Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160603

Numerical Solution Via a Singular Mixed Integral Equation in (2+1) Dimensional

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Received: 12 Jul. 2022, Revised: 2 Sep. 2022, Accepted: 18 Sep. 2022
Published online: 1 Nov. 2022

Abstract: In this paper, under certain conditions, the unique solution of a mixed integral equation (MIE) with a singular kernel in position and a continuous kernel in time, in $(2+1)$ dimensional is discussed and obtained in the space $L_2([a,b]\times[c,d])\times C[0,T], T<1$. After using a separation technique method, and Product Nystrom Method (PNM), we have a linear algebraic system (LAS) in two-dimensional with time coefficients. The convergence of the unique solution of the LAS is studied. In the end, and with the aid of Maple 18, many applications when a singular term of position kernel takes a logarithmic form and Carleman function are solved numerically. Moreover, the error is computed.

Keywords: Mixed integral equation, Fredholm integral equation, linear algebraic system, singular kernel, product Nystrom method, logarithmic kernel, Carleman function

1 Introduction

Integral equations of all kinds play a prominent role in solving many scientific problems in different sciences. For example, in mathematical physics, many natural phenomena have been explained by finding the spectral relationships of some singular integral equations, in different domains, see Abdou et al.,[1-3]. In two-dimensional problems, in thermoelasticity, for an elastic plate weakened by curvilinear holes, the problem can be turned into a kind of singular integro differential equations that can be solved using the integral equations method ,see (Ismail1 et al., [4], Hamza et al., Abdou et al., [6]). In two- dimensional problems, in quantum mechanics, see Rehab et al., [7, 8], and in contact and mixed mechanics problems, see ElBorai et al., [9, 10]. The numerical methods of integral equations play an important role in solving these kinds. More applications for using integral equations in some different sciences can be found in Popov[11], Rahman [12]and Alhazmi [14]. The tremendous development in the various sciences led the researchers of integral equations to search for analytical and numerical methods in one, two, or more dimensions to establish different solutions. Here, we mention some of these works. For example, Mirzaee et al., in [15, 16] used the collocation method to solve some two-dimensional MIE, in [15] and integral Volterra-Fredholm equations with continuous kernels in [16], respectively. The Toeplitz matrix method was used to discuss a solution to a nonlinear integral equation when the singular kernel takes different forms, see Abdou et al., [17, 18]. One of the famous methods is the orthogonal polynomial method, in which the researcher uses special functions and derives the required solution in the form of the convergent algebraic system; see Mirzaee et al., [19], Abdou et al., [20, 21]and Al-Bugami [22]. Also, the famous numerical methods that researchers have used to solve integral equations when the kernel is continuous, are the Adomian decomposition method and homotopy perturbation method, see Almousa et al., [23], Abdou et al., [24], Elzaki et al., [25] and Berenguer et al., [26]. Moreover, besides using these numerical methods, the researchers were interested in studying the error resulting from the use of these methods, as well as the error resulting from computer programs, for example, see Hetmaniok et al., [27, 28] and Abdou et al., [29, 30].

The importance of this research is evident in obtaining a single solution to a mixed integral equation in the space $L_2([a,b]\times[c,d])\times C[0,T], (T<1)$. It has been assumed that the kernel of position has a singularity. The researcher was able, under certain conditions, to prove the existence of a unique solution. Using Product Nystrom

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Method, it is possible to obtain an algebraic system for which a single solution has been studied. The convergence of this algebraic system was also studied. Using the mathematical programs Maple 18, it was possible to obtain numerical solutions for the algebraic system when the kernel of the position is in the form of a logarithmic form or Carleman function.

Consider in the space $L_2([a,b]\times[c,d])\times C[0,T],(T<1)$, the (2+1) DMIE,

$$\mu\Phi(x,y;t)=F(x,y;t)$$
$$+\lambda\int_0^t\int_c^d\int_a^b g(t,\tau)k(x-\zeta,y-\eta)\Phi(\zeta,\eta;\tau)d\zeta d\eta d\tau, \quad (1)$$

under the condition

$$\int_c^d\int_a^b\Phi(x,y,t)dxdy=M(t),t\in[0,T] \quad (2)$$

where $F(x,y;t)$ is a known function, μ is constant defines the kind of **IE**, λ is constant, may be complex and have physical meaning, $g(t,\tau)$ is a function of time and represents the kernel of Volterra integral term, while $k(x-\zeta,y-\eta)$ is the position kernel and have two singularities and $\Phi(x,y;t)$ is the unknown function.

In general, many different methods can be used to prove the existence of a unique solution of the integral equation with continuous or discontinuous kernel, see Abdou et al., [17,18,21].

2 The delayed Dalgaard and Strulik model

To discuss the existence of a unique solution of Eq. (1), in view of Banach fixed point theorem, we write (1) in the integral operator form

$$\bar{W}\Phi(x,y;t)=\frac{1}{\mu}F(x,y;t)+\frac{\lambda}{\mu}W\Phi(x,y;t), \quad (3)$$

$W\Phi(x,y;t)=\int_0^t\int_c^d\int_a^b g(t,\tau)k(x-\zeta,y-\eta)\Phi(\zeta,\eta;\tau)d\zeta d\eta d\tau.$
Then, assume the following conditions

(i) In $L_2([a,b]\times[c,d])$, the kernel of position satisfies

$$\left\{\int_c^d\int_c^d\left\{\int_a^b\int_a^b k^2(x-u,y-v)dxdu\right\}dydv\right\}^{\frac{1}{2}}\le C,$$

(C is constant).
(ii) The kernel of time $g(t,\tau)\in C[0,T]$ satisfies $|g(t,\tau)|\le M$, M is a constant. $\forall t\in[0,T],T<1$.

(iii) The function $F(x,y;t)$, with its partial derivatives with respect to x,y and t are continuous in $L_2([a,b]\times[a,b])\times C[0,T],T<1$, and for constant G, its norm is

$$\|F(x,y;t)\|_{L_2([a,b]\times[c,d])\times C[0,T]}=$$
$$\max_{0=t\le T<1}\int_0^t\left\{\int_c^d\int_a^b F(x,y;\tau)^2dxdy\right\}^{\frac{1}{2}}d\tau=G.$$

(iv) The unknown function $\Phi(x,y;t)$ behaves in $L_2([a,b]\times[c,d])\times C[0,T]$, as the free function $F(x,y,t)$ and its norm is defined as $\|\Phi(x,y;t)\|=Q$.

Theorem 1. The MIE (1) has an existence and unique solution, under the condition

$$|\lambda|MCT<|\mu|. \quad (4)$$

To prove the existence and uniqueness of the solution for the MIE (1) the following two lemmas must be proven:

Lemma 1. In the space $L_2([a,b]\times[c,d])\times C[0,T],T<1$, and under the conditions (i) $-($ iv), the operator $\bar{W}$ maps the space into itself.

Proof: In the light of (3), after using the conditions (ii) and (iii), and then applying Hölder inequality, we have

$$\|\bar{W}\Phi(x,y,t)\|\le\frac{G}{|\mu|}$$
$$+\frac{|\lambda|}{|\mu|}M\cdot\max_{0\le t\le T}|\int_c^d\int_c^d\left(\int_a^b\int_a^b|k(x-u,y-v)|^2dxdu\right)dydv|$$
$$\times\max_{0\le t\le T}\int_0^t\left\{\int_c^d\int_a^b|\Phi(\zeta,\eta,\tau)|^2d\zeta d\eta\right\}^{\frac{1}{2}}d\tau\|.$$

Then, in the light of the conditions (i) and (iv), we obtain

$$\|\bar{W}\Phi(x,y,t)\|\le\frac{G}{\|\mu\|}+\sigma\|\Phi(x,y,t)\|,$$
$$\left(\sigma=\left|\frac{\lambda}{\|\mu}\right|MCT\right). \quad (5)$$

From inequality (5) we deduce that the operator $\bar{W}$ maps the ball S_ρ into itself, where

$$\rho=\frac{G}{[|\mu|-|\lambda|MCT]}. \quad (6)$$

Since $\rho>0$, therefore we get $\sigma<1$. Moreover, the inequality (5) includes the limitation of the integral operator W, where

$$\|W\Phi(x,y;t)\|=\|W\|\|\Phi(x,y;t)\|\le\sigma\|\Phi(x,y;t)\|. \quad (7)$$

Lemma 2. Under the conditions (i), (ii) and (iv), $\bar{W}$ is a contraction in the space $L_2([a,b]\times[a,b])\times C[0,T]$.
Proof: Assume $\{\Phi_1(x,y;t),\Phi_2(x,y;t)\}\in L_2([a,b]\times[a,b])$ $\times C[0,T]$, then we have

$$\|\bar{W}\Phi_1(x,y;t)-\bar{W}\Phi_2(x,y;t)\|\le\frac{|\lambda|}{|\mu|}\left\|\int_0^t\int_c^d\int_a^b|g(t,\tau)\right.$$
$$\left.\times|k(x-\zeta,y-\eta)|\,|\Phi_1(\zeta,\eta;\tau)-\Phi_2(\zeta,\eta;\tau)|d\zeta d\eta d\tau\right\|.$$

With the aid of conditions (ii) and (iv), the above inequality becomes

$$\|\bar{W}\Phi_1(x,y;t)-\bar{W}\Phi_2(x,y;t)\|\le\frac{|\lambda|}{|\mu|}M\,\|k(x-\zeta,y-\eta)|$$
$$\left.\times\int_0^t\int_c^d\int_a^b|\Phi_1(\zeta,\eta;\tau)-\Phi_2(\zeta,\eta;\tau)|d\zeta d\eta d\tau\right\|.$$

Applying Hölder inequality, and then using condition (i), we finally get

$$\|\bar{W}\Phi_1(x,y;t)-\bar{W}\Phi_2(x,y;t)\|\le\sigma\,\|\Phi_1(x,y;t)-\Phi_2(x,y;t)\|. \quad (8)$$

The inequality (8) leads us to decide that :the integral operator $\bar{W}$ is continuous in the space of integration, and then $\bar{W}$ is a contraction operator, under the condition $\sigma<1$. So, from Banach fixed point theorem, $\bar{W}$ has a unique fixed point which is the unique solution of equation (3).

3 Convergence of solution

Lemma 3. Besides the conditions (i)-(iii), the infinite series $\sum_{i=0}^{\infty}\Psi(x,y;t)$ is uniformly convergent to a continuous solution function $\Phi(x,y;t)$

Proof: We construct the sequence of functions $\Phi_n(x,y;t)$ as

$$r\mu\Phi_n(x,y;t)=F(x,y;t)$$
$$+\lambda\int_0^t\int_c^d\int_a^b g(t,\tau)k(x-\zeta,y-\eta)\Phi_n(\zeta,\eta;\tau)d\zeta d\eta d\tau,$$
$$\{\Phi_0(x,y;t)=F(x,y;t)\}. \qquad (9)$$

For ease of manipulation, introduce

$$\Psi_n(x,y;t)=\Phi_n(x,y;t)-\Phi_{n-1}(x,y;t)$$
$$,\{\Psi_0(x,y;t)=F(x,y;t)\}, \qquad (10)$$

where

$$\Phi_n(x,y;t)=\sum_{i=0}^{n}\Psi_i(x,y;t), n=1,2,\ldots, \qquad (11)$$

Using the properties of the modulus, and then with the aid formula (10), we have

$$\|\Psi_n(x,y,t)\|\le\frac{|\lambda|}{|\mu|}M$$
$$\cdot\max_{0\le t\le T}\left|\int_c^d\int_c^d\left\{\int_a^b\int_a^b k^2(x-\zeta,y-\eta)dxd\zeta\right\}dyd\eta\right|$$
$$\times\|\max_{0\le\le\le T}\int_0^t\left\{\int_c^d\int_a^b\left|\Psi_{n-1}(\zeta,\eta;\tau)\right|^2 d\zeta d\eta\right\}^{\frac{1}{2}}d\tau\|.$$

Hence, we obtain

$$\|\Psi_n(x,y;t)\|_{L_2([a,b]\times[c,d])\times C[0,T]}\le\sigma\|\Psi_{n-1}(x,y,t)\|,$$
$$\left(\sigma=\left|\frac{\lambda}{\mu}\right|MCT\right). \qquad (12)$$

Using the conditions (i), (ii), and mathematical induction method, we get

$$\|\Psi_n(x,y;t)\|_{L_2([a,b]\times[a,b])\times C[0,T]}\le\alpha^n M,\left(\alpha=\left|\frac{\lambda}{\mu}\right|CT\right). \qquad (13)$$

This bound makes the sequence $\{\Psi_n(x,y;t)\}$ converges and then, the sequence $\{\Phi_n(x,y;t)\}$ converges. Hence, the infinite series

$$\Phi(x,y;t)=\sum_{i=0}^{\infty}\Psi_i(x,y;t),\forall t\in[0,T],$$

is uniformly convergent since the terms $\Psi_i(x,y;t)$ are dominated by α^i

4 Method of solution:

Assume the unknown and known functions, respectively take the forms

$$\Phi(x,y;t)=\varphi(x,y)A(t),\quad F(x,y;t)=f(x,y)A(t). \qquad (14)$$

Hence, the formula (1) yields,

$$\mu\varphi(x,y)=f(x,y)+\lambda(t)\int_c^d\int_a^b k(x-\zeta,y-\eta)\varphi(\zeta,\eta)d\zeta d\eta,$$
$$\left(\lambda(t)=\frac{1}{A(t)}\int_0^t g(t,\tau)A(\tau)d\tau;A(0)\neq0\right). \qquad (15)$$

Through the separation method, we were able to obtain directly, Fredholm integral equation in two-dimensional with coefficients time-related. With a scientific view of these coefficients, we find that they have become a time function of a time integral operator that can be explicitly calculated at any time points.

4.1 The numerical method for solving T-DFIE

In this section, the numerical solution of the T-DFIE of the second kind of equation (15) will be discussed using the PNM, see Delves [31] and Atkinson [32]. For this aim, consider the position kernel $k(x-\zeta,y-\eta)$ of (10) has two singularities within the range of integration. i.e, we can often factor out the singularity in $k(x-\zeta,y-\eta)$ by writing

$$k(x-\zeta,y-\eta)=\bar{k}(x,\zeta;y,\eta)p(x-\zeta,y-\eta). \qquad (16)$$

Here, $p(x-\zeta,y-\eta)$ and $\bar{k}(x,\zeta;y,\eta)$ are badly behaved and well behaved functions, respectively. Therefore, rewrite (15) to take the form

$$\mu\varphi(x,y)=f(x,y)$$
$$+\lambda(t)\int_c^d\int_a^b\bar{k}(x,\zeta;y,\eta)p(x-\zeta,y-\eta)\varphi(\zeta,\eta)d\zeta d\eta, \qquad (17)$$

Let $x=x_i, x_i=\zeta_i=a+ih,(0\le i\le J), h=\frac{b-a}{J}$, and $y=y_n=\eta_n=c+nh',(0\le n\le M),\ h'=\frac{d-c}{M}$. Then, approximate the integral term in (17), for even values of J and M, by product integration from of Simpson's rule, in the form

$$\int_c^d\int_a^b\bar{k}(x_i,\zeta;y_n,\eta)\,p(x_i-\zeta,y_n-\eta)\,\varphi(\zeta,\eta)d\zeta d\eta$$
$$=\sum_{m=0}^{(m=\frac{M-2}{2})}\sum_{j=0}^{(j=\frac{J-2}{2})}\int_{\eta_{2m}}^{\eta_{2m+2}}\int_{\zeta_{2j}}^{\zeta_{2j+2}}\bar{k}(x_i,\zeta;y_n,\eta)$$
$$\times p(x_i-\zeta,y_n-\eta)\,\varphi(\zeta,\eta)d\zeta d\eta. \qquad (18)$$

After this, approximate the nonsingular part of the integrand over each interval $[\zeta_{2j},\zeta_{2j+2}]$; $[\eta_{2m},\eta_{2m+2}]$ by the second degree Lagrange interpolation polynomial which interpolates it at the points $\zeta_{2j},\zeta_{2j+1},\zeta_{2j+2};\eta_{2m},\eta_{2m+1},\eta_{2m+2}$, to find

$$\int_c^d \int_a^b p(\zeta_i-\zeta;\eta_n-\eta)\bar{k}(\zeta_i,\zeta;\eta_n,\eta)\varphi(\zeta,\eta)d\zeta d\eta = \sum_{m=0}^{(M-2)/2}\sum_{j=0}^{(J-2)/2}\int_{\eta_{2m}}^{\eta_{2m+2}}\int_{\zeta_{2j}}^{\zeta_{2j+2}} p(\zeta_i-\zeta;\eta_n-\eta)$$
$$\left\{\frac{(\zeta_{2j+1}-\zeta)(\zeta_{2j+2}-\zeta)(\eta_{2m+1}-\eta)(\eta_{2m+2}-\eta)}{(2h^2)(2h'^2)}\varphi(\zeta_{2j};\eta_{2m})+\frac{(\zeta_{2j}-\zeta)(\zeta_{2j+2}-\zeta)(\eta_{2m}-\eta)(\eta_{2m+2}-\eta)}{(h^2)(h'^2)}\right.$$
$$\left.\times\varphi(\zeta_{2j+1};\eta_{2m+1})+\frac{(\zeta_{2j}-\zeta)(\zeta_{2j+1}-\zeta)(\eta_{2m}-\eta)(\eta_{2m+1}-\eta)}{(2h^2)(2h'^2)}\varphi(\zeta_{2j+2};\eta_{2m+2})\right\}d\zeta d\eta \tag{19}$$
$$=\sum_{m=0}^{M}\sum_{j=0}^{J} u_{n,m}w_{i,j}\bar{k}(x_i,\zeta_j;y_n,\eta_m)\varphi(\zeta_j,\eta_m),$$

where $\zeta_j = jh, \zeta_{j+1}-\zeta_j = h; \eta_m = mh', \eta_{m+1}-\eta_m = h'$ and the weight functions $u_{n,m}w_{i,j}$ are given by

$$u_{i,0}w_{n,0} = \frac{1}{4h^2h'^2}\int_{\eta_0}^{\eta_2}\int_{\zeta_0}^{\zeta_2} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_1-\zeta)(\zeta_2-\zeta)(\eta_1-\eta)(\eta_2-\eta)d\zeta d\eta$$
$$u_{i,2j+1}w_{n,2m+1} = \frac{1}{h^2h'^2}\int_{\eta_{2m}}^{\eta_{2m+2}}\int_{\zeta_{2j}}^{\zeta_{2j+2}} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{2j}-\zeta)(\zeta_{2j+2}-\zeta)(\eta_{2m}-\eta)(\eta_{2m+2}-\eta)d\zeta d\eta$$
$$u_{i,2j}w_{n,2m} = \frac{1}{4h^2h'^2}\int_{\eta_{2m-2}}^{\eta_{2m}}\int_{\zeta_{2j-2}}^{\zeta_{2j}} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{2j-2}-\zeta)(\zeta_{2j-1}-\zeta)(\eta_{2m-2}-\eta)(\eta_{2m-1}-\eta)d\zeta d\eta \tag{20}$$
$$+\frac{1}{4h^2h'^2}\int_{\eta_{2m}}^{\eta_{2m+2}}\int_{\zeta_{2j}}^{\zeta_{2j+2}} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{2j+1}-\zeta)(\zeta_{2j+2}-\zeta)(\eta_{2m+1}-\eta)(\eta_{2m+2}-\eta)d\zeta d\eta$$
$$u_{i,J}w_{n,M} = \frac{1}{4h^2h'^2}\int_{\eta_{M-2}}^{\eta_M}\int_{\zeta_{J-2}}^{\zeta_J} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{J-2}-\zeta)(\zeta_{J-1}-\zeta)(\eta_{M-2}-\eta)(\eta_{M-1}-\eta)d\zeta d\eta.$$

If we define the following notations

$$U_{j,m}(\zeta_i;\eta_n) = \frac{1}{4h^2h'^2}\int_{\eta_{2m-2}}^{\eta_{2m}}\int_{\zeta_{2j-2}}^{\zeta_{2j}} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{2j-2}-\zeta)(\zeta_{2j-1}-\zeta)(\eta_{2m-2}-\eta)(\eta_{2m-1}-\eta)d\zeta d\eta,$$
$$V_{j,m}(\zeta_i;\eta_n) = \frac{1}{4h^2h'^2}\int_{\eta_{2m-2}}^{\eta_{2m}}\int_{\zeta_{2j-2}}^{\zeta_{2j}} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{2j-1}-\zeta)(\zeta_{2j}-\zeta)(\eta_{2m-1}-\eta)(\eta_{2m}-\eta)d\zeta d\eta, \tag{21}$$
$$W_{j,m}(\zeta_i;\eta_n) = \frac{1}{4h^2h'^2}\int_{\eta_{2m-2}}^{\eta_{2m}}\int_{\zeta_{2j-2}}^{\zeta_{2j}} p(\zeta_i-\zeta;\eta_n-\eta)(\zeta_{2j-2}-\zeta)(\zeta_{2j}-\zeta)(\eta_{2m-2}-\eta)(\eta_{2m}-\eta)d\zeta d\eta,$$

we can rewrite (20) as the following

$$\begin{aligned} &V_{1,1}(\zeta_i;\eta_n) = u_{i,0}w_{n,0}, \quad 4W_{j+1,m+1}(\zeta_i;\eta_n) = u_{i,2j+1}w_{n,2m+1}\\ &U_{j,m}(\zeta_i;\eta_n)+V_{j+1,m+1}(\zeta_i;\eta_n) = u_{i,2j}w_{n,2m}\\ &U_{J/2,M/2}(\zeta_i;\eta_n) = u_{i,J}w_{n,M}.\end{aligned} \tag{22}$$

To rewrite the integral interval of (15) from [0,2], we assume $\zeta = \zeta_{2j-2}+\alpha h$, $(0\le\alpha\le 2)$; $\eta = \eta_{2m-2}+\beta h', (0\le\beta\le 2)$. And then letting $\zeta_i-\zeta_{2j-2} = (i-2j+2)h$; $\eta_n-\eta_{2m-2} = (n-2m+2)h'$. Finally, consider the following integral formula

$$\chi_k = \int_0^2\int_0^2 \alpha^k\beta^k p((z-\alpha)h,(g-\beta)h')\,d\alpha d\beta, \qquad (k=0,1,2; z=i-2j+2; g=n-2m+2). \tag{23}$$

Hence, the system (20) yields

$$\begin{aligned} &u_{i,0}w_{n,0} = \frac{hh'}{4}[2\chi_0(z;g)\ -3\chi_1(z;g)+\chi_2(z;g)], (z=i; g=n),\\ &u_{i,2j+1}w_{n,2m+1} = hh'[2\chi_1(z;g)-\chi_2(z;g)], (z=i-2j; g=n-2m),\\ &u_{i,2j}w_{n,2m} = \frac{hh'}{4}[\chi_2(z;g)-\chi_1(z;g)+2\chi_0(z-2;g-2)\\ &-3\chi_1(z-2;g-2)+\chi_2(z-2;g-2)],\ (z=i-2j+2; g=n-2m+1),\\ &u_{i,J}w_{n,M} = \frac{hh'}{4}[\chi_2(z;g)-\chi_1(z;g)].\end{aligned} \tag{24}$$

Now, in view of (24), rewrite (17) to obtain the following LAS:

$$\mu\varphi(x_i,y_n)-\lambda(t)\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\,\varphi(x_j,y_m)$$
$$=f(x_i,y_n),\forall t\in[0,T],T<1 \tag{25}$$

For all values of time $t\in[0,T],T<1$, the two- dimensional LAS (25) can be written in the following matrix form

$$(\mu-\lambda W\bar{W})H=K,\quad (\lambda=\lambda(t))$$

with the solution

$$\mathrm{H}=[\mu I-\lambda W\bar{W}]^{-1}\,\mathrm{K},\quad \det|\mu I-\lambda W\bar{W}|\neq 0, \tag{26}$$

where I is the identity matrix.

The formula (25), or its equivalent formula (26), represents an approximate solution of the MIE (I) in (2+1) - dimensional in the space of integration $L_2([a,b]\times[c,d])\times C[0,T],T<1$.

5 The existence of a unique solution of the LAS (25).

To prove the existence of a unique solution of (25), we write it in the operator sum form

$$\bar{E}\varphi(x_i,y_n)=E\varphi(x_i,y_n)+\frac{1}{\mu}f(x_i,y_n)$$
$$E\varphi(x_i,y_n)=\frac{\lambda(t)}{\mu}\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\,\varphi(x_j,y_m) \tag{27}$$

where

(1) $\mathrm{Sup}_{i;n}|f(x_i;y_n)|<Q_1$, (2) $\mathrm{Sup}_{i;n}\left|\bar{k}(x_i,x_j;y_n,y_m)\right|<\xi_1$, (Q Q_1,ξ_1- cons tants)

Then, the following lemma must hold.

Lemma 4 (without proof): If the badly kernel of equation (16) satisfies the conditions

(3) $k(x-\zeta;y-\eta)\in L_2([a,b]\times[c,d]),k(x-\zeta;y-\eta)=\bar{k}(x_i,x_j;y_n,y_m)\,p(x-\zeta;y-\eta)$

(4) $\lim_{x\to x';y\to y'}\|k(x-\zeta;y-\eta)-k(x'-\zeta;y'-\eta)\|\to 0,(x,x')\in[a,b];(y,y')\in[c,d]$,

then

(5) $\mathrm{Sup}\sum_{m=0}^{M}\sum_{j=0}^{J}\left|u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\right|<\xi_2,(\xi_2-$cons tan $t)$.

$$(6)\ \lim_{i\to i;n\to m'}\mathrm{Sup}_{j;m}\sum_{m=0}^{M}\sum_{j=0}^{J}\left|u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\right.$$
$$\left.-u_{n',m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\right|\to 0 \tag{28}$$

The first condition of (28) leads to decide that the operator sum $\tilde{\mathrm{E}}\varphi(x_i,y_n)$ is bounded, while the second condition leads to the continuity of $\tilde{\mathrm{E}}\varphi(x_i,y_n)$.

6 Convergence of the linear algebraic system:

To discuss the convergence solution of the system (25), we state the following:

Theorem 2. The LAS (25) for all values of time $t\in[0,T],T<1$, is convergent in the Banach space ℓ_2

Proof: We must prove that the infinite series $\sum_{s=0}^{\infty}\Psi_s(x_i;y_n)$ is uniformly convergent to a continuous function $\varphi(x;y)$. For this, we construct the following two sequence

$$\begin{aligned}
&f(x_i,y_n)=\mu\varphi_s(x_i,y_n)\\
&-\lambda(t)\textstyle\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\,\varphi_{s-1}(x_j,y_m),\\
&f(x_i,y_n)=\mu\varphi_{s-1}(x_i,y_n)\\
&-\lambda(t)\textstyle\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\,\varphi_{s-2}(x_j,y_m),\\
&f(x_i,y_n)=\mu\varphi_0(x_i,y_n).
\end{aligned} \tag{29}$$

It is convenient to introduce

$$\Psi_s(x_i,y_n)=\varphi_s(x_i,y_n)-\varphi_{s-1}(x_i,y_n) \tag{30}$$

with

Using (25) in (30), we get

$$\varphi_R(x_i,y_n)=\sum_{s=0}^{R}\Psi_s(x_i;y_n),\Psi_0(x_i;y_n)=f(x_i;y_n).$$

$$\Psi_s(x_i,y_n)=-\frac{\lambda(t)}{\mu}\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\,\Psi_{s-1}(x_i,y_n) \tag{31}$$

After using (30) and the properties of the modulus, the formula (31) gives

$$|\Psi_s(x_i,y_n)|\leq\left|\frac{\lambda(t)}{\mu}\right|\sum_{m=0}^{M}\sum_{j=0}^{J}\left|u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\right||\Psi_{s-1}(x_i,y_n)| \tag{32}$$

Using (27-1), (28-5) and with the aid of the mathematical induction method, the formula (32) becomes

$$\begin{aligned}
&\|\Psi_s(x_i,y_n)\|\leq\sigma^sQ_1,\|\Psi_s(x_i,y_n)\|=\mathrm{Sup}_{i,n}|\Psi_s(x_i,y_n)|;\\
&\sigma=\left|\frac{\lambda(t)}{\mu}\right|\xi<1,\xi=\max.(\xi_1,\xi_2).
\end{aligned} \tag{33}$$

Which makes the sequence $\{\Psi_s(x_i,y_n)\}$ converges under the condition $\sigma<1$ and hence, the sequence $\{\varphi_s(x_i,y_n)\}$ is uniformly converges. So we can write

$$\varphi(x_i,y_n)=\varphi_{R\to\infty}(x_i,y_n)=\sum_{s=0}^{\infty}\Psi_s(x_i;y_n). \tag{34}$$

Thus, the function $\varphi(x_i,y_n)$ satisfies the LAS of (29).

To prove that $\varphi(x_i,y_n)$ is the only unique solution, we assume another solution $\overline{\varphi(x_i,y_n)}$ satisfies equation (29), hence

$$\left|\varphi(x_i,y_n)-\overline{\varphi(x_i,y_n)}\right|\leq\left|\frac{\lambda(t)}{\mu}\right|\sum_{m=0}^{M}\sum_{j=0}^{J}\left|u_{n,m}w_{i,j}\bar{k}(x_i,x_j;y_n,y_m)\right|$$
$$\times\,\mathrm{Sup}_{i,n}\left|\varphi(x_i,y_n)-\overline{\varphi(x_i,y_n)}\right|$$

Finally, we have

$$\left\|\varphi\left(x_i,y_n\right)-\overline{\varphi\left(x_i,y_n\right)}\right\|\leq\sigma\left\|\varphi\left(x_i,y_n\right)-\overline{\varphi\left(x_i,y_n\right)}\right\|,(\sigma<1). \tag{35}$$

Since $\sigma<1$ then the inequality (35) is true only if $\varphi\left(x_i,y_n\right)=\overline{\varphi\left(x_i,y_n\right)}$ that is the solution of the system (25) is unique.

7 The equivalence between the LT-DFIE and LAS of PNM.

The equivalence between the algebraic system and the integral equation can be proved by the error equation. Or by finding the sequence of solutions and proving that they converge to the solution of the integral equation.

The estimated error of the method used can be calculated from the equation

$$R_{i,n}=\left|\int_c^d\int_a^b k(x-\zeta,y-\eta)\varphi(\zeta,\eta)d\zeta d\eta-\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}\left(x_i,x_j;y_n,y_m\right)\varphi\left(x_i,y_n\right)\right| \tag{36}$$

When $\text{Lim}_{i;j\to\infty}R_{i,n}\to 0$ and

$$\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}\left(x_i,x_j;y_n,y_m\right)\varphi\left(x_i,y_n\right)\to\int_c^d\int_a^b k(x-\zeta,y-\eta)\varphi(\zeta,\eta)d\zeta d\eta$$

The approximate solution of the LAS (25) is equivalent to the exact solution of (10) in the space $L_2([a,b]\times[c,d])\times C[0,T],T<1.$.

Theorem 3. If the sequence of the continuous functions $\{f^*\left(x_i,y_n\right)\}$ converges uniformly to the function $f(x,y)$ as $(i;n)\to\infty$, then the sequence of approximate solution $\{\varphi^*\left(x_i,y_n\right)\}$ convergences uniformly to the exact solution $\varphi(x,y)$ of equation (15)

Proof: The MIE (15) with its approximate solution gives

$\|\varphi(x,y)-\varphi^*\left(x_i,y_n\right)\|$

$$\leq\left|\frac{\lambda(t)}{\mu}\right|\int_c^d\int_a^b|k(x-\zeta,y-\eta)|\,\|\varphi(x,y)-\varphi^*\left(x_i,y_n\right)\|\,d\zeta d\eta+\left|\frac{1}{\mu}\right|\|f(x,y)-f*\left(x_i,y_n\right)\|$$

Then, after using the conditions (i) and (iii), we get

$$\|\varphi(x,y)-\varphi^*\left(x_i,y_n\right)\|\leq\left|\frac{1}{\mu-\lambda(t)AC}\right|\|f(x,y)-f*\left(x_i,y_n\right)\| \tag{37}$$

Finally, we have

$$\|\varphi(x,y)-\varphi^*\left(x_i,y_n\right)\|\to 0\text{ Since }\|f(x,y)-f*\left(x_i,y_n\right)\|\to 0\text{ as }(i;n)\to\infty$$

Also, the error can be determined using the following formula

$$R_q=\left[\varphi(x,y)-\left(\varphi^*\left(x_i,y_n\right)\right)_q\right]-\sum_{m=0}^{M}\sum_{j=0}^{J}u_{n,m}w_{i,j}\bar{k}\left(x_i,x_j;y_n,y_m\right)\left[\varphi(x,y)-\left(\varphi^*\left(x_i,y_n\right)\right)_q\right] \tag{38}$$

8 Applications and numerical results

In this section we applied **PNM**, to obtain the numerical solution of (15) when the kernel $k(x-u,y-v)$ has a singular term in a logarithmic form in $(2+1)$-dimensional.

Example 1-1. (Logarithmic kernel): Solve the integral equation:

$$(0.99)\Phi(x,y;t)-(0.2)\int_0^t\int_{-1}^1\int_{-1}^1\zeta^2\eta t\tau^2\ln|x-\zeta|\ln|y-\eta|\Phi(\zeta,\eta;\tau)d\zeta d\eta d\tau=F(x,y;t),-1\leq x;y\leq 1$$

$$\left(\text{E.S. }\ \Phi(x,y;t)=\tfrac{xy}{6}\left(1-2t^2\right)\right)\ (35)$$

Here **PNM** is used to get numerical solution for different two times $T=0.3,T=0.5$

Table (1): The results of logarithmic kernel of example1-1, when $N = 21$, 41and at $T = 0.3, T = 0.5$

N	x,y	$E.S$ $T=0.3.$	$Appr.\ S$ $T=0.3.$	$Error$ $T=0.3..$	$E.S$ $T=0.5$	$Appr.S$ $T=0.5.$	$Error$ $T=0.5.$
21	-1.00	2.0000E-02	1.960E-02	4.0015E-04	2.0001E-02	1.9553E-02	4.4675E-04
	-0.60	7.2000E-03	6.768E-03	4.3253E-04	7.2002E-03	6.59092E-03	6.0908E-04
	-0.20	8.0000E-04	7.4256E-04	5.7503E-05	8.0001E-04	5.67772E-04	2.3223E-04
	0.00	0.0000E+00	1.0060E-05	1.0061E-05	0.0001E+00	1.39496E-05	1.3956E-05
	0.20	8.0000E-04	7.4250E-03	5.7503E-05	8.0003E-04	9.88286E-04	1.8836E-04
	0.60	7.2000E-03	6.7675E-03	4.3253E-04	7.2002E-03	7.70094E-03	5.0095E-04
	1.00	2.0000E-02	1.9658E-02	4.0015E-04	2.0000E-02	2.04151E-02	4.1512E-04
41	-1.00	2.0000E-02	1.9572E-02	4.2861E-04	2.0004E-02	1.95713E-02	4.2870E-04
	-0.60	7.2000E-03	6.7366E-03	4.6339E-04	7.2200E-03	6.73661E-03	4.6340E-04
	-0.20	8.0000E-04	7.3580E-04	6.4203E-05	8.1003E-04	7.35797E-04	6.4210E-05
	0.00	0.0000E+00	4.7664E-06	4.7659E-06	0.0000E+00	4.76594E-06	4.7664E-06
	0.20	8.0000E-04	7.3585E-04	6.4204E-05	8.0000E-04	7.35795E-04	6.4205E-05
	0.60	7.20000E-03	6.7367E-03	4.6339E-04	7.20000E-03	6.73661E-03	4.6340E-04
	1.00	2.00000E-02	1.9573E-02	4.2861E-04	2.00000E-02	1.95713E-02	4.2870E-04

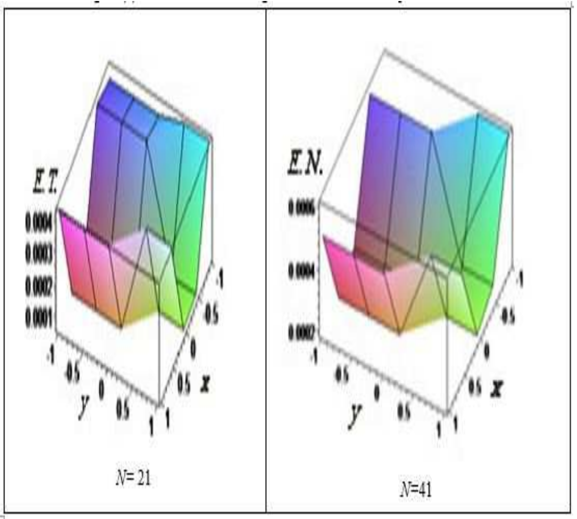

Figure (1): The error values of logarithmic kernel of example 1-1 at T = 0.3

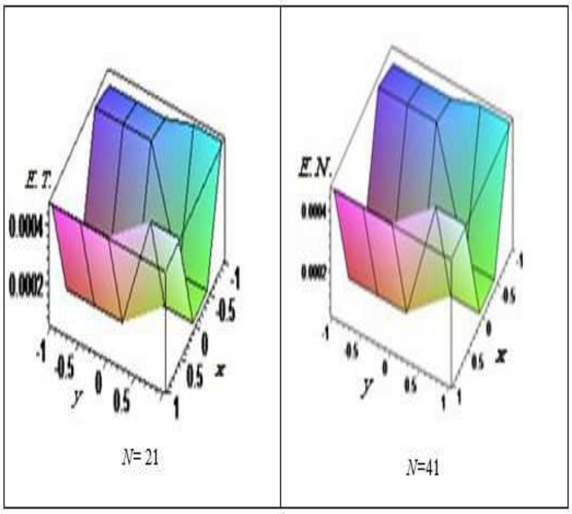

Figure (2): The error values of logarithmic kernel of example 1-1 at $T = 0.5$

Example (1-2): Solve the integral equation:

$$\Phi(x,y;t)-(0.01)\int_0^t\int_{-1}^1\int_{-1}^1\zeta^2\eta^2t^3\tau^2\ln|x-\zeta|\ln|y-\eta|\Phi(\zeta,\eta;\tau)d\zeta d\eta d\tau=F(x,y;t),-1\le x;y\le 1$$

$\left(E.S.\Phi(x,y;t)=\frac{xy}{50}\left(1-t^2\right)\right)$.

Table (2): The approximate solution and corresponding error of logarithmic kernel, of example 1-2 at $T=0.3, T=0.5$.

N	x,y	E.S. T=0.3	Appr.S T=0.3	Err. T=0.3	E. S T=0.7	Appr.S T=0.7.	Err. T=0.7
21	-1.00	2.00012E-02	1.95043E-02	6.97631E-04	2.00034E-02	1.93026E-02	6.97598E-04
	-0.60	7.23000E-03	6.48595E-03	7.54445E-03	7.23003E-03	6.17148E-03	1.02863E-03
	-0.20	8.10000E-04	6.99678E-04	1.10821E-04	8.10003E-04	4.24743E-04	3.75259E-04
	0.00	0.00000E+00	1.72848E-05	1.69848E-05	0.00000E+00	2.52805E-06	2.52807E-06
	0.20	8.10010E-04	6.99598E-04	1.10821E-04	8.10020E-04	1.14528E-03	3.45281E-04
	0.60	7.30003E-03	6.45595E-03	7.57045E-04	7.30013E-03	8.07501E-03	8.75012E-04
	1.00	2.10001E-02	1.94043E-02	6.97731E-04	2.10002E-02	2.07163E-02	7.16317E-04
41	-1.00	2.10001E-02	1.93560E-02	7.73950E-04	2.10001E-02	1.92560E-02	7.44002E-04
	-0.60	7.28000E-03	6.49441E-03	8.15587E-04	7.28001E-03	6.39441E-03	8.05591E-04
	-0.20	8.10001E-04	6.98093E-04	1.41906E-04	8.10001E-04	6.88093E-04	1.11917E-04
	0.00	0.00000E+00	7.89419E-06	7.83419E-06	0.00000E+00	7.80419E-06	7.81419E-06
	0.20	8.10005E-04	6.9193E-04	1.21916E-04	8.10005E-04	6.88096E-04	1.12907E-04
	0.60	7.20000E-03	6.44411E-03	8.15687E-04	7.20003E-03	6.45041E-03	8.15590E-04
	1.00	2.10000E-02	1.93560E-02	7.45950E-04	2.10002E-02	1.92563E-02	7.46000E-04

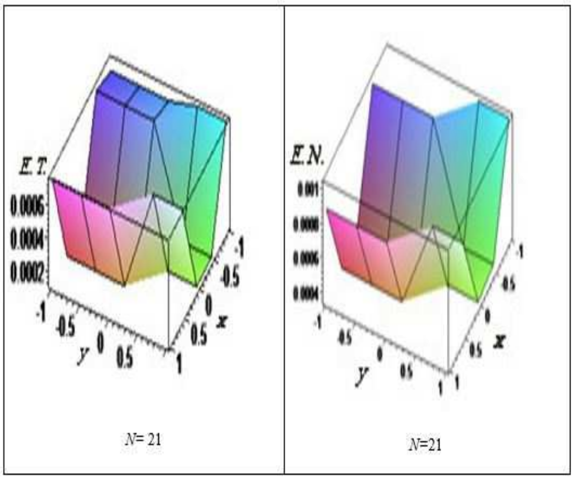

Figure (3): the error values of logarithmic kernel of example $1-1$ at $\mathrm{T}=0.3$

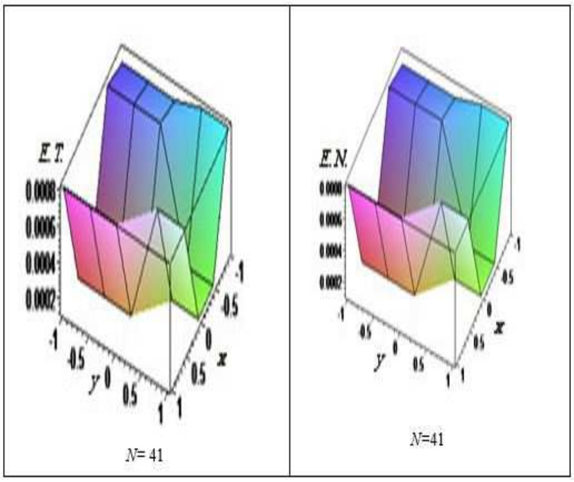

Figure (4): The error values of logarithmic kernel example 1-1 at $T=0.5$

From the above our results obtained, in general we note that:

1- If λ has a fixed value, the error is decreasing as well as N and time increase

2-If the value of N is fixed, the error values are increase.

3- The error has a maximum value at the ends when $x = y \simeq \pm 1$ and a minimum at the middle when $x = 0, y = 0$.

9 Application 2-1 (Carleman kernel)

Example (2-1): consider the mixed integral equation:

$$\mu \Phi(x,y;t) - \lambda \int_0^t \int_{-1}^1 \int_{-1}^1 \zeta^2 \eta t \tau^2 |x-\zeta|^{-v} \cdot |y-\eta|^{-v} \Phi(\zeta,\eta;\tau) d\zeta d\eta d\tau = F(x,y;t), -1 \leq x;y \leq 1$$
$$\left(\text{E.S. } \Phi(x,y;t) = \tfrac{xy}{6}\left(1-2t^2\right)\right)$$

. We divided the position interval by $N = 21, 41$ units. In the case (2-1) we assumed $v = 0.35, v = 0.37; \lambda = 0.039$. In the case (2-2) we considered $v = 0.38, v = 0.42; \lambda = 0.022$.

Case (2-1): $v = 0.35, v = 0.37; \lambda = 0.039, T = 0.4$

Table (3): The results for Carleman kernel, when $v = 0.35, v = 0.37; \lambda = 0.039, T = 0.4$

N	x; y	E. S. $\upsilon = 0.35$	Appr. S. $\upsilon = 0.35$	*Error* $\upsilon = 0.35$	E.S. $\upsilon = 0.37$	*Appr.S* $\upsilon = 0.37$.	*Error* $\upsilon = 0.37$
21	-1.00	2.0000E-02	1.9915E-02	9.0451E-05	2.0000E-02	2.00663E-02	6.6332E-05
	-0.60	7.2000E-03	7.0773E-03	1.2279E-04	7.2001E-03	7.27872E-03	7.8731E-05
	-0.20	8.0000E-04	8.2119E-04	2.1194E-05	8.0001E-04	9.86085E-04	1.8612E-04
	0.00	0.0000E+00	6.9637E-05	6.9637E-05	0.0000E+00	2.48415E-04	2.4842E-04
	0.20	8.0000E-04	8.2119E-04	2.1194E-05	8.0002E-04	1.11452E-03	3.1453E-04
	0.60	7.2000E-03	7.0772E-03	1.2282E-04	7.2001E-03	7.64174E-03	4.4175E-04
	1.00	2.0000E-02	1.9915E-02	9.0451E-05	2.1000E-02	2.05448E-02	5.4487E-04
41	-1.00	2.0000E-02	1.9866E-02	1.34723E-04	2.0000E-02	2.00220E-02	2.2000E-05
	-0.60	7.2000E-03	7.0215E-03	1.7851E-04	7.2100E-03	7.22301E-03	2.3010E-05
	-0.20	8.0000E-04	7.9398E-04	6.02225E-06	8.010E-04	9.58869E-04	1.5887E-04
	0.00	0.0000E+00	3.2347E-05	3.23367E-05	0.0000E+00	2.11115E-04	2.1112E-04
	0.20	8.0000E-04	7.9398E-04	6.02225E-06	8.010E-04	1.08730E-03	2.8731E-04
	0.60	7.2000E-03	7.0215E-03	1.78509E-04	7.2001E-03	7.58603E-03	3.8604E-04
	1.00	2.0000E-02	1.9866E-02	1.34713E-04	2.0000E-02	2.05005E-02	5.0050E-04

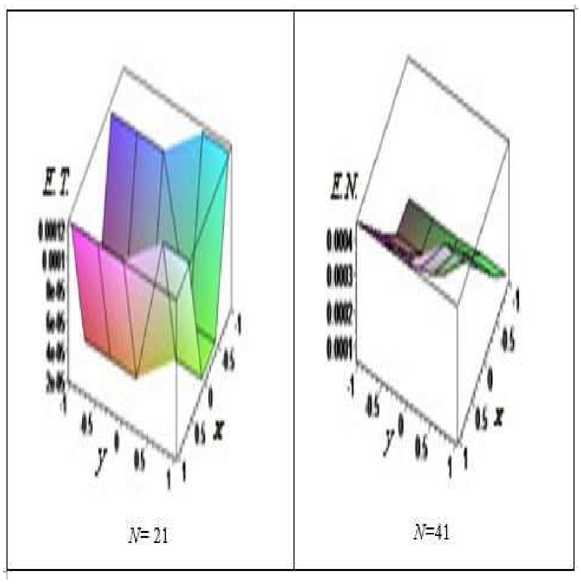

Figure (5): The error values when $v = 0.35, \quad \lambda = 0.039, T = 0.4$.

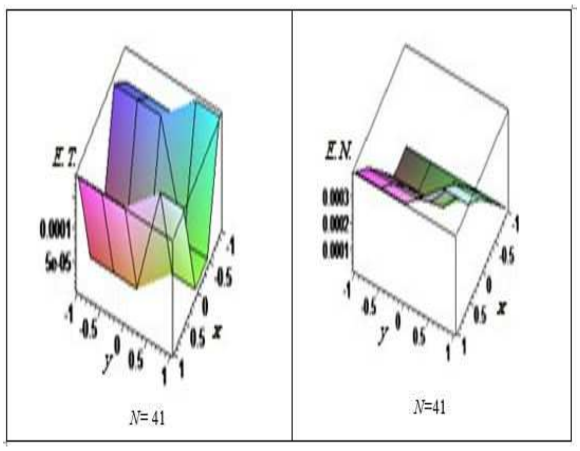

Figure (6): The error values when $v = 0.37; \lambda = 0.039, T = 0.4$

Example (2-2): consider the mixed integral equation:

$$\mu\Phi(x,y;t) - \lambda \int_0^t \int_{-1}^{1} \int_{-1}^{1} \zeta^2\eta^2 t^3 \tau^2 |x-\zeta|^v |y-\eta|^v$$
$$\times \Phi(\zeta,\eta;\tau) d\zeta d\eta d\tau = F(x,y;t), -1 \le x; y \le 1$$
$$\text{(E.S. } \Phi(x,y;t) = \tfrac{xy}{50}\left(1-\mathrm{t}^2\right))$$

Case (2-2) $v = 0.38, v = 0.42; \lambda = 0.022, T = 0.4$:

Table (3): The results for Carleman kernel, when $v = 0.38, v = 0.42; \lambda = 0.022, T = 0.4$

N	x, y	Exact sol. υ = 0.38	Appr. sol. υ = 0.38.	Err. T υ = 0.38.	Exact sol. υ = 0.42	Appr. sol υ = 0.42.	Err. υ = 0.42.
21	-1.00	2.0001E-02	1.9897E-02	1.0398E-04	2.0001E-02	2.0201E-02	2.0163E-04
	-0.60	7.2001E-03	7.0734E-03	1.2666E-04	7.2101E-03	7.4385E-03	2.3851E-04
	-0.20	8.0001E-04	8.4658E-04	4.6459E-05	8.0101E-04	1.1678E-03	3.6483E-04
	0.00	0.0000E+00	1.1760E-04	1.1707E-04	0.0000E+00	4.4094E-04	4.4094E-04
	0.20	8.0000E-04	8.4658E-04	4.6469E-05	8.0100E-04	1.3347E-03	5. 3476E-04
	0.60	7.2001E-03	7.0734E-03	1.2667E-04	7.2021E-03	7.8839E-03	6.8396E-04
	1.00	2.0001E-02	1.9897E-02	1.0398E-04	2.0002E-02	2.1294E-02	8.3492E-04
41	-1.00	2.0001E-02	1.9852E-02	1.6795E-04	2.0003E-02	2. 1393E-02	1.4300E-04
	-0.60	7.2000E-03	6.9923E-03	2.0860E-04	7.2001E-03	7.4124E-03	1.51240E-04
	-0.20	8.0000E-04	8.0297E-04	1.9983E-06	8.0001E-04	1.1231E-03	3.2319E-04
	0.00	0.0000E+00	5.5886E-05	5.4886E-05	0.0000E+00	3.7932E-04	3.7932E-04
	0.20	8.0001E-04	8.2997E-04	1.9983E-06	8.0002E-04	1.2800E-03	4.8009E-04
	0.60	7.3000E-03	6.9927E-03	2.0860E-04	7.3001E-03	7.8178E-03	6.1780E-04
	1.00	2.0000E-02	1.9842E-02	1.6795E-04	2.0001E-02	2.1685E-02	7.7580E-04

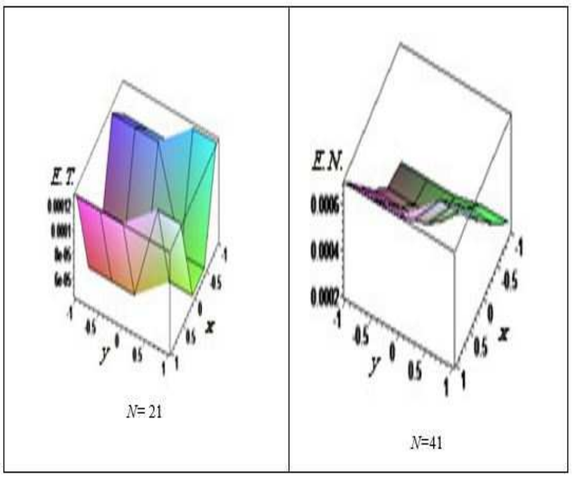

Figure (7): The error values when $v = 0.38, \lambda = 0.022, T = 0.4$

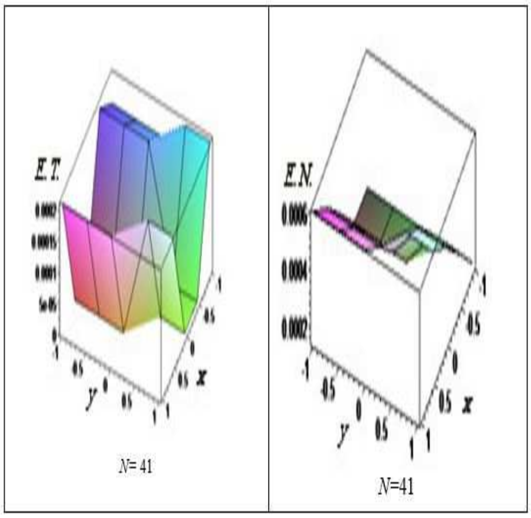

Figure (8): The error values when $v = 0.42; \lambda = 0.022, T = 0.4$

From the above our results obtained, in general, we note that:

1- In the theory of elasticity, the relation between μ, v, λ are given by $\lambda = \frac{2\mu v}{1-2v}$ where v is called Poisson ratio, and λ, μ are called Lame constants 2- If the values of λ and v are fixed, the error values are decrease as well as N increase for the two different materials $(v_1 = 0.42, v_2 = 0.38), (v_1 = 0.37, v_2 = 0.35)$, 3- As x and y are increasing in $[-1,1]$, the error has a maximum at the two ends $x = y \simeq \pm 1$ and minimum at the middle when $x, y = 0$.

4-If the values of N are fixed, the error values increase with the increasing of v and λ, for each materials

10 Conclusions

1- The importance of the mixed integral equation (1) appears in different sciences, whether in basic sciences or in plant diseases, which treat cracks in bodies of all kinds in two dimensions. Its importance becomes clear if these cracks are linked to time.
2- By studying numerical methods for solving singular integral equations, it can be concluded that the PNM is one of the best methods for dealing numerically. As the singular part in the kernel turns into an integral system in one period [0, 2], and during this period all the integrals can be calculated in a simple and easy way.
3- In general, as the time increases the error is also increases.
4- The maximum error at the ends of the points, while the minimum error at the origin.

Acknowledgement

The researcher extends all thanks and gratitude to Prof. Dr. M. A. Abdou for his important scientific advice and guidance while working on this research.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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