilde{\chi}_B) = \widetilde{\chi}_{f^{-1}(B)}$,
3) $\widetilde{\varphi}^{-1}_{uf}(\widetilde{\chi}_{G_K}) = \widetilde{\chi}_(\varphi^{-1}_{uf}(G_K))$,
4) $\varphi^{-1}_{uf}(\widetilde{S}(g_K)) = \widetilde{S}(\widetilde{\varphi}^{-1}_{uf}(g_K))$.

Proof. It is obvious.

Theorem 4.*(1) Let $\widetilde{\varphi}_{uf} : (X,\delta,E) \to (Y,\eta,K)$ be a FSC-map, then $f : (X,\tau_\delta) \to (Y,\tau_\eta)$ is a continuous map.*

(2) Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous map, then $\widetilde{\varphi}_{uf} : (X,\delta_\tau,E) \to (Y,\delta_\sigma,K)$ is a FSC- map.

Proof. For the case (1). Let $\widetilde{\varphi}_{uf} : (X,\delta,E) \to (Y,\eta,K)$ be a FSC-map and let $B \in \tau_\eta$, then $\widetilde{\chi}_B \in \eta$. By hypothesis and from (2) of the above lemma $\widetilde{\varphi}^{-1}_{uf}(\widetilde{\chi}_B) = \widetilde{\chi}_{f^{-1}(B)}$, then by (2) of Theorem (2) $f^{-1}(B) \in \tau_\sigma$. The result holds.
The proof of the case (2) is obtained from the Lemma (2) and Theorem (2).

Theorem 5.*(1) Let $\widetilde{\varphi}_{uf} : (X,\delta,E) \to (Y,\eta,K)$ be a FSC-map, then $\varphi_{uf} : (X,\tau^*_\Delta,E) \to (Y,\sigma^*_\Delta,K)$ is a SC-map.*

(2) Let $\varphi_{uf} : (X,\tau^*,E) \to (Y,\sigma^*,K)$ be a SC-map, then $\widetilde{\varphi}_{uf} : (X,\delta_\Delta,E) \to (Y,\eta_\Delta,K)$ is a FSC-map.

Proof. Follows from (3) of Lemma (2) and Theorem (3).

Theorem 6.*(1) Let $\widetilde{\varphi}_{uf} : (X,\delta,E) \to (Y,\eta,K)$ be a FSC-map, then $\varphi_{uf} : (X,\tau^*_\delta,E) \to (Y,\tau^*_\eta,K)$ is a SC-map.*

(2) Let $\varphi_{uf} : (X,\tau^*,E) \to (Y,\sigma^*,K)$ be a SC-map, then $\widetilde{\varphi}_{uf} : (X,\delta_{\tau^*},E) \to (Y,\delta_{\sigma^*},K)$ is a FSC-map.

Proof. Follows from (4) of Lemma(2) and Proposition(1).

3 One Category of FST-spaces

We know that the FSTSs and FSC-maps between them forms category, denoted by ***FST***, the topological spaces with continuous maps between them forms category, denoted by ***Top***, and the STSs with SC-maps between them forms category denoted by ***STop***.

In the next theorem, for the categories ***Top*** and ***FST***, we define the following two functors.

Theorem 7.*For the categories **Top** and **FST**, we define:*
*i) **F:FST** $\to$ **Top** as follows $\forall(X,\delta,E) \in$ **FST**, $\boldsymbol{F}(X,\delta,E) = (X,F(\delta))$ and $\boldsymbol{F}(\widetilde{\varphi}_{uf}) = f\ \forall\widetilde{\varphi}_{uf}$, where $\boldsymbol{F}(\delta) = \tau_\delta = \{A \subset X : \widetilde{\chi}_A \in \delta\}$,*

*ii) **D** : **Top** $\to$ **FST** as follows*
*$\forall(X,\tau) \in$ **Top** ,$\boldsymbol{D}(X,\tau) = (X,D(\tau),E)$ and*

$\boldsymbol{D}(f) = \widetilde{\varphi}_{uf}$ $\forall f$, *where* $\boldsymbol{D}(\tau) = \delta_\tau = \{\widetilde{\chi}_A \in FSS(X,E) : A \in \tau\}$. *Then* $\boldsymbol{F}$ *and* $\boldsymbol{D}$ *are functors.*

Proof. The proof of case i)follows from(2) of Theorem (2) and (1) of Theorem (4) and in similar the proof of the case ii) follows from (1) of Theorem (2) and (2) of Theorem (4).

Definition 9. *For the category* $\boldsymbol{FST}$, *we define two subcategories as follows:*
1) $\delta_\tau - \boldsymbol{FST}$*:full subcategory of* $\boldsymbol{FST}$ *whose objects are all FSTSs in the form* (X,δ_τ,E) *where* $\delta_\tau = \{\widetilde{\chi}_A \in FSS(X,E) : A \in \tau\}, (X,\tau) \in \boldsymbol{Top}$ *and whose morphisms are all FSC-maps.*
2) $\widetilde{\chi} - \boldsymbol{FST}$*:full subcategory of* $\boldsymbol{FST}$ *whose objects are all FSTSs in the form* $(X,\widetilde{\chi}_\delta,E)$ *where* $\widetilde{\chi}_\delta = \{\widetilde{\chi}_A \in \delta : A \subset X\}$ *for any* $(X,\delta,E) \in \boldsymbol{FST}$ *and whose morphisms are all FSC-maps.*

Theorem 8. *Let the restriction* $\boldsymbol{F}^* : \delta_\tau - \boldsymbol{FST} \to \boldsymbol{Top}$ *of the functor* $\boldsymbol{F} : \boldsymbol{FST} \to \boldsymbol{Top}$, *then the functor* $\boldsymbol{D} : \boldsymbol{Top} \to \boldsymbol{FST}$ *is a left adjoint to the functor* $\boldsymbol{F}$.

Proof. By using universal property, for any $(X,\tau) \in \boldsymbol{Top}$ and a continuous map $I_X : (X,\tau) \to \boldsymbol{FD}(X,\tau) = (X,\tau)$. To show that I_X is an $\boldsymbol{F}$-universal map. Consider the restriction $\boldsymbol{F}^* : \delta_\tau - \boldsymbol{FST} \to \boldsymbol{Top}$ of $\boldsymbol{D}$ to $\delta_\tau - \boldsymbol{FST}$ and a continuous map $f : (X,\tau) \to \boldsymbol{F}^*(Y,\delta^*_{\tau^*},K) = (Y,\boldsymbol{F}^*(\delta^*_{\tau^*}))$ $\forall (Y,\delta^*_{\tau^*},K) \in \delta_\tau - \boldsymbol{FST}$. Now we need only to check that $\widetilde{\varphi}_{uf} : \boldsymbol{D}(X,\tau) = (X,\boldsymbol{D}(\tau),E) \to (Y,\delta^*_{\tau^*},K)$ is a FSC-map. So let $\widetilde{\chi}_B \in \delta^*_{\tau^*} \Longrightarrow B \in \tau^* = \boldsymbol{F}^*(\delta^*_{\tau^*})$. Since $f : (X,\tau) \to \boldsymbol{F}^*(Y,\delta^*_{\tau^*},K) = (Y,\boldsymbol{F}^*(\delta^*_{\tau^*}))$ is continuous, then $f^{-1}(B) \in \tau \Longrightarrow \widetilde{\chi}_{f^{-1}(B)} = f^{-1}(\widetilde{\chi}_B) \in \delta_\tau = \boldsymbol{D}(\tau)$. Therefore I_X is an $\boldsymbol{F}$-universal map for (X,τ) in $\boldsymbol{Top}$. So the result holds.

Theorem 9. *The categories* $\boldsymbol{Top}$ *and* $\delta_\tau - \boldsymbol{FST}$ *are isomorphic.*

Proof. First, let us define $\boldsymbol{P} : \boldsymbol{Top} \to \delta_\tau - \boldsymbol{FST}$ as follows: $\boldsymbol{P}(X,\tau) = (X,\delta_\tau,E)$ and $\boldsymbol{P}(f) = \widetilde{\varphi}_{uf}$ $\forall (X,\tau) \in \boldsymbol{Top}$ with a continuous map f. Consider the restriction $\boldsymbol{F}^* : \delta_\tau - \boldsymbol{FST} \to \boldsymbol{Top}$ of the functor $\boldsymbol{D}$, then $\boldsymbol{P}, \boldsymbol{F}^*$ are functors and so, the result follows from the fact that $\boldsymbol{F}^*\boldsymbol{P}(X,\tau) = (X,\tau)$ for any $(X,\tau) \in \boldsymbol{Top}$ and $\boldsymbol{PF}^*(X,\delta.E) = (X,\delta,E)$ for any $(X,\delta,E) \in \delta_\tau - \boldsymbol{FST}$.

Theorem 10. *The categories* $\boldsymbol{Top}$ *and* $\widetilde{\chi} - \boldsymbol{FST}$ *are isomorphic.*

Proof. The proof is similar of that in the above theorem.

Now from Theorem (9) and Theorem (10) we get the following result.

Corollary 1. *The categories* $\boldsymbol{Top}$, $\widetilde{\chi} - \boldsymbol{FST}$ *and* $\delta_\tau - \boldsymbol{FST}$ *are isomorphic.*

Theorem 11. *The category* $\widetilde{\chi} - \boldsymbol{FST}$ *is a bireflective full subcategory of the category* $\boldsymbol{FST}$.

Proof. It is clear that $\widetilde{\chi} - \boldsymbol{FST}$ is a full subcategory of $\boldsymbol{FST}$. Let $(X,\delta,E) \in \boldsymbol{FST}$ and let $\delta^* = \{\widetilde{\chi}_A \in \delta : A \subset X\}$, then we can verify that $(X,\delta^*,E) \in \widetilde{\chi} - \boldsymbol{FST}$ and $I_X : (X,\delta,E) \to (X,\delta^*,E)$ is a FSC-map. Now to show that I_X is a bireflection of (X,δ,E) in $\widetilde{\chi} - \boldsymbol{FST}$. Consider $(Y,\eta,K) \in \widetilde{\chi} - \boldsymbol{FST}$ and a FSC-map $\widetilde{\varphi}_{fu} : (X,\delta,E) \to (Y,\eta,K)$. We need only to check that $\widetilde{\varphi}_{fu} : (X,\delta^*,E) \to (Y,\eta,K)$ is a FSC-map. So let $h_K \in \eta$. Since $(Y,\eta,K) \in \widetilde{\chi} - \boldsymbol{FST}$, then $h_K = \widetilde{\chi}_B$, also $\widetilde{\varphi}_{fu} : (X,\delta,E) \to (Y,\eta,K)$ is a FSC-map, then $\widetilde{\varphi}^{-1}_{uf}(h_K) = \widetilde{\varphi}^{-1}_{uf}(\widetilde{\chi}_B) = \widetilde{\chi}_{f^{-1}(B)} \in \delta$ and so, $\widetilde{\chi}_{f^{-1}(B)} \in \delta^*$. Therefore $\widetilde{\varphi}_{fu} : (X,\delta,E) \to (Y,\eta,K)$ is a FSC-map. Hence the result holds.

From the above theorem and Corollary (1) we get the next result.

Corollary 2. *The categories* $\boldsymbol{Top}$ *and* $\delta_\tau - \boldsymbol{FST}$ *are bireflective full subcategories of the category* $\boldsymbol{FST}$.

Definition 10. *For any* $(X,\delta,E) \in \boldsymbol{FST}$ *and any FSC-map* $\widetilde{\varphi}_{fu}$ *such that* $\boldsymbol{G}(X,\delta,E) = (X,\boldsymbol{G}(\delta),E)$ *and* $\boldsymbol{G}(\widetilde{\varphi}_{fu}) = \widetilde{\varphi}_{fu}$, *where* $\boldsymbol{G}(\delta)$ *is a FST on X generated by the subbase* $\delta \cup \{\widetilde{\alpha}_E : \alpha \in I\}$. *We define the full subcategory* $\widetilde{\alpha} - \boldsymbol{FST}$ *of* $\boldsymbol{FST}$ *whose objects are all FSTSs in the form* $(X,\boldsymbol{G}(\delta),E)$ *with morphisms are all FSC-maps.*

From the above definition, one can show the following theorem.

Theorem 12. *For the categories* $\widetilde{\alpha} - \boldsymbol{FST}$ *and* $\boldsymbol{FST}$, *we have:*
1) $\boldsymbol{G} : \boldsymbol{FST} \to \widetilde{\alpha} - \boldsymbol{FST}$ *is functor,*
2) $\boldsymbol{R} : \widetilde{\alpha} - \boldsymbol{FST} \to \boldsymbol{FST}$ *is an inclusion functor.*

Theorem 13. *The category* $\widetilde{\alpha} - \boldsymbol{FST}$ *is a bicoreflective full subcategory of the category* $\boldsymbol{FST}$.

Proof. Clearly $\widetilde{\alpha} - \boldsymbol{FST}$ is a full subcategory of $\boldsymbol{FST}$. Let $(X,\delta,E) \in FST$ and $\delta^* = G(\delta)$ which is defined as in Definition (10), then we can verify that $(X,\delta^*,E) \in \widetilde{\alpha} - \boldsymbol{FST}$ and $I_X : (X,\delta^*,E) \to (X,\delta,E)$ is a bicoreflection of (X,δ,E) with respect to $\widetilde{\alpha} - \boldsymbol{FST}$. Hence $\widetilde{\alpha} - \boldsymbol{FST}$ is a bicoreflective in $\boldsymbol{FST}$.

In the following, we study the relationship between the categories $\boldsymbol{FST}$ and $\boldsymbol{STop}$, for this reason we define the following functors.

Theorem 14. *For the categories* $\boldsymbol{STop}$ *and* $\boldsymbol{FST}$, *we define:*
(1) $\boldsymbol{F}_1 : \boldsymbol{FST} \to \boldsymbol{STop}$ *as follows. For any* (X,δ,E) *in* $\boldsymbol{FST}$, $\boldsymbol{F}_1(X,\delta,E) = (X,\boldsymbol{F}_1(\delta),E)$ *and* $\boldsymbol{F}_1(\widetilde{\varphi}_{fu}) = \varphi_{uf}$ $\forall \widetilde{\varphi}_{fu}$ *in* $\boldsymbol{FST}$ *where,* $\boldsymbol{F}_1(\delta) = \tau^*_\Delta = \{F_E \in SS(X,E) : \widetilde{\chi}_{F_E} \in \delta\}$,

(2) $\boldsymbol{D}_1 : \boldsymbol{STop} \to \boldsymbol{FST}$ *as follows,*
for any $(X,\delta,E) \in \boldsymbol{STop}, \boldsymbol{D}_1(X,\delta,E) = (X,\boldsymbol{D}_1(\delta),E)$ *and* $\boldsymbol{D}_1(\varphi_{uf}) = \widetilde{\varphi}_{fu}\ \forall \varphi_{uf}$ *in* $\boldsymbol{STop}$ *where,* $\boldsymbol{D}_1(\tau) = \delta_\Delta = \{\widetilde{\chi}_{F_E} \in FSS(X,E) : F_E \in \tau\}$. *Then* $\boldsymbol{F}_1$ *and* $\boldsymbol{D}_1$ *are functors.*

Proof. The proof for the cases (1), (2) follows from the Theorems (3) and Theorem (5).

In similar, by using Proposition (1) and Theorem (6) one can show the following theorem.

Theorem 15.*For the categories* $\boldsymbol{STop}$ *and* $\boldsymbol{FST}$*, we define:*
(1) $\boldsymbol{F}_2 : \boldsymbol{FST} \to \boldsymbol{STop}$ *as follows,*
for any $(X,\delta,E) \in \boldsymbol{STop}, \boldsymbol{F}_2(X,\delta,E) = (X,\boldsymbol{F}_2(\delta),E)$ *and* $\boldsymbol{F}_2(\widetilde{\varphi}_{uf}) = \varphi_{uf}\ \forall \widetilde{\varphi}_{uf}$ *in* $\boldsymbol{FST}$ *where* $\boldsymbol{F}_2(\delta) = \delta^*_\tau = \{\widetilde{S}(f_E) \in SS(X,E) : f_E \in \delta\}$,

(2) $\boldsymbol{D}_2 : \boldsymbol{STop} \to \boldsymbol{FST}$ *as follows,*
for any $(X,\tau,E) \in \boldsymbol{STop}, \boldsymbol{D}_2(X,\tau,E) = (X,\boldsymbol{D}_2(\tau),E)$ *and* $\boldsymbol{D}_2(\varphi_{uf}) = \widetilde{\varphi}_{uf}\ \forall \varphi_{uf}$ *in* $\boldsymbol{STop}$*, where* $\boldsymbol{D}_2(\tau) = \delta^*_\tau = \{f_E \in FSS(X,E) : \widetilde{S}(f_E) \in \tau\}$. *Then* $\boldsymbol{F}_2$ *and* $\boldsymbol{D}_2$ *are functors.*

Definition 11.*For the category* $\boldsymbol{FST}$ *we define the subcategories as:*
(1) $\delta_\Delta - \boldsymbol{FST}$*:full subcategory of* $\boldsymbol{FST}$ *whose objects are all FSTSs in the form* (X,δ_Δ,E) *where,* $\delta_\Delta = \{\widetilde{\chi}_{F_E} \in FSS(X,E) : F_E \in \tau\}, (X,\tau,E) \in \boldsymbol{STop}$ *and whose morphisms are all FSC-maps.*

(2) $[\delta] - \boldsymbol{FST}$*:full subcategory of* $\boldsymbol{FST}$ *whose objects are all FSTSs in the form* $(X,[\delta],E)$ *where,* $[\delta] = \{\widetilde{\chi}_{F_E} \in \delta : F_E \in SS(X,E)\}$*, for any* $(X,\delta,E) \in \boldsymbol{FST}$ *and whose morphisms are all FSC-maps.*

(3) $\delta^*_\tau - \boldsymbol{FST}$*:full subcategory of* $\boldsymbol{FST}$ *whose objects are all FSTSs in the form* (X,δ^*_τ,E) *where* $\delta^*_\tau = \{f_E \in FSS(X,E) : \widetilde{S}(f_E) \in \tau\}$*, for any* $(X,\tau,E) \in \boldsymbol{STop}$ *and whose morphisms are all FSC-maps.*

Theorem 16.*Let the restriction* $\boldsymbol{F}^*_1 : \delta_\Delta - \boldsymbol{FST} \to STop$ *of the functor*
$\boldsymbol{F}_1 : \boldsymbol{FST} \to \boldsymbol{STop}$*, then the functor*
$\boldsymbol{D}_1 : \boldsymbol{STop} \to \boldsymbol{FST}$ *is a left adjoint to the functor* $\boldsymbol{F}_1$.

Proof. The proof is analogous to that of the Theorem (8).

Theorem 17.*The categories* $\boldsymbol{STop}$ *and* $\delta_\Delta - \boldsymbol{FST}$ *are isomorphic.*

Proof. The proof follows from the fact that, for any $(X,\tau,E) \in \boldsymbol{STop}, \boldsymbol{F}^*_1\boldsymbol{D}_1(X,\tau,E) = (X,\tau,E)$ and $D_1F^*_1(X,\delta,E) = (X,\delta,E)$ for any $(X,\delta,E) \in \delta_\Delta - FST$, where $\boldsymbol{F}^*_1 : \delta_\Delta - FST \to \boldsymbol{STop}$ is the restriction of the functor $\boldsymbol{F}_1$.

Finally, the applications of the above results can be found in different directions, where the calculations contain uncertainties and nonclassical treatments [40]-[47]. A number of theories have been proposed for dealing with uncertainties in an efficient way like theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets [8].

4 Conclusion

In this work, we have presented a categorical framework for fuzzy soft topological spaces $\boldsymbol{FST}$ and defined some of their subcategories such as $\boldsymbol{Top}$, $\boldsymbol{STop}$, $\delta_\tau - \boldsymbol{FST}, \widetilde{\chi} - \boldsymbol{FST}$, and $\widetilde{\alpha} - \boldsymbol{FST}$. Also, we have constructed some functors between them and revealed some results and relations of FS-maps. Moreover, the relationships between $\boldsymbol{FST}$ and their subcategories have been studied.

As a future work, we extend this work to some generalizations of fuzzy soft topological spaces $\boldsymbol{FST}$ such as supra fuzzy soft and infra fuzzy soft structures.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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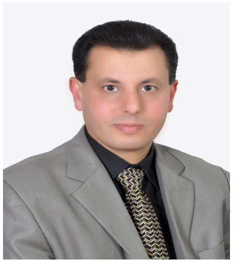
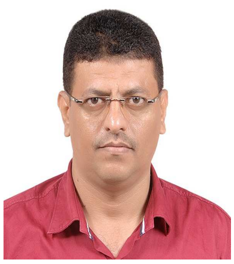
S. Saleh is Professor of pure Mathematics at Hodeidah University-Yemen, currently, he is a researcher fellow at Cihan University-Erbil, Iraq. His research interests are in the areas of pure and applied mathematics including Topology, Fuzzy topology, Fuzzy Mathematics, soft and fuzzy soft theory and it's applications. He has published research articles in reputed international journals of mathematical sciences. He is referee and editor of some mathematical journals.



Radwan Abu-Gdairi is an assistant professor of Mathematics at Department of Mathematics, Faculty of Science, Zarqa University, Jordan. His research interests are in the areas of pure and applied mathematics including Topology, Fuzzy topology, Rough set theory and it's applications. He has published research articles in reputed international journals of mathematical sciences. He is referee of some mathematical journals.

Tareq M. Al-shami is an assistant professor of Mathematics at Sana'a University, Yemen. He received his M. SC. degree and Ph.D. degree from the Faculty of Science, Mansoura University in 2016 and 2021, respectively. He has authored/co-authored over 123 scientific papers in top-ranked international journals and conference proceedings. He is referee and editor of several international journals in the frame of pure and applied mathematics. His research interests include general topology and its extensions, Menger spaces, soft set theory, rough set theory, Fuzzy set theory and their applications to decision-making problems. He received Obada-Prize for postgraduate students in Feb 2019.



Mohammed S. Abdo is an Assistant Professor of pure Mathematics, Hodeidah University, Yemen. He recently completed his Ph.D., in Mathematics, from Dr. BAMU University, India. He does research in Nonlinear Analysis in Applied Mathematics and Mathematical Modeling. His current project is "Theory of Fractional Differential Equations and Fractional Calculus". He has a particular interest in mathematical models describing biological and medical phenomena.