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Hadamard-Fejér type inequalities related to φ_m-convex function

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Abstract: We will bring up a new generalization φ_m-convex function for the convex function. We give some basic properties for this notion. Furthermore, we set down proofs of Hermite-Hadamard type and Hermite-Hadamard-Fejér type integral inequalities for this notion.

Keywords: Convex function, $\varphi-$convex function, $m-$convex function, φ_m-convex function, Hermite-Hadamard type inequalities and Hermite-Hadamard-Fejér type integral inequalities.

1 Introduction

Throughout the paper, we will use the symbol "κ" for convex function (and its generalizations). Let $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function of two real variables unless we shall specify otherwise.

In the present section, we give some basic definitions and inequalities, which already exist in the literature, we use them through the paper. In Section 2, we investigate four forms of already defined $\varphi-$convex function [6]. By using one of the four forms and $m-$convex function [6], we introduce our new notion φ_m-convex function, which is a generalization of convex, $\phi-$convex and $m-$convex function. Let's see later. The remaining sections are clear by their title.

The following definition [1, 2], is the base of the literature:
$\kappa : A \subset \mathbb{R} \to \mathbb{R}$ is known as ***convex function*** if,

$$\kappa(ru+(1-r)v) \leq r\kappa(u)+(1-r)\kappa(v) \tag{1}$$

for every $u, v \in A$ *and* $r \in [0,1]$.

An inequality [3, 4], which is very basic and fundamental for the literature:
If $\kappa : A \subset \mathbb{R} \to \mathbb{R}$ is convex function and $p, q \in A$ with $p < q$. Then

$$\kappa\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q \kappa(u)\,\mathrm{d}u \leq \frac{\kappa(p)+\kappa(q)}{2} \tag{2}$$

is called ***Hermite-Hadamard inequality***.

Another inequality [5], which is the generalization of above inequality (2) was derived in the year 1905 by Leopold Fejér, as the following:
If $\kappa : [p,q] \subset \mathbb{R} \to \mathbb{R}$ is a convex function, and $\chi : [p,q] \to \mathbb{R}$ is symmetric about $\frac{p+q}{2}$, integrable and non-negative. Then

$$\kappa\left(\frac{p+q}{2}\right)\int_p^q \chi(u)\,\mathrm{d}u \leq \frac{1}{q-p}\int_p^q \kappa(u)\chi(u)\,\mathrm{d}u \leq \frac{\kappa(p)+\kappa(q)}{2}\int_p^q \chi(u)\,\mathrm{d}u \tag{3}$$

is known as ***Hermite-Hadamard-Fejér inequality***.

G. Toader [6] , generalize the convex function as $m-$convex function, in the year 1984, as the following:
$\kappa : [0,q) \subset \mathbb{R} \to \mathbb{R}$, $q > 0$ be an $m-$***convex function*** if,

$$\kappa(ru+m(1-r)v) \leq r\kappa(u)+m(1-r)\kappa(v) \tag{4}$$

holds for every $u, v \in [0,q)$ and $r, m \in [0,1]$.

M. Eshaghi Gordji, M. Rostamian Delavar, M. De La Sen [7], generalize convex function as $\varphi-$convex function, in the year 2016, as the following:
let $A \subseteq \mathbb{R}$ and $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function of two real variables then, a function $\kappa : A \to \mathbb{R}$ is called $\varphi-$***convex*** if,

$$\kappa(ru+(1-r)v) \leq \kappa(v)+r\varphi\Big(\kappa(u),\kappa(v)\Big) \tag{5}$$

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for every $u,v \in A$ and $r \in [0,1]$. Note that $\eta-$convex function [10] and $\varphi-$convex function [8] are the same notions. So we can also termed φ_m-convex function as η_m-convex function.

2 Basic definitions.

In this section, we investigate different forms of $\varphi-$convex function. One of these forms is required for our new generalization. We make some important remarks and examples for our new notion φ_m-convex function.

Now we give the following four forms of $\varphi-$convex function, or it can be defined in the following four different ways.

Definition 1. *Let $A \subseteq \mathbb{R}$, then a function $\kappa : A \to \mathbb{R}$ is called $\varphi-$**convex** if,*

$$\kappa(ru+(1-r)v) \leq r\kappa(u)+(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big) \quad (6)$$

$$\kappa(ru+(1-r)v) \leq (1-r)\kappa(v)+r\varphi\Big(\kappa(u),\kappa(v)\Big) \quad (7)$$

$$\kappa(ru+(1-r)v) \leq \kappa(v)+r\varphi\Big(\kappa(u),\kappa(v)\Big) \quad (8)$$

$$\kappa(ru+(1-r)v) \leq \kappa(u)+(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big) \quad (9)$$

for every $r \in [0,1]$ and for every $u,v \in A$. Above inequality (8) is defined in [7].

The above definitions will become classical convex functions if we take:
$\varphi(u,v) = v$ in (6)
$\varphi(u,v) = u$ in (7)
$\varphi(u,v) = u-v$ in (8)
$\varphi(u,v) = v-u$ in (9).

Remark. All four definitions are similar to each other. Let's see. If we set $\varphi(u,v) = \eta(u,v)+\kappa(u)$ in the inequality (6), we get inequality (9) and if we set $\varphi(u,v) = \eta(u,v)+\kappa(v)$ in inequality (7), we get inequality (8), where $\eta(u,v)$ is another function of two real variables.

Definition 2. *If we take equalities in place of inequalities in Definition 1, we get $\varphi-$**affine functions**, for all $r,u,v \in \mathbb{R}$. Clearly, we can also get classical affine functions.*

We give one example to illustrate the $\varphi-$convex function (6).

Example 1. Let $\kappa(u) = u^2$ which is convex. If $\varphi(u,v) = 2v+u$, then κ is $\varphi-$ convex.

Solution.

$$\begin{aligned}\kappa(ru+(1-r)v) &= (ru+(1-r)v)^2\\ &= r^2u^2+(1-r)^2v^2+r(1-r)2uv\\ &\leq ru^2+(1-r)v^2+(1-r)(u^2+v^2)\\ &= ru^2+(1-r)(u^2+2v^2)\\ &= r\kappa(u)+(1-r)\varphi(\kappa(u),\kappa(v))\end{aligned}$$

which shows κ is $\varphi-$convex.

Now we give different forms of the $\varphi-$quasi convex function.

Definition 3. *Let $A \subseteq \mathbb{R}$, then a function $\kappa : A \to \mathbb{R}$ is called $\varphi-$**quasi convex** if,*

$$\kappa(ru+(1-r)v) \leq max\Big\{\kappa(u),\varphi\Big(\kappa(u),\kappa(v)\Big)\Big\} \quad (10)$$

$$\kappa(ru+(1-r)v) \leq max\Big\{\kappa(v),\varphi\Big(\kappa(u),\kappa(v)\Big)\Big\} \quad (11)$$

$$\kappa(ru+(1-r)v) \leq max\Big\{\kappa(v),\kappa(v)+\varphi\Big(\kappa(u),\kappa(v)\Big)\Big\} \quad (12)$$

$$\kappa(ru+(1-r)v) \leq max\Big\{\kappa(u),\kappa(u)+\varphi\Big(\kappa(u),\kappa(v)\Big)\Big\} \quad (13)$$

for every $r \in [0,1]$ and for every $u,v \in A$.

The above definitions will become classical quasi convex function if we take:
$\varphi(u,v) = v$ in (10)
$\varphi(u,v) = u$ in (11)
$\varphi(u,v) = u-v$ in (12)
$\varphi(u,v) = v-u$ in (13).

Definition 4. *If we reverse the inequalities in Definition 1 and Definition 3 then we get $\varphi-$**concave** and $\varphi-$**quasi concave** functions.*

Through the rest of this paper, let $[0,q] = I \subset \mathbb{R}, q > 0, [0,+\infty) = J \subset \mathbb{R}$ and $m,r \in [0,1]$, unless we specify otherwise.

Now we come to our main concern and construct the φ_m-convex function.

Definition 5. *$\kappa : I \to \mathbb{R}$ is φ_m-**convex function** with respect to non-negative φ if,*

$$\kappa(ru+m(1-r)v) \leq r\kappa(u)+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big) \quad (14)$$

for every $u,v \in I$ and for every $r \in (0,1)$.

We are denoting the set of all φ_m-convex functions as a class $C_{\varphi_m}(q)$.

If we choose $\varphi(u,v) = v$, we come to $m-$convexity (4).

If we choose $m = 1$, we come to $\varphi-$convexity (6) (actually $\varphi-$convexity [8]) for the interval I.

Definition 6. *If we reverse the inequality in Definition 5, then we get φ_m-**concave function**.*

We give one example to illustrate our φ_m-convex function.

Example 2. let $\kappa(u) = u^2$ which is convex. If $\varphi(u,v) = 2v+u$ then, κ is φ_m- convex.

Solution.

$$
\begin{aligned}
\kappa(ru+m(1-r)v) & \\
&= (ru+m(1-r)v)^2 \\
&= r^2u^2+m^2(1-r)^2v^2+rm(1-r)2uv \\
&\leq ru^2+m(1-r)v^2+m(1-r)(u^2+v^2) \\
&= ru^2+m(1-r)(u^2+2v^2) \\
&= r\kappa(u)+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big).
\end{aligned}
$$

Which shows κ is φ_m- convex.

3 Some properties for φ_m-convex function

In the present section, we shall give some basic properties for our notion of the φ_m- convex function. We first give various conditions for the function φ. We use these concepts often in our results.

Definition 7. *We say that φ is,*

(i) additive if, $\varphi(u_1,v_1)+\varphi(u_2,v_2)=\varphi(u_1+u_2,v_1+v_2)$ for all $u_1,u_2,v_1,v_2\in\mathbb{R}$.

(ii) non-negatively homogeneous if, $\varphi(\beta u,\beta v)=\beta\varphi(u,v)$ for all $u,v\in\mathbb{R}$ and $\beta\geq 0$.

(iii) non-negatively linear if, it satisfies conditions (ii) and (i).

The following is a trivial fact of calculus for the function of two variables:

Let limit of u_n and v_n exists in $\mathbb{R}$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ is continuous, then

$$\lim_{n\to\infty} f(u_n,v_n)=f(\lim_{n\to\infty}u_n,\lim_{n\to\infty}v_n).$$

The incoming results and corresponding proofs are mostly inspired by [7,8].

Proposition 1. *Consider $\kappa,\chi:I\to\mathbb{R}$ be two φ_m-convex functions and φ is additive, then $\kappa+\chi:I\to\mathbb{R}$ will be φ_m-convex function.*

Proof.

$$
\begin{aligned}
&(\kappa+\chi)(ru+m(1-r)v) \\
&= \Big[\kappa(ru+m(1-r)v)\Big]+\big[\chi(ru+m(1-r)v)\big] \\
&\leq \Big[r\kappa(u)+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big)\Big]+ \\
&\Big[r\chi(u)+m(1-r)\varphi\Big(\chi(u),\chi(v)\Big)\Big] \\
&= r\Big[\kappa(u)+\chi(u)\Big]+ \\
&m(1-r)\Big[\varphi\Big(\kappa(u),\kappa(v)\Big)+\varphi\Big(\chi(u),\chi(v)\Big)\Big] \\
&= r\Big[(\kappa+\chi)(u)\Big]+ \\
&m(1-r)\Big[\varphi\Big(\kappa(u)+\chi(u),\kappa(v)+\chi(v)\Big)\Big] \\
&= r\Big[(\kappa+\chi)(u)\Big]+ \\
&m(1-r)\Big[\varphi\Big((\kappa+\chi)(u),(\kappa+\chi)(v)\Big)\Big].
\end{aligned}
$$

Hence the result.

Proposition 2. *The function $\beta\kappa:I\to\mathbb{R}$ is φ_m-convex for any $\beta\geq 0$, if $\kappa:I\to\mathbb{R}$ be φ_m-convex function and φ is non negatively homogeneous function.*

Proof.

$$
\begin{aligned}
&(\beta\kappa)(ru+m(1-r)v) \\
&= \beta\left[\kappa(ru+m(1-r)v)\right] \\
&\leq \beta\Big[r\kappa(u)+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big)\Big] \\
&= r(\beta\kappa)(u)+m(1-r)\beta\varphi\Big(\kappa(u),\kappa(v)\Big) \\
&= r(\beta\kappa)(u)+m(1-r)\varphi\Big((\beta\kappa)(u),(\beta\kappa)(v)\Big).
\end{aligned}
$$

Hence the result.

Theorem 1. *The function $\kappa=\sum_{i=1}^{n}\beta_i\kappa_i:I\to\mathbb{R}$ is φ_m-convex for $\beta_i\geq 0, i=1,2,\ldots,n$ if $\kappa_i:I\to\mathbb{R}, i=1,2,\ldots,n$ be φ_m-convex functions, such that φ is non-negatively linear.*

Proof.

$$
\begin{aligned}
&(\sum_{i=1}^{n}\beta_i\kappa_i)(ru+m(1-r)v) \\
&= \sum_{i=1}^{n}[\beta_i\{\kappa_i(ru+m(1-r)v)\}] \\
&\leq \sum_{i=1}^{n}\Big[\beta_i\Big(r\kappa_i(u)+m(1-r)\varphi\Big(\kappa_i(u),\kappa_i(v)\Big)\Big)\Big] \\
&= r\sum_{i=1}^{n}\beta_i\kappa_i(u)+m(1-r)\sum_{i=1}^{n}\beta_i\varphi\Big(\kappa_i(u),\kappa_i(v)\Big) \\
&= r\sum_{i=1}^{n}\beta_i\kappa_i(u)+m(1-r)\varphi\Big(\sum_{i=1}^{n}\beta_i\kappa_i(u),\sum_{i=1}^{n}\beta_i\kappa_i(v)\Big).
\end{aligned}
$$

Hence the result.

Proposition 3. *The function $\kappa := \max_{i=1}^{n} \kappa_i$ is φ_m-convex function if, $\kappa_i : J \to \mathbb{R}, i = 1,2,...,n$ are all φ_m-convex functions.*

Proof.

$$\begin{aligned}
&\kappa_i(ru+m(1-r)v)\\
&\leq r\kappa_i(u)+m(1-r)\varphi(\kappa_i(u),\kappa_i(v))\\
&\leq r\max_{i=1}^{n}\kappa_i(u)+m(1-r)\varphi\Big(\max_{i=1}^{n}\kappa_i(u),\max_{i=1}^{n}\kappa_i(v)\Big),\\
&= r\kappa(u)+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big).
\end{aligned}$$

Hence the result.

Proposition 4. *Let $\kappa_n : J \to \mathbb{R}$ be a sequence of φ_m-convex functions, φ is continuous and for all $n \geq N \in \mathbb{N}, \kappa_n(u) \to \kappa(u)$, (on J), then κ is φ_m-convex function.*

Proof. Each κ_n is φ_m-convex function implies, for all $u, v \in J$

$$\begin{aligned}
&\kappa_n(ru+m(1-r)v) \leq r\kappa_n(u)\\
&+m(1-r)\varphi\Big(\kappa_n(u),\kappa_n(v)\Big)
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\kappa_n(ru+m(1-r)v) \leq r\lim_{n\to\infty}\kappa_n(u)\\
&+m(1-r)\lim_{n\to\infty}\varphi\Big(\kappa_n(u),\kappa_n(v)\Big)
\end{aligned}$$

$$\begin{aligned}
&\kappa(ru+m(1-r)v) \leq r\kappa(u)\\
&+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big).
\end{aligned}$$

Hence the result.

Proposition 5. *The function $\chi o\kappa$ will be φ_m-convex if, $\kappa : I \to \mathbb{R}$ be $m-$convex function and $\chi : A \subseteq \kappa(I) \to \mathbb{R}$ be non decreasing φ_m-convex function.*

Proof.

$$\begin{aligned}
&\kappa(ru+m(1-r)v) \leq r\kappa(u)+m(1-r)\kappa(v)\\
&\chi\big[\kappa(ru+m(1-r)v)\big] \leq \chi\big[r\kappa(u)\\
&+m(1-r)\kappa(v)\big]
\end{aligned}$$

$$\begin{aligned}
&(\chi o\kappa)\big(ru+m(1-r)v\big) \leq r\chi\big[\kappa(u)\big]\\
&+m(1-r)\varphi\Big(\chi\big[\kappa(u)\big],\chi\big[\kappa(v)\big]\Big)
\end{aligned}$$

$$\begin{aligned}
&(\chi o\kappa)\big(ru+m(1-r)v\big) \leq r(\chi o\kappa)(u)\\
&+m(1-r)\varphi\Big((\chi o\kappa)(u),(\chi o\kappa)(v)\Big).
\end{aligned}$$

Hence the result.

4 Hermite-Hadamard type inequalities.

The incoming theorems follow ideas from [7–9]. The following theorem gives boundedness of φ_m-convex function and will be used in Theorem 4.

Theorem 2. *The function κ will be bounded on $[p,q] \subseteq I$ for $p,q \in I$ with $p < q$ if, $\kappa : I \to \mathbb{R}$ be φ_m-convex function, such that φ is bounded from above on $\kappa(I) \times \kappa(I)$.*

Proof. Firstly,
As every $u \in [p,q]$ can be represented as $u = rp+(1-r)q$ for $r \in [0,1]$, then

$$\begin{aligned}
&\kappa(rp+(1-r)q)\\
&\qquad \leq r\kappa(p)+m(1-r)\varphi\Big(\kappa(p),\kappa\Big(\frac{q}{m}\Big)\Big)\\
&\qquad \leq \kappa(p)+M_\varphi
\end{aligned}$$

Which is upper bound for κ and M_φ is upper bound for function φ.
Secondly, Let

$$\begin{aligned}
&\frac{p+q}{2}-r = u \in [p,q], \text{ then}\\
&\kappa\Big(\frac{p+q}{2}\Big) = \kappa\Big(\frac{p+q}{4}+\frac{r}{2}+\frac{p+q}{4}-\frac{r}{2}\Big)\\
&= \kappa\Big(\frac{1}{2}\Big[\frac{p+q}{2}-r\Big]+\frac{1}{2}\Big[\frac{p+q}{2}+r\Big]\Big)\\
&\leq \frac{1}{2}\kappa\Big(\frac{p+q}{2}-r\Big)\\
&+\Big(\frac{m}{2}\Big)\varphi\Bigg(\kappa\Big(\frac{p+q}{2}-r\Big),\kappa\Bigg(\frac{\frac{p+q}{2}+r}{m}\Bigg)\Bigg)\\
&\leq \frac{1}{2}\kappa\Big(\frac{p+q}{2}-r\Big)+\Big(\frac{m}{2}\Big)M_\varphi\\
&\kappa\Big(\frac{p+q}{2}\Big)-\Big(\frac{m}{2}\Big)M_\varphi \leq \frac{1}{2}\kappa\Big(\frac{p+q}{2}-r\Big)\\
&2\kappa\Big(\frac{p+q}{2}\Big)-mM_\varphi \leq \kappa(u),
\end{aligned}$$

which is lower bound for κ and M_φ is upper bound for function φ.
Hence κ is bounded on $[p,q] \subseteq I$.

Now we are going to establish Hermite-Hadamard type inequalities.

Theorem 3. *If $\kappa \in L_1[p,q]$ and $\kappa : I \to \mathbb{R}$ be φ_m-convex function, such that φ is bounded from above on $\kappa(I) \times \kappa(I)$. For $p,q \in I$ with $p < q$, then we have*

$$\begin{aligned}
&\frac{1}{q-p}\int_p^q \kappa(u)\,\mathrm{d}u \leq \min\Bigg\{\\
&\kappa(p)\int_0^1 r\,\mathrm{d}r+m\varphi\Big(\kappa(p),\kappa\Big(\frac{q}{m}\Big)\Big)\int_0^1(1-r)\,\mathrm{d}r\ , \qquad (15)\\
&\kappa(q)\int_0^1 r\,\mathrm{d}r+m\varphi\Big(\kappa(q),\kappa\Big(\frac{p}{m}\Big)\Big)\int_0^1(1-r)\,\mathrm{d}r\Bigg\}.
\end{aligned}$$

*Proof.*Since κ is φ_m-convex, we have

$$\kappa(ru+m(1-r)v)\leq r\kappa(u)+m(1-r)\varphi\Big(\kappa(u),\kappa(v)\Big)$$

, which gives:

$$\kappa(rp+(1-r)q)\leq r\kappa(p)+m(1-r)\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)$$

and

$$\kappa(rq+(1-r)p)\leq r\kappa(q)+m(1-r)\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)$$

integrating on [0,1], we have

$$\int_0^1 \kappa(rp+(1-r)q)\,\mathrm{d}r \leq \kappa(p)\int_0^1 r\,\mathrm{d}r+m\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\int_0^1(1-r)\,\mathrm{d}r$$

and

$$\int_0^1 \kappa(rq+(1-r)p)\,\mathrm{d}r \leq \kappa(q)\int_0^1 r\,\mathrm{d}r+m\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)\int_0^1(1-r)\,\mathrm{d}r$$

this implies,

$$\frac{1}{q-p}\int_p^q \kappa(u)\,\mathrm{d}u\leq \kappa(p)\int_0^1 r\,\mathrm{d}r+m\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\int_0^1(1-r)\,\mathrm{d}r. \tag{16}$$

and

$$\frac{1}{q-p}\int_p^q \kappa(u)\,\mathrm{d}u\leq \kappa(q)\int_0^1 r\,\mathrm{d}r+m\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)\int_0^1(1-r)\,\mathrm{d}r. \tag{17}$$

However,

$$\int_0^1 \kappa(rp+(1-r)q)\,\mathrm{d}r=\int_0^1 \kappa(rq+(1-r)p)\,\mathrm{d}r =\frac{1}{q-p}\int_p^q \kappa(u)\,\mathrm{d}u$$

Hence we get Inequality (15).

Hermite-Hadamard type inequality for $m-$convex function in [9] will be obtain after putting $\varphi(u,v)=v$ in above Theorem 3.

Theorem 4.*If $\kappa\in L_1[p,q]$ and $\kappa: I\to\mathbb{R}$ be φ_m-convex function, such that φ is bounded from above on $\kappa(I)\times\kappa(I)$. For $p,q\in I$ with $p<q$, then we have*

$$2\kappa\left(\frac{p+q}{2}\right)-mM_\varphi\leq\frac{1}{q-p}\int_p^q\kappa(u)\mathrm{d}u\leq \frac{\kappa(p)+\kappa(q)}{2}\int_0^1 r\,\mathrm{d}r+\frac{m}{2}M_\varphi\int_0^1(1-r)\,\mathrm{d}r$$

, where M_φ is upper bound for function φ.

*Proof.*Firstly,
κ is bounded by Theorem 2.
Secondly,
Adding inequalities (16) and (17) we get,

$$\frac{1}{q-p}\int_p^q\kappa(u)\,\mathrm{d}u\leq\frac{\kappa(p)+\kappa(q)}{2}\int_0^1 r\,\mathrm{d}r+$$

$$\frac{m}{2}\left[\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)+\phi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)\right]\int_0^1(1-r)\,\mathrm{d}r$$

$$\leq\frac{\kappa(p)+\kappa(q)}{2}\int_0^1 r\,\mathrm{d}r+\frac{m}{2}M_\varphi\int_0^1(1-r)\,\mathrm{d}r$$

, where M_φ is upper bound for the function φ.
Thirdly,

For $p,\ q\in I$ we know $\frac{p+q}{2}\in I$

$$\kappa\left(\frac{p+q}{2}\right)=\kappa\left(\frac{rp+(1-r)q+(1-r)p+rq}{2}\right) =\kappa\left(\frac{1}{2}(rp+(1-r)q)+\frac{m}{2}\frac{(1-r)p+rq}{m}\right)$$

$$\leq\frac{1}{2}\kappa\Big(rp+(1-r)q\Big)+\left(\frac{m}{2}\right)\varphi\left(\kappa\Big(rp+(1-r)q\Big),\kappa\left(\frac{(1-r)p+rq}{m}\right)\right)$$

$$\leq\frac{1}{2}\kappa(rp+(1-r)q)+\left(\frac{m}{2}\right)M_\varphi$$

Where M_ϕ is upper bound for function φ.
On integrating in the interval $(0,1)$ we get,

$$\kappa\left(\frac{p+q}{2}\right)\leq\frac{1}{2(q-p)}\int_p^q\kappa(u)\,\mathrm{d}u+\left(\frac{m}{2}\right)M_\varphi$$

$$2\kappa\left(\frac{p+q}{2}\right)-mM_\varphi\leq\frac{1}{q-p}\int_p^q\kappa(u)\,\mathrm{d}u.$$

Hermite-Hadamard type inequality for the $\varphi-$convex function [7] defined on interval I will be obtain after putting $m=1$ in Theorem 4 and we obtain classical Hermite-Hadamard Inequality (2) after putting $m=1$, $\varphi(u,v)=v$, in Theorem 4.

5 Hermite-Hadamard-Fejér type inequalities.

Now, we prove Hermite-Hadamard-Fejér type inequalities, an inspiration from [8].

Theorem 5. *For $p,q \in [p,q]$ with $p < q$ and with $\kappa \in L_1([p,q])$. Suppose $\kappa : [p,q] \subset J \to J$ be φ_m-convex function, and $\chi : [p,q] \to \mathbb{R}$ is non-negative, integrable and symmetric about $\frac{p+q}{2}$, then we have*

$$\int_p^q \kappa(u)\chi(u)\,du \le \frac{\kappa(p)+\kappa(q)}{2}\int_p^q \left(\frac{q-u}{q-p}\right)\chi(u)\,du + \frac{m}{2}\left[\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)+\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\right] \int_p^q \left(\frac{u-p}{q-p}\right)\chi(u)\,du.$$

Proof. Since χ is integrable and symmetric about $\frac{p+q}{2}$ and κ and χ are real non-negative functions, then we get

$$\int_p^q \kappa(u)\chi(u)\,du = \frac{1}{2}\left[\int_p^q \kappa(u)\chi(u)du + \int_p^q \kappa(p+q-u)\chi(p+q-u)\,du\right]$$

$$= \frac{1}{2}\int_p^q [\kappa(u)+\kappa(p+q-u)]\chi(u)\,du$$

$$= \frac{1}{2}\int_p^q \left[\kappa\left(p\left(\frac{q-u}{q-p}\right) + q\left(\frac{u-p}{q-p}\right)\right) + \kappa\left(p\left(\frac{u-p}{q-p}\right)+q\left(\frac{q-u}{q-p}\right)\right)\right]\chi(u)\,du$$

$$\le \frac{1}{2}\int_p^q \left[\left(\frac{q-u}{q-p}\right)\kappa(p) + m\left(\frac{u-p}{q-p}\right)\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right) + m\left(\frac{u-p}{q-p}\right)\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right) + \left(\frac{q-u}{q-p}\right)\kappa(q)\right]\chi(u)\,du$$

$$= \frac{\kappa(p)+\kappa(q)}{2}\int_p^q \left(\frac{q-u}{q-p}\right)\chi(u)\,du + \frac{m}{2}\left[\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right) + \varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)\right]\int_p^q \left(\frac{u-p}{q-p}\right)\chi(u)\,du.$$

Hence we get our require result.

Hermite-Hadamard-Fejér type inequality for $m-$convex function [6], will be obtain after putting $\varphi(u,v) = v$. i.e;

$$\int_p^q \kappa(u)\chi(u)du \le \frac{\kappa(p)+\kappa(q)}{2}\int_p^q \left(\frac{q-u}{q-p}\right)\chi(u)\,du + \frac{m}{2}\left[\kappa\left(\frac{p}{m}\right)+\kappa\left(\frac{q}{m}\right)\right]\int_p^q \left(\frac{u-p}{q-p}\right)\chi(u)\,du.$$

Corollary 1. *With the same conditions of above Theorem 5 and if $\chi(u) = 1$, we have*

$$\frac{1}{q-p}\int_p^q \kappa(u)\,du \le \frac{1}{2}(\kappa(p)+\kappa(q)) + m\left[\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)+\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\right].$$

Proof. Putting $\chi(u) = 1$ in Theorem 5, we get

$$\int_p^q \kappa(u)\,du \le \frac{\kappa(p)+\kappa(q)}{2}\int_p^q \left(\frac{q-u}{q-p}\right)du + \frac{m}{2}\left[\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)+\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\right]\int_p^q \left(\frac{u-p}{q-p}\right)du.$$

then from Jensen Inequality,

$$\frac{1}{q-p}\int_p^q \kappa(u)\,du \le \frac{\kappa(p)+\kappa(q)}{2}\frac{1}{q-p}\int_p^q \left(\frac{q-u}{q-p}\right)du + \frac{m}{2}\left[\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right)+\varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\right]\frac{1}{q-p}\int_p^q \left(\frac{u-p}{q-p}\right)du$$

$$= \frac{\kappa(p)+\kappa(q)}{2} + \frac{m}{2}\left[\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right) + \varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\right]$$

$$= \frac{\kappa(p)+\kappa(q)}{2} + \frac{m}{2}\left[\varphi\left(\kappa(q),\kappa\left(\frac{p}{m}\right)\right) + \varphi\left(\kappa(p),\kappa\left(\frac{q}{m}\right)\right)\right],$$

which is right side of Hermite-Hadamard type inequality Theorem 4.

Theorem 6. *For $p,q \in [p,q]$ with $p < q$, $\kappa \in L_1([c,d])$, where $c = min\{\frac{p}{m}, p\}$, $d = max\{q, \frac{q}{m}\}$. Suppose $\kappa : [p,q] \subset J \to J$ be φ_m-convex function and $\chi : [p,q] \to \mathbb{R}$ is non-negative, integrable and symmetric* *about* $\frac{p+q}{2}$. *Then we get*

$$\kappa\left(\frac{p+q}{2}\right)\int_p^q \chi(u)\,\mathrm{d}u \le \left(\frac{1}{2}\right)\int_p^q \kappa(u)\chi(u)\,\mathrm{d}u + m\left(\frac{1}{2}\right)\int_p^q \varphi\left(\kappa(p+q-u),\kappa\left(\frac{u}{m}\right)\right)\chi(u)\,\mathrm{d}u.$$

*Proof.*We have,

$$\begin{aligned}
&\kappa\left(\frac{p+q}{2}\right)\int_p^q \chi(u)\,\mathrm{d}u \\
&= \int_p^q \kappa\left(\frac{p+q-u+u}{2}\right)\chi(u)\,\mathrm{d}u \\
&\le \int_p^q \left[\left(\frac{1}{2}\right)\kappa(p+q-u) + \left(\frac{m}{2}\right)\varphi\left(\kappa(p+q-u),\kappa\left(\frac{u}{m}\right)\right)\right]\chi(u)\,\mathrm{d}u \\
&= \left(\frac{1}{2}\right)\int_p^q \kappa(p+q-u)\chi(u)\mathrm{d}u + \left(\frac{m}{2}\right)\int_p^q \varphi\left(\kappa(p+q-u),\kappa\left(\frac{u}{m}\right)\right)\chi(u)\,\mathrm{d}u \\
&= \left(\frac{1}{2}\right)\int_p^q \kappa(p+q-u)\chi(p+q-u)\,\mathrm{d}u + \left(\frac{m}{2}\right)\int_p^q \varphi\left(\kappa(p+q-u),\kappa\left(\frac{u}{m}\right)\right)\chi(u)\,\mathrm{d}u \\
&= \left(\frac{1}{2}\right)\int_p^q \kappa(u)\chi(u)\,\mathrm{d}u + \left(\frac{m}{2}\right)\int_p^q \varphi\left(\kappa(p+q-u),\kappa\left(\frac{u}{m}\right)\right)\chi(u)\,\mathrm{d}u.
\end{aligned}$$

Hermite-Hadamard-Fejér type inequality for $m-$convex function [6] will be obtain after putting $\varphi(u,v)=v$, i.e:

$$\kappa\left(\frac{p+q}{2}\right)\int_p^q \chi(u)\,\mathrm{d}u \le \frac{1}{2}\int_p^q \kappa(u)\chi(u)\,\mathrm{d}u + \frac{m}{2}\int_p^q \kappa\left(\frac{u}{m}\right)\chi(u)\,\mathrm{d}u.$$

We may expect that the notions and results which, we have used in this paper may inspire and encourage the interested readers to derive and explore some new notions and results, in various fields of mathematics and other sciences.

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