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On Orthogonal Special Class of Caterpillars Squares

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Abstract: Orthogonal Double Cover (ODC) is a set $\mathscr{G}$ of $2n$ subgraphs of a complete bipartite graph $K_{n,n}$ of a graph G such that each edge in graph $K_{n,n}$ appears once in both subgraphs of set $\mathscr{G}$, and all subgraphs are isomorphic to graph G. we aim to construct two graph squares by a new engineering method that uses two induced starter functions to find the ODC of $K_{n,n}$. we also compose ODC from small to obtain a larger ODC. Starting from ODC $\mathscr{F}$ of $K_{q,q}$ by qK_2 we replace each point with n new points and each edge with the ODC of $K_{n,n}$ to obtain the ODC of $K_{qn,qn}$ by Some disjoint caterpillar unions, where $q,n \in \mathbb{Z}^+$.

Keywords: Orthogonal double cover, Edge decomposition, Orthogonal graph squares

1 Interdiction

In this paper, we will use of the usual notation:

Nomenclature	
$K_{m,n}$	The complete bipartite graph with partition sets of sizes m and n.
$D\cup F$	The disjoint union of D and F.
$D\cup^* F$	The joint union of D and F in one vertex.
$K_{1,n} \equiv S_n$	The star on $n+1$ vertices and n edges.
sG	s disjoint copies of G.
P_{m+1}	The path with $m+1$ vertices and m edges.
$C_r(n_1,n_2,\ldots,n_r)$	The caterpillar (tree) obtained from the path $P_r = x_1x_2\cdots x_r$ by joining vertex x_i to n_i new vertices; $i=\{1,2,\ldots,r\}$

The vertices of a complete bipartite graph $K_{n,n}$ are marked by elements of $\mathbb{Z}_n \times \mathbb{Z}_2$, where $\mathbb{Z}_n = \{0,1,2,...,n-1\}$ is an additive group of Order n, such that $\{x_r,y_r\} \notin E(K_{n,n})$ for $x,y \in \mathbb{Z}_n$ and fixed $r \in \mathbb{Z}_2$. It will be later shown that these groups can be used to construct an ODC of $K_{n,n}$. If there is no risk of confusion write $(x,y) \equiv xy$ instead of $\{x_0,y_1\}$ for edges between vertices x_0,y_1. For construction, we need the order of the elements of $\mathbb{Z}_n$.

Let $\mathscr{G} = \{G_0,...,G_{n-1},\ F_0,F_1,...,F_{n-1}\}$ be the set of $2n$ subgraphs (called pages) of $K_{n,n}$. $\mathscr{G}$ is called an *Orthogonal Double Cover* (ODC) of $K_{n,n}$ if:

(i)Every edge of $K_{n,n}$ is exactly on one page of $\{G_0,...,G_{n-1}\}$ and exactly on one page of $\{F_0,F_1,...,F_{n-1}\}$.

(ii)For $i,j \in \{0,1,2,...,n-1\}$ and $i \neq j$:

$$|E(G_i)\cap E(G_j)| = |E(F_i)\cap E(F_j)| = 0$$

and

$$|E(G_i)\cap E(F_j)| = 1.$$

If all edges in $\mathscr{G}$ are isomorphic to a graph G, then $\mathscr{G}$ is called the ODC by G. Obviously, G must have exactly n edges. The original purpose of obtaining ODC stems from the question posed by Demetrovics et al. [6] on minimal databases, and a question raised by Hering and Rosenfeld [3] on the organization of statistical testing programs. The ODC by G has been considered for several graph families: short cycles [1], clique graphs [2], trees [5], small graphs [8]. A survey on this topic can be found in [4].

El-Shanawany et al. [8] presents a basic definitions that usually relies on half-starter vectors.

Below, we give a formal basic definitions of $K_{n,n}$ subgraph induced by a function on the additive group $\mathbb{Z}_n$.

Definition 1.*Let G_f be a subgraph of $K_{n,n}$ induced by the function $f:\mathbb{Z}_n \to \mathbb{Z}_n$. Then G_f is called $f-$starter if*

$$E(G_f) = \{(f(i), f(i)+i) : i \in \mathbb{Z}_n\}. \qquad (1)$$

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Definition 2. *Let G be a $f-$starter subgraph of $K_{n,n}$, and let $x,i \in \mathbb{Z}_n$. Then the graph $G_f + x$ with $E(G_f + x) = \{(f(i)+x, f(i)+i+x) : (f(i), f(i)+i) \in E(G_f)\}$ is called the $(x,f)-$translate of G_f.*

Definition 3. *If G is a $f-$ starter subgraph of $K_{n,n}$, then the union of all translates of G_f forms an edge decomposition of $K_{n,n}$ i.e. $E(K_{n,n}) = \cup_{x\in\mathbb{Z}_n} E(G_f + x)$.*

In the following, we give the formal basic definitions of a G-square over additive group $\mathbb{Z}_n$.

Definition 4. *Let G be a subgraph of $K_{n,n}$. A square matrix M of order n is called an G-square if every element in $\mathbb{Z}_n$ occurs exactly n times, and the graphs G_i, $i \in \mathbb{Z}_n$ with*

$$E(G_i) = \{(x,y) : M(x,y) = i; x,y \in \mathbb{Z}_n\}, \qquad (2)$$

are isomorphic to a subgraph G.

Definition 5. *Two square M_0, M_1 of order n are said to be orthogonal if for any order pair (a,b), there is exactly one positive (x,y) for $M_0(x,y) = a$, and $M_1(x,y) = b$.*

That is, the two graph squares have the property that, when superimposed, every ordered pair occurs exactly once.

For a subgraph G_f of $K_{n,n}$ with n edges, the subgraph G_g induced by the function g with $E(G_g) = \{y_0x_1 : x_0y_1 \in E(G_f)\}$ is called *symmetric subgraph* of G_f.

Definition 6. *Caterpillar graph is a tree with central path and the ended vertices with degree 1.*

We will give an example that will illustrate the above definitions.

Example 1. Let $G_f \simeq C_6(0,0,0,0,0,2)$ be a caterpillar subgraph of $K_{7,7}$ such that $f-$starter subgraph G_f induced by the function $f : \mathbb{Z}_7 \to \mathbb{Z}_7$ defined as follows

$$f(i) = \begin{cases} 0\,; & i = 0,2 \\ 4\,; & i = 5,6 \\ 2\,; & i = 1,3,4 \end{cases} \quad .; i \in \mathbb{Z}_7$$

Note that, every edge in the subgraph G_f formed from equation 1 as follows, $E(G_f) = \{(f(0), f(0)+0), (f(1), f(1)+1), (f(2), f(2)+2), (f(3), f(3)+3), (f(4), f(4)+4), (f(5), f(5)+5), (f(6), f(6)+6)\} = \{(0,0), (2,3), (0,2), (2,5), (2,6), (4,2), (4,3)\}$. as shown in Figure 1, then $(x,f)-$ translates is form an edge decomposition as shown in Figure 2, where $x \in \mathbb{Z}_7$ which is associated with the $C_6(0,0,0,0,2)$-square as follows by using the equation 2

$$M = \begin{bmatrix} 0&5&0&5&5&3&3 \\ 4&1&6&1&6&6&4 \\ 5&5&2&0&2&0&0 \\ 1&6&6&3&1&3&1 \\ 2&2&0&0&4&2&4 \\ 5&3&3&1&1&5&3 \\ 4&6&4&4&2&2&6 \end{bmatrix}, \quad M^T = \begin{bmatrix} 0&4&5&1&2&5&4 \\ 5&1&5&6&2&3&6 \\ 0&6&2&6&0&3&4 \\ 5&1&0&3&0&1&4 \\ 5&6&2&1&4&1&2 \\ 3&6&0&3&2&5&2 \\ 3&4&0&1&4&3&6 \end{bmatrix}.$$

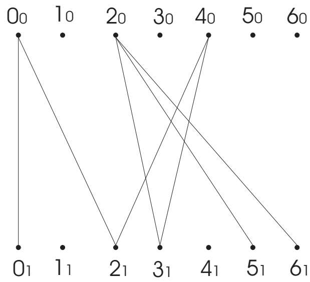

Fig. 1: The subgraph $G_f \simeq C_6(0,0,0,0,0,2)$ induced by the f-starter w.r.t $\mathbb{Z}_7$.

2 Main result

In this particular section we are especially interested by making extensions of the small ingredient *ODCs* of $K_{n,n}$ in the theorem 4 by using the Latin squares to get larger *ODCs* of $K_{qn,qn}$.

Theorem 1. *(see [7])There exists ODC of $K_{n,n}$ by G if and only if there exist two orthogonal G-squares of order n.*

Theorem 2. *(see [9]) Let n be a positive integer and let f and g be starter functions of a subgraphs G_f and G_g of $K_{n,n}$, where $g(i) = f(i)+i, i \in \mathbb{Z}_n$,then there exist two orthogonal squares M_f and M_g of order n defined as*

$$(M_f(a,b), M_f^T(a,b)) = (a - f(b-a), b - f(a-b)); \text{ where } a,b \in \mathbb{Z}_n. \qquad (3)$$

Theorem 3. *(see [7]) Assume that there exist symmetric starters ODCs $\mathscr{G}_l$ of $K_{n,n}$ by G_l for $l = 0,1,....,m-1$. Furthermore, assume that there exists an ODC of $K_{m,m}$ by mK_2, which is generated by a symmetric starter. Then there exists a symmetric $(G_0 \cup G_1 \cup .... \cup G_{m-1})-$square of an ODC of $K_{mn,mn}$.*

Theorem 4. *(see [10]) Let n and m be integers such that $2 \le m \le 10$ and $m \le n$. Then there is an ODC of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$.*

Theorem 5. *Let $q \ge 3$ be a prime number, and n,m be integers such that $5 \le m \le 10$ and $m \le n$. Then there is an ODC of $K_{qn,qn}$ by $qC_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}}, n-m)$.*

Proof. To prove that theorem we need to have two *ODCs*. The first one, we got it from the Latin square (see [7]) when there exist *ODC* of $K_{q,q}$ by qK_2 with qK_2-square defined as follows

$$L_0(i,j) = [i+j], \text{ and } L_1(i,j) = [2i+j]$$

where q is a prime number and $i,j \in \mathbb{Z}_q$. The second *ODC* we get it from theorem 3 which it prove the existence of an *ODC* of $K_{n,n}$ by $C_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}}, n-m)$ where n,m are

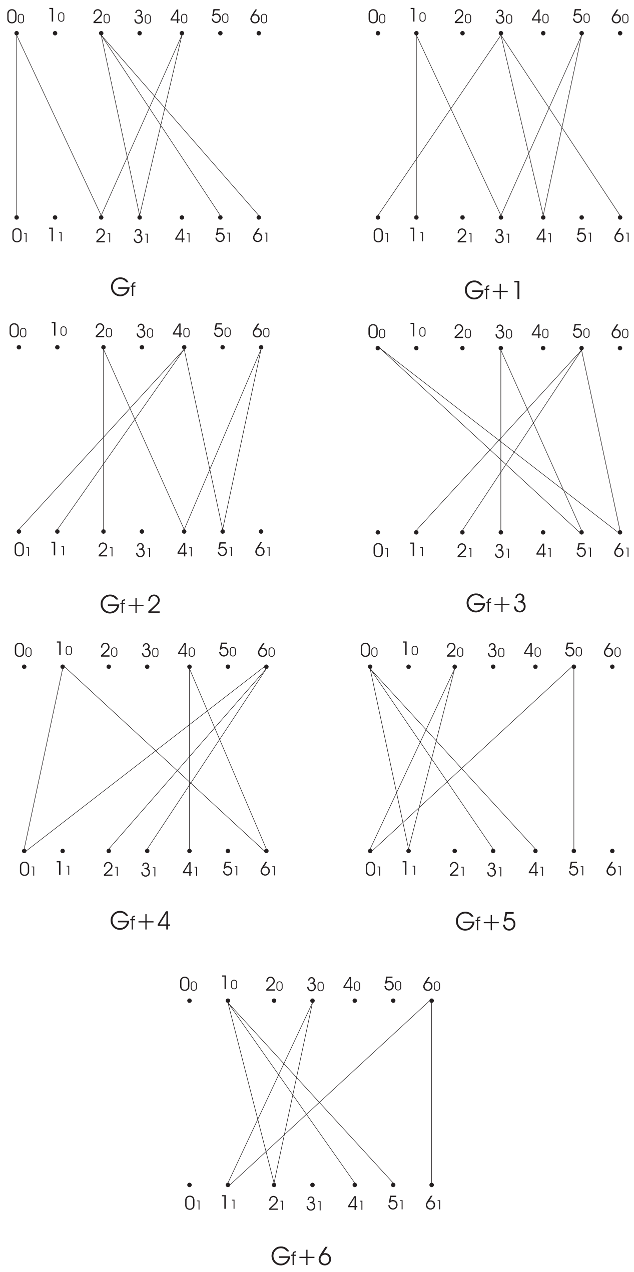

Fig. 2: Edge decomposition of the subgraph $G_f \simeq C_6(0,0,0,0,0,2)$ of $K_{7,7}$.

positive integers such that $m \le n$; and according to the theorem 1 and the theorem 2 the *ODC* of $K_{n,n}$ has $C_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}},n-m)$-square defined as equation 3.

Now, we can make the combination of the two *ODCs* of $K_{n,n}$ and $K_{q,q}$ according to the theorem 3 and we getting two $qC_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}},n-m)$-squares of order qn by superimposing the matrices M_f with L_0 and M_f^T with L_1 as follows

$$S(r,t) = [n(i+j)+a-f_1(b-a)], \text{ and} \qquad (4)$$
$$S^*(r,t) = [n(2i+j)+b-f_1(a-b)].$$

where the elements $r,t \in \mathbb{Z}_{qn}$ defined as follows

$$r = ni+a, \text{ and } t = nj+b.$$

It is easily to verify that the order pair $(S(r,t),S^*(r,t))$ is orthogonal and form an *ODC* of $K_{qn,qn}$. Then we will prove that the pages obtained from each entry y in $\mathbb{Z}_{qn}$ is isomorphic to $qC_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}},n-m)$ such that $S(r,t) = y = n(i+j)+x$ where $x \in \mathbb{Z}_n$, $i,j \in \mathbb{Z}_p$. Also, a similar argument can be applied to the pages in $S^*(r,t)$.

1.At $m=5$, their exist an *ODC* of $K_{n,n}$ by $C_6(\underbrace{0,\ldots,0}_{5\text{-times}},n-5) \equiv P_6 \cup^* S_{n-5}$ defined with the starter function f_1 as follows

$$f_1(i) = \begin{cases} 0\,; & i=0,2 \\ 4\,; & i=2,n-2\,.;i \in \mathbb{Z}_n \\ 2\,; & otherwise \end{cases}$$

with $C_6(\underbrace{0,\ldots,0}_{5\text{-times}},n-5)$-square $(M_{f_1}(a,b),M^*_{f_1}(a,b))$ defined as equation as follows

$$M_{f_1}(a,b) = a-f_1(b-a),\ M^*_{f_1}(a,b) = b-f_1(a-b).$$

In that *ODC* the pages obtained from each entry $x \in \mathbb{Z}_n$ such that $M_{f_1}(a,b) = x$ is isomorphic to $C_6(0,0,0,0,0,n-5)$. Also, a similar argument can be applied to the pages in $M^*_{f_1}(a,b)$, so from the definition of the caterpillar we know that $C_6(\underbrace{0,\ldots,0}_{5\text{-times}},n-5)$ is consist of two part the first one is a path P_6 of length 5 with the 6 vertices as: $(x)_1,(x)_0,(2+x)_1,(4+x)_0,(3+x)_1,(2+x)_0$, and the second part is the star as: $\{(2+x)_0,(\alpha+x)_1\}$; such that $5 \le \alpha \le n-1$.
So, the *ODC* of $K_{qn,qn}$ is isomorphic to $q(C_6(\underbrace{0,\ldots,0}_{5\text{-times}},n-6))$ because the page y is isomorphic to q paths of length 5 with 6 vertices as follows: $(x+nj)_1,(x+ni)_0,(2+x+nj)_1,(4+x+ni)_0,(3+x+nj)_1,(2+x+ni)_0$, and isomorphic to q stars of length $n-5$ as follows: $\{(2+x+ni)_0,(\alpha+x+nj)_1\}$; such that $nj+5 \le \alpha \le n(1+j)-1$.
Hence there exist an *ODC* of $K_{qn,qn}$ by $q(P_6 \cup S_{n-5}) \equiv qC_6(\underbrace{0,\ldots,0}_{5\text{-times}},n-5)$.

2.At $m=6$, their exist an *ODC* of $K_{n,n}$ by $C_7(\underbrace{0,\ldots,0}_{6\text{-times}},n-6) \equiv P_7 \cup^* S_{n-6}$ defined with the starter function f_2 as follows

$$f_2(i)=\begin{cases} 2 & ; \quad i=0 \\ n-1 & ; i=1,n-1 \\ 0 & ; i=2,n-2 \\ n-i-1 & ; \ otherwise \end{cases} .; i\in\mathbb{Z}_n$$

where the pages obtained from each entry $x\in\mathbb{Z}_n$ such that $M_{f_2}(a,b)=x$ is isomorphic to $C_7(\underbrace{0,\dots,0}_{6\text{-times}},n-6)$ as follows: $(x)_1,(n-1+x)_0,(n-2+x)_1,(x)_0,(2+x)_1,(2+x)_0,(n-1+x)_1,\ (n-1+x)_1,(\alpha+x)_0$ where $3\le\alpha\le n-4$.
Then the pages y is isomorphic to $C_7(\underbrace{0,\dots,0}_{6\text{-times}},n-6)$ with the following vertices $(nj+x)_1,\ (n(1+i)-1+x)_0,(n(1+j)-2+x)_1,(ni+x)_0,(2+nj+x)_1,(ni+2+x)_0,(n(1+i)-1+x)_0$ and $(n(1+j)-1+x)_1,(\alpha+ni+x)_0$ where $ni+3\le\alpha\le n(1+i)-4$.

3.At $m=7$, their exist an *ODC* of $K_{n,n}$ by $C_8(\underbrace{0,\dots,0}_{7\text{-times}},n-7)\equiv P_8\cup^* S_{n-7}$ defined with the starter function f_3 as follows

$$f_3(i)=\begin{cases} 0\ ; & i=0,3 \\ 1\ ; & i=1,n-1 \\ 6\ ; & i=n-3,n-2 \\ 2\ ; & otherwise \end{cases} .; i\in\mathbb{Z}_n$$

where the pages obtained from each entry $x\in\mathbb{Z}_n$ such that $M_{f_3}(a,b)=x$ is isomorphic to $C_8(\underbrace{0,\dots,0}_{7\text{-times}},n-7)$ as follows: $(2+x)_1,(1+x)_0,(x)_1,(x)_0,(3+x)_1,\ (n-3+x)_0,(4+x)_1,(2+x)_0$, and $(2+x)_0,(\alpha+x)_1$ where $6\le\alpha\le n-2$.
Then the pages y is isomorphic to $C_8(\underbrace{0,\dots,0}_{7\text{-times}},n-7)$ with the following vertices $(2+nj+x)_1,(1+ni+x)_0,(nj+x)_1,(ni+x)_0,(3+nj+x)_1,(n(1+j)-3+x)_0,(4+nj+x)_1,(2+ni+x)_0$ and $(2+ni+x)_0,(\alpha+nj+x)_1$ where $nj+6\le\alpha\le n(1+j)-2$.

4.At $m=8$, their exist an *ODC* of $K_{n,n}$ by $C_9(\underbrace{0,\dots,0}_{8\text{-times}},n-8)\equiv P_9\cup^* S_{n-8}$ defined with the starter function f_4 as follows

$$f_4(i)=\begin{cases} 0\ ; & i=0,2 \\ 4\ ; & i=1,n-2 \\ 3\ ; & i=3,n-3 \\ 6\ ; & i=n-1 \\ 2\ ; & otherwise \end{cases} .; i\in\mathbb{Z}_n$$

where the pages obtained from each entry $x\in\mathbb{Z}_n$ such that $M_{f_4}(a,b)=x$ is isomorphic to $C_9(\underbrace{0,\dots,0}_{8\text{-times}},n-8)$ as follows: $(n-4+x)_0,(5+x)_1,(4+x)_0,(2+x)_1,(x)_0,(x)_1,(3+x)_0,(n-4+x)_1,(2+x)_0$ and $(2+x)_0,(\alpha+x)_1$ where $7\le\alpha\le n-2$.
Then the pages y is isomorphic to $C_9(\underbrace{0,\dots,0}_{8\text{-times}},n-8)$ with the following vertices $(n(i+1)-4+x)_0,(nj+5+x)_1,(ni+4+x)_0,(nj+2+x)_1,(ni+x)_0,(nj+x)_1,(ni+3+x)_0,(n(j+1)-4+x)_1,\ (ni+2+x)_0$ and $(ni+2+x)_0,(nj+\alpha+x)_1$ where $nj+7\le\alpha\le n(1+j)-2$.

5.At $m=9$, their exist an *ODC* of $K_{n,n}$ by $C_{10}(\underbrace{0,\dots,0}_{9\text{-times}},n-9)\equiv P_{10}\cup^* S_{n-9}$ defined with the starter function f_5 as follows

$$f_5(i)=\begin{cases} 0\ ; & i=0,4 \\ 1\ ; & i=1,n-1 \\ 4\ ; & i=2,n-2 \\ 8\ ; & i=n-4,n-3 \\ 2\ ; & otherwise \end{cases} .; i\in\mathbb{Z}_n$$

where the pages obtained from each entry $x\in\mathbb{Z}_n$ such that $M_{f_5}(a,b)=x$ is isomorphic to $C_{10}(\underbrace{0,\dots,0}_{9\text{-times}},n-9)$ as follows: $(n-5+x)_1,(4+x)_0,(2+x)_1,(1+x)_0,(x)_1,(x)_0,(4+x)_1,(n-3+x)_0,(5+x)_1,(2+x)_0$, and $(2+x)_0,(\alpha+x)_1$ where $7\le\alpha\le n-3$.
Then the pages y is isomorphic to $C_{10}(\underbrace{0,\dots,0}_{9\text{-times}},n-8)$ with the following vertices $(n(j+1)-5x)_1,(ni+4+x)_0,(nj+2+x)_1,(ni+1+x)_0,(nj+x)_1,\ (ni+x)_0,(nj+4+x)_1,(n(i+1)-3+x)_0,(nj+5+x)_1,(ni+2+x)_0$ and $(ni+2+x)_0,(nj+\alpha+x)_1$ where $nj+7\le\alpha\le n(1+j)-3$.

6.At $m=10$, their exist an *ODC* of $K_{n,n}$ by $C_{11}(\underbrace{0,\dots,0}_{10\text{-times}},n-10)\equiv P_{11}\cup^2 S_{n-10}$ defined with the starter function f_6 as follows

$$f_6(i)=\begin{cases} 0\ ; & i=0,4 \\ 1\ ; & i=1,n-1 \\ 4\ ; & i=2 \\ 5\ ; & i=3,n-3 \\ 8\ ; & i=n-4,n-2 \\ 3\ ; & otherwise \end{cases} .; i\in\mathbb{Z}_n$$

where the pages obtained from each entry $x\in\mathbb{Z}_n$ such that $M_{f_5}(a,b)=x$ is isomorphic to $C_{11}(\underbrace{0,\dots,0}_{10\text{-times}},n-10)$ as follows: $(4+x)_0,(6+x)_1,(n-4+x)_0,(4+x)_1,(x)_0,(x)_1,(1+x)_0,(2+x)_1,(5+x)_0,(n-4+x)_1,\ (3+x)_0$, and $(3+x)_0,(\alpha+x)_1$ where $9\le\alpha\le n-1$.

Then the pages y is isomorphic to $C_{11}(\underbrace{0,\ldots,0}_{10\text{-times}},n-10)$ with the following vertices $(ni+4+x)_0,(nj+6+x)_1,(n(1+i)-4+x)_0,(nj+4+x)_1,(ni+x)_0,(nj+x)_1,\ (ni+1+x)_0,(nj+2+x)_1,(ni+5+x)_0,(n(1+j)-4+x)_1,(ni+3+x)_0$, and $(ni+3+x)_0,(nj+\alpha+x)_1$ where $nj+9\leq\alpha\leq n(1+j)-1$.

As mentioned in the above theorem there exist *ODC* of $K_{qn,qn}$ by $qC_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}},n-m)$ and we proved it in six cases. So, we will give examples for each of these previous cases where $q=3$ every time as follows.

For $m=5$, $n=7$, and in case 1 described by the example 1, then the combination has two squares of order 21 from equation 4 as follows

$$S_{f_1}=\begin{bmatrix}
0&5&0&5&5&3&3&7&12&7&12&12&10&10&14&19&14&19&19&17&17\\
4&1&6&1&6&6&4&11&8&13&8&13&13&11&18&15&20&15&20&20&18\\
5&5&2&0&2&0&0&12&12&9&7&9&7&7&19&19&16&14&16&14&14\\
1&6&6&3&1&3&1&8&13&13&10&8&10&8&15&20&20&17&15&17&15\\
2&2&0&0&4&2&4&9&9&7&7&11&9&11&16&16&14&14&18&16&18\\
5&3&3&1&1&5&3&12&10&10&8&8&12&10&19&17&17&15&15&19&17\\
4&6&4&4&2&2&6&11&13&11&11&9&9&13&18&20&18&18&16&16&20\\
7&12&7&12&12&10&10&14&19&14&19&19&17&17&0&5&0&5&5&3&3\\
11&8&13&8&13&13&11&18&15&20&15&20&20&18&4&1&6&1&6&6&4\\
12&12&9&7&9&7&7&19&19&16&14&16&14&14&5&5&2&0&2&0&0\\
8&13&13&10&8&10&8&15&20&20&17&15&17&15&1&6&6&3&1&3&1\\
9&9&7&7&11&9&11&16&16&14&14&18&16&18&2&2&0&0&4&2&4\\
12&10&10&8&8&12&10&19&17&17&15&15&19&17&5&3&3&1&1&5&3\\
11&13&11&11&9&9&13&18&20&18&18&16&16&20&4&6&4&4&2&2&6\\
14&19&14&19&19&17&17&0&5&0&5&5&3&3&7&12&7&12&12&10&10\\
18&15&20&15&20&20&18&4&1&6&1&6&6&4&11&8&13&8&13&13&11\\
19&19&16&14&16&14&14&5&5&2&0&2&0&0&12&12&9&7&9&7&7\\
15&20&20&17&15&17&15&1&6&6&3&1&3&1&8&13&13&10&8&10&8\\
16&16&14&14&18&16&18&2&2&0&0&4&2&4&9&9&7&7&11&9&11\\
19&17&17&15&15&19&17&5&3&3&1&1&5&3&12&10&10&8&8&12&10\\
18&20&18&18&16&16&20&4&6&4&4&2&2&6&11&13&11&11&9&9&13
\end{bmatrix}$$

$$S^*_{f_1}=\begin{bmatrix}
0&5&0&5&5&3&3&7&12&7&12&12&10&10&14&19&14&19&19&17&17\\
4&1&6&1&6&6&4&11&8&13&8&13&13&11&18&15&20&15&20&20&18\\
5&5&2&0&2&0&0&12&12&9&7&9&7&7&19&19&16&14&16&14&14\\
1&6&6&3&1&3&1&8&13&13&10&8&10&8&15&20&20&17&15&17&15\\
2&2&0&0&4&2&4&9&9&7&7&11&9&11&16&16&14&14&18&16&18\\
5&3&3&1&1&5&3&12&10&10&8&8&12&10&19&17&17&15&15&19&17\\
4&6&4&4&2&2&6&11&13&11&11&9&9&13&18&20&18&18&16&16&20\\
7&12&7&12&12&10&10&14&19&14&19&19&17&17&0&5&0&5&5&3&3\\
11&8&13&8&13&13&11&18&15&20&15&20&20&18&4&1&6&1&6&6&4\\
12&12&9&7&9&7&7&19&19&16&14&16&14&14&5&5&2&0&2&0&0\\
8&13&13&10&8&10&8&15&20&20&17&15&17&15&1&6&6&3&1&3&1\\
9&9&7&7&11&9&11&16&16&14&14&18&16&18&2&2&0&0&4&2&4\\
12&10&10&8&8&12&10&19&17&17&15&15&19&17&5&3&3&1&1&5&3\\
11&13&11&11&9&9&13&18&20&18&18&16&16&20&4&6&4&4&2&2&6\\
14&19&14&19&19&17&17&0&5&0&5&5&3&3&7&12&7&12&12&10&10\\
18&15&20&15&20&20&18&4&1&6&1&6&6&4&11&8&13&8&13&13&11\\
19&19&16&14&16&14&14&5&5&2&0&2&0&0&12&12&9&7&9&7&7\\
15&20&20&17&15&17&15&1&6&6&3&1&3&1&8&13&13&10&8&10&8\\
16&16&14&14&18&16&18&2&2&0&0&4&2&4&9&9&7&7&11&9&11\\
19&17&17&15&15&19&17&5&3&3&1&1&5&3&12&10&10&8&8&12&10\\
18&20&18&18&16&16&20&4&6&4&4&2&2&6&11&13&11&11&9&9&13
\end{bmatrix}$$

For $m=6$, $n=8$ there exist two orthogonal $C_7(\underbrace{0,\ldots,0}_{7\ times},2)$-squares of order 8 in case 2 defined as follows

$$M_{f_2}=\begin{bmatrix}
6&1&0&4&5&6&0&1\\
2&7&2&1&5&6&7&1\\
2&3&0&3&2&6&7&0\\
1&3&4&1&4&3&7&0\\
1&2&4&5&2&5&4&0\\
1&2&3&5&6&3&6&5\\
6&2&3&4&6&7&4&7\\
0&7&3&4&5&7&0&5
\end{bmatrix},M^T_{f_2}=\begin{bmatrix}
6&2&2&1&1&1&6&0\\
1&7&3&3&2&2&2&7\\
0&2&0&4&4&3&3&3\\
4&1&3&1&5&5&4&4\\
5&5&2&4&2&6&6&5\\
6&6&6&3&5&3&7&7\\
0&7&7&7&4&6&4&0\\
1&1&0&0&0&5&7&5
\end{bmatrix}.$$

then the combination has two squares of order 24 from equation 4.

For $m=7$, $n=9$ there exist two orthogonal $C_8(\underbrace{0,\ldots,0}_{7\ times},2)$-squares of order 9 in case 3 defined as follows

$$M_{f_3}=\begin{bmatrix}
0&8&7&0&7&7&3&3&8\\
0&1&0&8&1&8&8&4&4\\
5&1&2&1&0&2&0&0&5\\
6&6&2&3&2&1&3&1&1\\
2&7&7&3&4&3&2&4&2\\
3&3&8&8&4&5&4&3&5\\
6&4&4&0&0&5&6&5&4\\
5&7&5&5&1&1&6&7&6\\
7&6&8&6&6&2&2&7&8
\end{bmatrix},M^T_{f_3}=\begin{bmatrix}
0&0&5&6&2&3&6&5&7\\
8&1&1&6&7&3&4&7&6\\
7&0&2&2&7&8&4&5&8\\
0&8&1&3&3&8&0&5&6\\
7&1&0&2&4&4&0&1&6\\
7&8&2&1&3&5&5&1&2\\
3&8&0&3&2&4&6&6&2\\
3&4&0&1&4&3&5&7&7\\
8&4&5&1&2&5&4&6&8
\end{bmatrix}.$$

then the combination has two squares of order 27 from equation 4.

For $m=8$, $n=10$ there exist two orthogonal $C_9(\underbrace{0,\ldots,0}_{7\ times},2)$-squares of order 10 in case 4 defined as follows

$$M_{f_4}=\begin{bmatrix}
0&6&0&7&8&8&8&7&6&4\\
5&1&7&1&8&9&9&9&8&7\\
8&6&2&8&2&9&0&0&0&9\\
0&9&7&3&9&3&0&1&1&1\\
2&1&0&8&4&0&4&1&2&2\\
3&3&2&1&9&5&1&5&2&3\\
4&4&4&3&2&0&6&2&6&3\\
4&5&5&5&4&3&1&7&3&7\\
8&5&6&6&6&5&4&2&8&4\\
5&9&6&7&7&7&6&5&3&9
\end{bmatrix},M^T_{f_4}=\begin{bmatrix}
0&5&8&0&2&3&4&4&8&5\\
6&1&6&9&1&3&4&5&5&9\\
0&7&2&7&0&2&4&5&6&6\\
7&1&8&3&8&1&3&5&6&7\\
8&8&2&9&4&9&2&4&6&7\\
8&9&9&3&0&5&0&3&5&7\\
8&9&0&0&4&1&6&1&4&6\\
7&9&0&1&1&5&2&7&2&5\\
6&8&0&1&2&2&6&3&8&3\\
4&7&9&1&2&3&3&7&4&9
\end{bmatrix}.$$

then the combination has two squares of order 30 from equation 4.

For $m=9$, $n=11$ there exist two orthogonal $C_{10}(\underbrace{0,\ldots,0}_{8\ times},2)$-squares of order 11 in case 5 defined as follows

$$M_{f_5} = \begin{bmatrix} 0 & 10 & 7 & 9 & 0 & 9 & 9 & 3 & 3 & 7 & 10 \\ 0 & 1 & 0 & 8 & 10 & 1 & 10 & 10 & 4 & 4 & 8 \\ 9 & 1 & 2 & 1 & 9 & 0 & 2 & 0 & 0 & 5 & 5 \\ 6 & 10 & 2 & 3 & 2 & 10 & 1 & 3 & 1 & 1 & 6 \\ 7 & 7 & 0 & 3 & 4 & 3 & 0 & 2 & 4 & 2 & 2 \\ 3 & 8 & 8 & 1 & 4 & 5 & 4 & 1 & 3 & 5 & 3 \\ 4 & 4 & 9 & 9 & 2 & 5 & 6 & 5 & 2 & 4 & 6 \\ 7 & 5 & 5 & 10 & 10 & 3 & 6 & 7 & 6 & 3 & 5 \\ 6 & 8 & 6 & 6 & 0 & 0 & 4 & 7 & 8 & 7 & 4 \\ 5 & 7 & 9 & 7 & 7 & 1 & 1 & 5 & 8 & 9 & 8 \\ 9 & 6 & 8 & 10 & 8 & 8 & 2 & 2 & 6 & 9 & 10 \end{bmatrix},$$

$$M_{f_5}^T = \begin{bmatrix} 0 & 0 & 9 & 6 & 7 & 3 & 4 & 7 & 6 & 5 & 9 \\ 10 & 1 & 1 & 10 & 7 & 8 & 4 & 5 & 8 & 7 & 6 \\ 7 & 0 & 2 & 2 & 0 & 8 & 9 & 5 & 6 & 9 & 8 \\ 9 & 8 & 1 & 3 & 3 & 1 & 9 & 10 & 6 & 7 & 10 \\ 0 & 10 & 9 & 2 & 4 & 4 & 2 & 10 & 0 & 7 & 8 \\ 9 & 1 & 0 & 10 & 3 & 5 & 5 & 3 & 0 & 1 & 8 \\ 9 & 10 & 2 & 1 & 0 & 4 & 6 & 6 & 4 & 1 & 2 \\ 3 & 10 & 0 & 3 & 2 & 1 & 5 & 7 & 7 & 5 & 2 \\ 3 & 4 & 0 & 1 & 4 & 3 & 2 & 6 & 8 & 8 & 6 \\ 7 & 4 & 5 & 1 & 2 & 5 & 4 & 3 & 7 & 9 & 9 \\ 10 & 8 & 5 & 6 & 2 & 3 & 6 & 5 & 4 & 8 & 10 \end{bmatrix}.$$

then the combination has two squares of order 33 from equation 4.

For $m = 10$, $n = 12$ there exist two orthogonal $C_{11}(\underbrace{0,\ldots,0}_{9\ times},2)$-squares of order 12 in case 6 defined as follows

$$M_{f_6} = \begin{bmatrix} 0 & 11 & 8 & 7 & 0 & 9 & 9 & 9 & 4 & 7 & 4 & 11 \\ 0 & 1 & 0 & 9 & 8 & 1 & 10 & 10 & 10 & 5 & 8 & 5 \\ 6 & 1 & 2 & 1 & 10 & 9 & 2 & 11 & 11 & 11 & 6 & 9 \\ 10 & 7 & 2 & 3 & 2 & 11 & 10 & 3 & 0 & 0 & 0 & 7 \\ 8 & 11 & 8 & 3 & 4 & 3 & 0 & 11 & 4 & 1 & 1 & 1 \\ 2 & 9 & 0 & 9 & 4 & 5 & 4 & 1 & 0 & 5 & 2 & 2 \\ 3 & 3 & 10 & 1 & 10 & 5 & 6 & 5 & 2 & 1 & 6 & 3 \\ 4 & 4 & 4 & 11 & 2 & 11 & 6 & 7 & 6 & 3 & 2 & 7 \\ 8 & 5 & 5 & 5 & 0 & 3 & 0 & 7 & 8 & 7 & 4 & 3 \\ 4 & 9 & 6 & 6 & 6 & 1 & 4 & 1 & 8 & 9 & 8 & 5 \\ 6 & 5 & 10 & 7 & 7 & 7 & 2 & 5 & 2 & 9 & 10 & 9 \\ 10 & 7 & 6 & 11 & 8 & 8 & 8 & 3 & 6 & 3 & 10 & 11 \end{bmatrix},$$

$$M_{f_6}^T = \begin{bmatrix} 0 & 0 & 6 & 10 & 8 & 2 & 3 & 4 & 8 & 4 & 6 & 10 \\ 11 & 1 & 1 & 7 & 11 & 9 & 3 & 4 & 5 & 9 & 5 & 7 \\ 8 & 0 & 2 & 2 & 8 & 0 & 10 & 4 & 5 & 6 & 10 & 6 \\ 7 & 9 & 1 & 3 & 3 & 9 & 1 & 11 & 5 & 6 & 7 & 11 \\ 0 & 8 & 10 & 2 & 4 & 4 & 10 & 2 & 0 & 6 & 7 & 8 \\ 9 & 1 & 9 & 11 & 3 & 5 & 5 & 11 & 3 & 1 & 7 & 8 \\ 9 & 10 & 2 & 10 & 0 & 4 & 6 & 6 & 0 & 4 & 2 & 8 \\ 9 & 10 & 11 & 3 & 11 & 1 & 5 & 7 & 7 & 1 & 5 & 3 \\ 4 & 10 & 11 & 0 & 4 & 0 & 2 & 6 & 8 & 8 & 2 & 6 \\ 7 & 5 & 11 & 0 & 1 & 5 & 1 & 3 & 7 & 9 & 9 & 3 \\ 4 & 8 & 6 & 0 & 1 & 2 & 6 & 2 & 4 & 8 & 10 & 10 \\ 11 & 5 & 9 & 7 & 1 & 2 & 3 & 7 & 3 & 5 & 9 & 11 \end{bmatrix}.$$

then the combination has two squares of order 36 from equation 4.

3 Conclusion

In this paper, we got a larger ODC of $K_{qn,qn}$ by making combination of two $ODCs$ first one is a caterpillar of $K_{n,n}$ and the second one is the one factorization ODC of Latin squares of $K_{q,q}$ and we proved six cases of ODC of $K_{qn,qn}$ by $qC_{m+1}(\underbrace{0,\ldots,0}_{m\text{-times}},n-m)$; where $5 \leq m \leq 10$.

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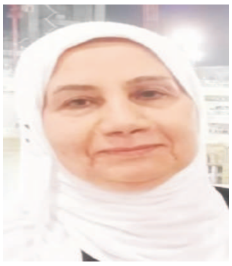

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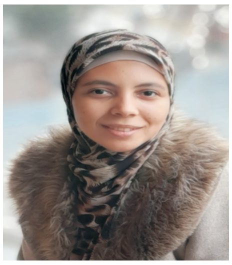

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