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Dynamic Study of a Delayed Fractional-Order $\mathcal{SEIR}$ Epidemic Model with General Incidence and Treatment Functions

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Abstract: This study examines the global features of the $\mathcal{SEIR}$ epidemic model in its fractional-order version with time delay. General functions are considered to govern the infection transmission rate, and the rate at which diseased individuals are removed from the infected class. First, we form the proposed model in the Caputo case and perform fundamental mathematical analysis of the model solutions, such as checking for non-negativity and boundedness. The basic reproduction number $\mathcal{R}_0$ is then provided after computing the equilibrium points. Following that, sufficient criteria for the global stability of each equilibrium are checked using the relevant Lyapunov functions. It is shown that the characteristics of these general functions, along with the basic reproduction number $\mathcal{R}_0$, impact the model's global features. Finally, a numerical simulation is presented to show the viability and effectiveness of the derived analytical conclusions. According to the results, the system's enhanced dynamic behavior and larger stability regions in equilibria demonstrate the influence of incorporating the time delay and fractional-order.

Keywords: Fractional epidemic model; Time delay; General incidence; Global stability

1 Introduction

In our world, infectious diseases are a constant threat to humanity, bringing panic and disaster if not controlled. The emergence and re-emergence of infectious diseases has become a major global issue, motivating scientists from a variety of disciplines to work quickly to better understand the mechanisms of disease transmission. One of the most important areas of mathematical modelling is the study of disease propagation. Infectious disease modelling is a methodology used to investigate disease transmission mechanisms, forecast the path of future outbreaks, and identify the most effective control methods. Mathematical models are commonly employed in epidemiology to analyze the dynamics of diseases and epidemics. It seeks to capture key elements that influence the epidemic progression and can predict how a disease will disseminate across time and space [1,2,3,4,5,6].

As a result, many researchers have applied various forms of mathematical models to explore the dynamical behaviour of infectious diseases [7,8]. Several infectious diseases, including measles, SARS, AIDS, tuberculosis, and COVID-19, possess an incubation or latent phase when a person contracts an infection but is not yet contagious. This delay can be simulated by creating a separate group named exposed, in which the susceptible person stays for a specified period of time before migrating to the infected group [9,10,11,12].

Epidemic models with time delays have piqued researchers' interest. The need for time delays in the epidemic models stems from the need to reflect the effects of these factors (latent, incubation periods, temporary immunity). Several epidemic models have been developed with infectious force in the latent phase, since some models require a limited amount of time to collect information and take action. Adding some previous history to the system in the models is a more realistic approach and incorporating time delays into models is the best technique to simulate such processes. Some authors

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emphasize the significance of including a time delay in these epidemic models to consider the incubation period, which leads to the use of delay differential equations [9, 13, 14].

In recent decades, there has been a great deal of interest in applying fractional calculus theory to model biological systems mathematically. It has been shown that a fractional-order derivative is a useful tool in epidemiology. Fractional differentiation to an arbitrary order is more general than classical differentiation and integration. Because it naturally includes both nonlocal and memory effects, it's a good fit for modelling epidemic transmission. For these reasons, many scientists have begun to use fractional differential equations (FDEs) to study epidemiological models [15, 16, 17, 18, 19].

Time delay was recently introduced to a fractional-order epidemic model by numerous researchers [12, 20, 21]. Fractional derivatives and time delay in the model exhibit similar features. Hence, the combination of these two forms of memory in epidemiological models will certainly enhance the model dynamics. Recently, the authors in [22] investigated a fractional $\mathcal{SI}$ epidemiological model with time delay. The dynamics of $\mathcal{SIR}$ epidemiological model with a fractional-order were studied in [23] using time delay and generic incidence rate functionals. Furthermore, Ilhem et al [24] studied the qualitative behaviour of a fractional $\mathcal{SEIR}$ model that includes a time delay and a generic incidence rate function.

Following previous studies, we present in this work a delayed $\mathcal{SEIR}$ epidemic model as a system of FDEs that includes generalized functions of nonlinear incidence and treatment rates. The model we present is a generalization of many of the previous models mentioned above.

The following is how the remaining manuscript is structured. The model according to Caputo's definition and its basic properties are described in Section 2. The existence of both infection-free and endemic equilibria in terms of the basic reproduction number is investigated in Section 3. Section 4 examines the global dynamics of the Caputo model, which are dependent on the basic reproduction number under some assumptions. After Section 5 provides the numerical simulations, Section 6 wraps up with a brief comment.

2 Model description

In light of the foregoing, we propose a delayed $\mathcal{SEIR}$ epidemiological model in which the entire population $\mathcal{N}(t)$ is separated into four categories $\mathcal{S}(t)$, $E(t)$, $\mathcal{I}(t)$ and $R(t)$ representing the susceptible, exposed, infectious, recovered individuals, respectively, at time t. The model as a system of FDEs comprising time delay and two general nonlinear terms of the form $\mathcal{J}(\mathcal{S},\mathcal{I})$ and $¥(\mathcal{I})$, respectively, is depicted as described below.

$$\begin{aligned} D^\alpha \mathcal{S}(t) &= \Delta - \mathcal{J}(\mathcal{S}(t),\mathcal{I}(t)) - d\mathcal{S}(t), \\ D^\alpha E(t) &= e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau)) - (\rho+d)E(t), \\ D^\alpha \mathcal{I}(t) &= \rho E(t) - (\gamma+\delta+d)\mathcal{I}(t) - ¥(\mathcal{I}(t)), \\ D^\alpha R(t) &= \gamma\mathcal{I}(t) + ¥(\mathcal{I}(t)) - dR(t). \end{aligned} \tag{1}$$

D^α stands for the fractional derivative of Caputo, $\alpha \in (0,1]$ symbolizes the differential operator order and the initial conditions of system (1) are

$$\begin{aligned} \mathcal{S}(\xi) = \Psi_1(\xi),\ E(\xi) = \Psi_2(\xi),\ \mathcal{I}(\xi) = \Psi_3(\xi), \\ R(\xi) = \Psi_4(\xi), \quad \xi \in [-\tau, 0], \end{aligned} \tag{2}$$

where, $\Psi_j \in C([-\tau,0],R)$ such that $\Psi_i(0) > 0$ for $j = 1,2,3,4$.

The equation parameters in system (1) are all positive constants with biological meanings, detailed as follows: $\Delta = d\mathcal{N}$ is the population recruitment rate, d indicates the population's natural mortality rate, γ is the recovery rate in contaminated people, δ is the disease-related mortality rate, and exposed individuals become infectious at rate ρ. The term $0 < e^{-m\tau} \leq 1$ denotes the population's survival rate in the latent phase, whereas the time delay $\tau \geq 0$ denotes the time it takes for them to become infectious at time t, m is a positive constant. The functions $\mathcal{J}(\mathcal{S},\mathcal{I})$ and $¥(\mathcal{I})$ represent nonlinear general forms of incidence and treatment rates of infection, respectively. For any $\mathcal{S} > 0$ and $\mathcal{I} > 0$, we suppose that $\mathcal{J}(\mathcal{S},\mathcal{I})$ and $¥(\mathcal{I})$ are continuously differentiable, increase monotonically and meet the following criteria:

(C_1) $\mathcal{J}(\mathcal{S},\mathcal{I}) > 0$ and $\mathcal{J}(\mathcal{S},0) = \mathcal{J}(0,\mathcal{I}) = 0$ for any $\mathcal{S} > 0, \mathcal{I} > 0$.
(C_2) $\mathcal{J}'_{\mathcal{S}}(\mathcal{S},\mathcal{I}) > 0$, $\mathcal{J}'_{\mathcal{I}}(\mathcal{S},\mathcal{I}) > 0$, $\mathcal{J}'_{\mathcal{S}}(\mathcal{S},0) = 0$ and $\mathcal{J}'_{\mathcal{I}}(\mathcal{S},0) > 0$ for any $\mathcal{S} > 0, \mathcal{I} > 0$.
(C_3) $\left(\dfrac{\mathcal{J}(\mathcal{S},\mathcal{I})}{\mathcal{I}}\right)'_{\mathcal{I}} \leq 0$ for any $\mathcal{S} > 0, \mathcal{I} > 0$.
(C_4) $¥(\mathcal{I}) \geq 0$, $¥(0) = 0$ and $¥'(\mathcal{I}) > 0$ for any $\mathcal{I} \geq 0$.

2.1 Properties of solutions

The non-negativity and boundedness of system (1) solutions are discussed in this subsection.

Lemma 1 *System (1) solutions with the initial conditions (2) are non-negative, bounded and enters some compact attracting set Υ such that*

$$\Upsilon = \{(\mathcal{S},E,\mathcal{I},R) \in R^4_{+0}, 0 \leq \mathcal{S},E,\mathcal{I},R \leq \frac{\Delta}{d}\}.$$

Proof. For system (1) and from conditions C_1 and C_4, it is clear that

$$\begin{aligned} &D^{\alpha}\mathcal{S}(t)|_{\mathcal{S}=0} = \Delta > 0,\\ &D^{\alpha}E(t)|_{E=0} = e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau)) \geq 0,\\ &D^{\alpha}\mathcal{I}(t)|_{\mathcal{I}=0} = \rho E(t) \geq 0,\\ &D^{\alpha}R(t)|_{R=0} = \gamma\mathcal{I}(t) + \yen(\mathcal{I}(t)) \geq 0,\\ &\quad \text{for any } \ \mathcal{S}, E, \mathcal{I} \geq 0. \end{aligned}$$

By Lemmas 2.5 and 2.6 in [25], we have $\mathcal{S}(t), E(t), \mathcal{I}(t), R(t) \geq 0$ for any $t \geq 0$.

To demonstrate the boundedness, let

$$\mathcal{N}(t) = e^{-m\tau}\mathcal{S}(t-\tau) + E(t) + \mathcal{I}(t) + R(t),$$

taking the fractional-order derivative of $\mathcal{N}(t)$ and from system (1), then

$$\begin{aligned} D^{\alpha}\mathcal{N}(t) =& e^{-m\tau}D^{\alpha}\mathcal{S}(t-\tau) + D^{\alpha}E(t) + D^{\alpha}\mathcal{I}(t)\\ &+ D^{\alpha}R(t)\\ =& e^{-m\tau}\Big(\Delta - d\mathcal{S}(t-\tau)\Big) - \delta\mathcal{I}(t)\\ &- d(E(t) + \mathcal{I}(t) + R(t))\\ \leq& \Delta - d\mathcal{N}(t). \end{aligned}$$

Solving the last inequality, we get

$$\mathcal{N}(t) \leq \Big(\mathcal{N}(0) - \frac{\Delta}{d}\Big)E_{\alpha}(-dt^{\alpha}) + \frac{\Delta}{d},$$

$E_{\alpha}(-dt^{\alpha})$ is the Mittag-Leffer function of parameter α [26,27], satisfying $0 \leq E_{\alpha}(-dt^{\alpha}) \leq 1$ for all $t > 0$, then

$$\mathcal{N}(t) \leq \Big(\mathcal{N}(0) - \frac{\Delta}{d}\Big) + \frac{\Delta}{d},$$

$\mathcal{N}(0) = \mathcal{S}(0) + E(0) + \mathcal{I}(0) + R(0)$, This lead to

$$\limsup_{t\to+\infty}\mathcal{N}(t) \leq \frac{\Delta}{d}.$$

Therefore, we conclude that system (1) solutions are bounded. Furthermore, the set

$$\Upsilon = \{(\mathcal{S}, E, \mathcal{I}, R) \in R^4_{+0}, 0 \leq \mathcal{S}, E, \mathcal{I}, R \leq \frac{\Delta}{d}\}$$

is a non-negative attracting set for system (1), and this completes the proof of Lemma 1.□

3 Equilibria and basic reproduction number $\mathcal{R}_0$

The model equilibria and the criteria for their existence are now derived. To compute them, we put $D^{\alpha}\mathcal{S}(t) = D^{\alpha}E(t) = D^{\alpha}\mathcal{I}(t) = D^{\alpha}R(t) = 0$. Then,

$$\begin{aligned} &0 = \Delta - \mathcal{J}(\mathcal{S}(t), \mathcal{I}(t)) - d\mathcal{S}(t),\\ &0 = e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau), \mathcal{I}(t-\tau)) - (\rho + d)E(t),\\ &0 = \rho E(t) - (\gamma + \delta + d)\mathcal{I}(t) - \yen(\mathrm{I}(t)),\\ &0 = \gamma\mathcal{I}(t) + \yen(\mathcal{I}(t)) - dR(t). \end{aligned} \tag{3}$$

(i) If $\mathcal{I} = 0$ for system (3), then conditions (C_1) and (C_4) imply that $E = 0$, $R = 0$ and $\mathcal{S} = \frac{\Delta}{d}$. As a result, system (1) admits an infection-free equilibrium $\mathcal{E}_0 = (\mathcal{S}^0, 0, 0, 0) = (\frac{\Delta}{d}, 0, 0, 0)$, and this equilibrium exists for all parameter values.

(ii) If $\mathcal{I} \neq 0$ and from system (3), we get

$$\begin{aligned} \Delta - d\mathcal{S} &= \mathcal{J}(\mathcal{S}, \mathcal{I}) = e^{m\tau}(\rho + d)E\\ &= \frac{e^{m\tau}(\rho + d)}{\rho}[(\gamma + \delta + d)\mathcal{I} + \yen(\mathcal{I})]. \end{aligned} \tag{4}$$

From Eq. (4), we obtain

$$\mathcal{S} = G(\mathcal{I}) = \frac{\Delta}{d} - \frac{e^{m\tau}(\rho + d)}{\rho d}[(\gamma + \delta + d)\mathcal{I} + \yen(\mathcal{I})]. \tag{5}$$

Let us build the next function for $\mathcal{I}$ on the interval $[0, \mathcal{I}_0]$ by replacing $\mathcal{S} = G(\mathcal{I})$ with $\mathcal{I}$ in Eq. (4), and $\mathcal{I}_0$ be the unique positive solution of $G(\mathcal{I}) = 0$ such that $\Delta\rho = e^{m\tau}(\rho + d)[(\gamma + \delta + d)\mathcal{I}_0 + \yen(\mathcal{I}_0)]$.

$$\begin{aligned} \aleph(\mathcal{I}) =& \mathcal{J}(\mathcal{S}, \mathcal{I}) - \frac{e^{m\tau}(\rho + d)}{\rho}[(\gamma + \delta + d)\mathcal{I} + \yen(\mathcal{I})]\\ =& \mathcal{J}\Big(\frac{\Delta\rho - e^{m\tau}(\rho + d)[(\gamma + \delta + d)\mathcal{I} + \yen(\mathcal{I})]}{\rho d}, \mathcal{I}\Big)\\ &- \frac{e^{m\tau}(\rho + d)}{\rho}[(\gamma + \delta + d)\mathcal{I} + \yen(\mathcal{I})]. \end{aligned} \tag{6}$$

It is obvious from conditions (C_1) and (C_4) that

$$\aleph(0) = 0, \quad \aleph(\mathcal{I}_0) = \mathcal{J}(0, \mathcal{I}_0) - \Delta = -\Delta < 0. \tag{7}$$

Moreover, since $\aleph(\mathcal{I})$ is continuously differentiable, we get

$$\begin{aligned} \aleph'(0) =& \lim_{\mathcal{I}\to 0+}\frac{\aleph(\mathcal{I}) - \aleph(0)}{\mathcal{I} - 0}\\ =& \mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0, 0)\\ &- \frac{e^{m\tau}(\rho + d)}{\rho}[(\gamma + \delta + d) + \yen'(0)]\\ &- \frac{e^{m\tau}(\rho + d)}{\rho d}[(\gamma + \delta + d) + \yen'(0)]\mathcal{J}'_{\mathcal{S}}(\mathcal{S}^0, 0). \end{aligned} \tag{8}$$

Condition (C_2) implies that $\mathcal{J}'_{\mathcal{S}}(\mathcal{S}^0, 0) = 0$, then

$$\begin{aligned} \aleph'(0) =& \mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0, 0) - \frac{e^{m\tau}(\rho + d)}{\rho}[(\gamma + \delta + d) + \yen'(0)]\\ =& \frac{e^{m\tau}(\rho + d)}{\rho}[(\gamma + \delta + d) + \yen'(0)]\\ &\times\Big(\frac{\rho e^{-m\tau}}{(\rho + d)[(\gamma + \delta + d) + \yen'(0)]}\mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0, 0) - 1\Big). \end{aligned}$$

We also obtain from condition (C_4) that $¥'(0) > 0$. Therefore, if

$$\frac{\rho e^{-m\tau}}{(\rho+d)[(\gamma+\delta+d)+¥'(0)]}\mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0,0) > 1,$$

then, $\aleph'(0) > 0$ and there exists $\mathcal{I}^* \in (0,\mathcal{I}_0)$ such that $\aleph(\mathcal{I}^*) = 0$. From condition (C_4), $¥(\mathcal{I})$ increases strictly monotonically, thus

$$\begin{aligned}&e^{m\tau}(\rho+d)[(\gamma+\delta+d)\mathcal{I}^* + ¥(\mathcal{I}^*)]\\&\quad < e^{m\tau}(\rho+d)[(\gamma+\delta+d)\mathcal{I}_0 + ¥(\mathcal{I}_0)].\end{aligned}$$

Therefore,

$$\mathcal{S}^* = \frac{\Delta}{d} - \frac{e^{m\tau}(\rho+d)}{\rho d}[(\gamma+\delta+d)\mathcal{I}^* + ¥(\mathcal{I}^*)].$$

It follows from system (3) that $E^* = \frac{1}{\rho}[(\gamma+\delta+d)\mathcal{I}^* + ¥(\mathcal{I}^*)]$, $R^* = \frac{1}{d}[\gamma I^* + ¥(\mathcal{I}^*)]$. As a result, system (1) admits an endemic equilibrium $\mathcal{E}_1 = (\mathcal{S}^*, E^*, \mathcal{I}^*, R^*)$, which exists if

$$\frac{\rho e^{-m\tau}}{(\rho+d)[(\gamma+\delta+d)+¥'(0)]}\mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0,0) > 1.$$

Now, using the next generation matrix approach [28], we calculate the basic reproduction number $\mathcal{R}_0$ for the proposed model (1).

Letting $\mathcal{X} = (\mathcal{S}, E, \mathcal{I})^T$, then system (1) can be expressed as

$$D^\alpha(\mathcal{X}) = \varphi(\mathcal{X}) - \psi(\mathcal{X}),$$

$$\varphi(\mathcal{X}) = \begin{pmatrix} 0 \\ e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau)) \\ 0 \end{pmatrix},$$

$$\psi(\mathcal{X}) = \begin{pmatrix} -\Delta + \mathcal{J}(\mathcal{S}(t),\mathcal{I}(t)) + d\mathcal{S}(t) \\ (\rho+d)E(t) \\ -\rho E(t) + (\gamma+\delta+d)\mathcal{I}(t) + ¥(\mathcal{I}(t)) \end{pmatrix}.$$

The specific matrices $\mathcal{F}$ for new infection terms and $\mathcal{V}$ for the other terms are defined by

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ e^{-m\tau}\mathcal{J}'_{\mathcal{S}}(\mathcal{S},\mathcal{I}) & 0 & e^{-m\tau}\mathcal{J}'_{\mathcal{I}}(\mathcal{S},\mathcal{I}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} \mathcal{J}'_{\mathcal{S}}(\mathcal{S},\mathcal{I}) + d & 0 & \mathcal{J}'_{\mathcal{I}}(\mathcal{S},\mathcal{I}) \\ 0 & \rho+d & 0 \\ 0 & -\rho & (\gamma+\delta+d)+¥'(\mathcal{I}) \end{pmatrix}.$$

For system (1), at the infection-free equilibrium $\mathcal{E}_0 = (S^0,0,0,0)$, the next generation matrix $\mathcal{FV}^{-1}$ is

$$\mathcal{FV}^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\rho e^{-m\tau}\mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0,0)}{(\rho+d)[\gamma+\delta+d+¥'(0)]} & \frac{e^{-m\tau}\mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0,0)}{[\gamma+\delta+d+¥'(0)]} \\ 0 & 0 & 0 \end{pmatrix}.$$

$\mathcal{R}_0$ is the spectral radius of the last matrix, hence

$$\mathcal{R}_0 = \rho(\mathcal{FV}^{-1}) = \frac{\rho e^{-m\tau}\mathcal{J}'_{\mathcal{I}}(\mathcal{S}^0,0)}{(\rho+d)[\gamma+\delta+d+¥'(0)]}.$$

The value of $\mathcal{R}_0$ indicates the likelihood of the epidemic occurring. It computes the average number of new infections brought about by a single sick individual in a community of susceptible individuals. Considering the study presented above, we reach the following conclusion:

Lemma 2 *Assume that C_1-C_4 are hold, then there is a positive threshold parameter $\mathcal{R}_0$ for system (1) such that:*
(i) *If $\mathcal{R}_0 < 1$, then only infection-free equilibrium $\mathcal{E}_0$ exists.*
(ii) *If $\mathcal{R}_0 > 1$, then the infection-free equilibrium $\mathcal{E}_0$ in addition to the endemic equilibrium $\mathcal{E}_1$ persist.*

4 Global dynamics

This section covers the investigation of stability outcomes in the global case by constructing suitable Lyapunov functions. Because the variable $R(t)$ is absent from the equations for $D^\alpha\mathcal{S}(t)$, $D^\alpha E(t)$ and $D^\alpha\mathcal{I}(t)$, we can simplify the analysis of the behaviours of system (1) solutions by using the following sub-system:

$$\begin{aligned}D^\alpha\mathcal{S}(t) &= \Delta - \mathcal{J}(\mathcal{S}(t),\mathcal{I}(t)) - d\mathcal{S}(t),\\ D^\alpha E(t) &= e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau)) - (\rho+d)E(t),\\ D^\alpha\mathcal{I}(t) &= \rho E(t) - (\gamma+\delta+d)\mathcal{I}(t) - ¥(\mathcal{I}(t)).\end{aligned} \tag{9}$$

Theorem 1 *Assume that conditions (C_1) - (C_4) are hold. For any $\tau > 0$, the equilibrium $\mathcal{E}_0 = (\mathcal{S}^0,0,0)$ of system (9) is asymptotic stable globally if $\mathcal{R}_0 \le 1$.*

Proof. Consider a Lyapunov function $P_0(\mathcal{S}(t),E(t),\mathcal{I}(t))$ of the form:

$$\begin{aligned}P_0(t) =& \mathcal{S} - \mathcal{S}^0 - \int_{\mathcal{S}^0}^{\mathcal{S}} \lim_{\mathcal{I}\to 0^+}\frac{\mathcal{J}(\mathcal{S}^0,\mathcal{I})}{\mathcal{J}(\Im,\mathcal{I})}d\Im + e^{m\tau}E\\ &+ \Big(\frac{\rho+d}{\rho}\Big)e^{m\tau}\mathcal{I} + \int_0^\tau \mathcal{J}(\mathcal{S}(t-\theta),\mathcal{I}(t-\theta))d\theta.\end{aligned} \tag{10}$$

By conditions (C_1)-(C_4), we note that $P_0(\mathcal{S},E,\mathcal{I}) > 0$ for all $\mathcal{S},E,\mathcal{I} > 0$, where $\theta \in [-\tau,0]$ and $P_0 = 0$ at $\mathcal{E}_0 = (\mathcal{S}^0,0,0)$.

The fractional differentiation of $P_0(t)$ with respect to t along the system (9) solutions yields

$$\begin{aligned}D^\alpha P_0(t) =&\Big(1-\lim_{\mathcal{I}\to 0^+}\frac{\mathcal{J}(\mathcal{S}^0,\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big)D^\alpha\mathcal{S}(t)\\&+e^{m\tau}D^\alpha E(t)+\frac{\rho+d}{\rho}e^{m\tau}D^\alpha\mathcal{I}(t)\\&+\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))-\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))\\=&\Big(1-\lim_{\mathcal{I}\to 0^+}\frac{\mathcal{J}(\mathcal{S}^0,\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big)\\&\times[\varDelta-\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))-d\mathcal{S}(t)]\\&+e^{m\tau}[e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))-(\rho+d)E(t)]\\&+\frac{\rho+d}{\rho}e^{m\tau}[\rho E(t)-(\gamma+\delta+d)\mathcal{I}(t)-\yen(\mathcal{I}(t))]\\&+\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))-\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau)).\end{aligned}$$

Cancelling the same terms and applying the infection-free equilibrium condition $\varDelta = d\mathcal{S}^0$, then

$$\begin{aligned}D^\alpha P_0(t) =&\Big(1-\lim_{\mathcal{I}\to 0^+}\frac{\mathcal{J}(\mathcal{S}^0,\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big)[d\mathcal{S}^0-d\mathcal{S}(t)]\\&+\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))\lim_{\mathcal{I}\to 0^+}\frac{\mathcal{J}(\mathcal{S}^0,\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\\&-\frac{\rho+d}{\rho}e^{m\tau}[(\gamma+\delta+d)\mathcal{I}(t)+\yen(\mathcal{I}(t))]\\=&d\mathcal{S}^0\Big(1-\frac{\mathcal{J}'_\mathcal{I}(\mathcal{S}^0,0)}{\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0)}\Big)\Big(1-\frac{\mathcal{S}(t)}{\mathcal{S}^0}\Big)\\&+\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))\frac{\mathcal{J}'_\mathcal{I}(\mathcal{S}^0,0)}{\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0)}\\&-\frac{\rho+d}{\rho}e^{m\tau}\mathcal{I}(t)\Big[(\gamma+\delta+d)+\frac{\yen(\mathcal{I}(t))}{\mathcal{I}(t)}\Big].\end{aligned}$$

From conditions (C_3) and (C_4), we have

$$\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{I}(t)}\le\lim_{I\to 0^+}\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{I}(t)}=\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0),$$

and hence,

$$\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))\le\mathcal{I}(t)\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0),$$

furthermore,

$$\yen'(0)\le\frac{\yen(\mathcal{I}(t))}{\mathcal{I}(t)},\quad\text{with respect to }\ \mathcal{I}>0,$$

it follows that,

$$\begin{aligned}D^\alpha P_0(t)\le&d\mathcal{S}^0\Big(1-\frac{\mathcal{J}'_\mathcal{I}(\mathcal{S}^0,0)}{\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0)}\Big)\Big(1-\frac{\mathcal{S}(t)}{\mathcal{S}^0}\Big)\\&-\frac{\rho+d}{\rho}e^{m\tau}\mathcal{I}(t)[(\gamma+\delta+d)+\yen'(0)]\\&+\mathcal{I}(t)\mathcal{J}'_\mathcal{I}(\mathcal{S}^0,0)\\=&d\mathcal{S}^0\Big(1-\frac{\mathcal{J}'_\mathcal{I}(\mathcal{S}^0,0)}{\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0)}\Big)\Big(1-\frac{\mathcal{S}(t)}{\mathcal{S}^0}\Big)\\&+\frac{\rho+d}{\rho}e^{m\tau}\mathcal{I}(t)[\gamma+\delta+d+\yen'(0)](\mathcal{R}_0-1).\end{aligned}$$

From (C_1), (C_2), and the monotonicity of $\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))$ with respect to $\mathcal{S}$, then

$$\Big(1-\frac{\mathcal{J}'_\mathcal{I}(\mathcal{S}^0,0)}{\mathcal{J}'_\mathcal{I}(\mathcal{S}(t),0)}\Big)\Big(1-\frac{\mathcal{S}(t)}{\mathcal{S}^0}\Big)\le 0.$$

Thus, $\mathcal{R}_0\le 1$ ensures that $D^\alpha P_0(t)\le 0$ for any $\mathcal{S}(t),\mathcal{I}(t)>0$ and $D^\alpha P_0(t)=0$ holds only for $\mathcal{S}=\mathcal{S}^0$, $\mathcal{I}=0$. It is simple to demonstrate that the largest invariant set of $D^\alpha P_0(t)=0$ is the singleton $\mathcal{E}_0$. The Lyapunov Lasalle theorem for FDEs [29] states that $\mathcal{E}_0$ is asymptotic stable globally for any $\tau>0$. Now, this proof is complete.□

To investigate the next theorem we need the following conditions:

(C_5)

$$\frac{\mathcal{I}}{\mathcal{I}^*}\le\frac{\mathcal{J}(\mathcal{S},\mathcal{I})}{\mathcal{J}(\mathcal{S},\mathcal{I}^*)}\quad for\ \ \mathcal{I}\in(0,\mathcal{I}^*),$$

$$\frac{\mathcal{J}(\mathcal{S},\mathcal{I})}{\mathcal{J}(\mathcal{S},\mathcal{I}^*)}\le\frac{\mathcal{I}}{\mathcal{I}^*}\quad for\ \ \mathcal{I}\ge\mathcal{I}^*,$$

(C_6)

$$\frac{\yen(\mathcal{I})}{\yen(\mathcal{I}^*)}\le\frac{\mathcal{I}}{\mathcal{I}^*}\quad for\ \ \mathcal{I}\in(0,\mathcal{I}^*),$$

$$\frac{\mathcal{I}}{\mathcal{I}^*}\le\frac{\yen(\mathcal{I})}{\yen(\mathcal{I}^*)}\quad for\ \ \mathcal{I}\ge\mathcal{I}^*.$$

To keep things simple and mathematically convenient, we'll utilise the following function: $Z:R>0\to R\ge 0$ as $Z(\upsilon)=1-\upsilon+\ln(\upsilon)$, we see that $Z(\upsilon)\le 0$ for any $\upsilon>0$.

Theorem 2 *Assume that conditions (C_1) - (C_6) are hold. For any $\tau>0$, if the endemic equilibrium $\mathcal{E}_1=(\mathcal{S}^*,E^*,\mathcal{I}^*,R^*)$ of system (9) exists, then it is asymptotic stable globally.*

Proof. Consider a Lyapunov function $P_1(\mathcal{S}(t),E(t),\mathcal{I}(t))$ of the form:

$$\begin{aligned}P_1(t)=&\mathcal{S}(t)-\mathcal{S}^*-\int_{\mathcal{S}^*}^{\mathcal{S}(t)}\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\xi,\mathcal{I}^*)}d\xi\\&+e^{m\tau}\Big(E(t)-E^*-E^*\ln\frac{E(t)}{E^*}\Big)\\&+\frac{\rho+d}{\rho}e^{m\tau}\Big(\mathcal{I}(t)-\mathcal{I}^*-\mathcal{I}^*\ln\frac{\mathcal{I}(t)}{\mathcal{I}^*}\Big)\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\int_0^\tau\Bigg(\frac{\mathcal{J}(\mathcal{S}(t-\theta),\mathcal{I}(t-\theta))}{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}-1\\&-\ln\frac{\mathcal{J}(\mathcal{S}(t-\theta),\mathcal{I}(t-\theta))}{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\Bigg)d\theta.\end{aligned}\tag{11}$$

By conditions (C_1)-(C_4), we note that $P_1(\mathcal{S},E,\mathcal{I})>0$ for all $\mathcal{S},E,\mathcal{I}>0$, where $\theta\in[-\tau,0]$ and $P_1=0$ at $\mathcal{E}_1=(\mathcal{S}^*,E^*,\mathcal{I}^*)$.

The fractional differentiation of $P_1(t)$ with respect to t along the system (9) solutions yields

$$\begin{aligned}D^{\alpha}P_1(t)=&\Big(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)[\varDelta-\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))-d\mathcal{S}(t)]\\&+e^{m\tau}\Big(1-\frac{E^*}{E(t)}\Big)\\&\times[e^{-m\tau}\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))-(\rho+d)E(t)]\\&+\frac{\rho+d}{\rho}e^{m\tau}\Big(1-\frac{\mathcal{I}^*}{\mathcal{I}(t)}\Big)\\&\times[\rho E(t)-(\gamma+\delta+d)\mathcal{I}(t)-\yen(\mathcal{I}(t))]\\&+\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))-\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\ln\frac{\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}.\end{aligned}$$

Applying the following endemic equilibrium conditions

$$\begin{aligned}\varDelta=&d\mathcal{S}^*+J(\mathcal{S}^*,\mathcal{I}^*),\\\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)=&(\rho+d)e^{m\tau}E^*,\\\rho E^*=&(\gamma+\delta+d)\mathcal{I}^*+\yen(\mathcal{I}^*),\end{aligned}\tag{12}$$

and cancelling the same terms, we have

$$\begin{aligned}D^{\alpha}P_1(t)=&\Big(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)[d\mathcal{S}^*-d\mathcal{S}(t)]+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\\&\times\left(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}+\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\right)\\&-\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))\frac{E^*}{E(t)}+(\rho+d)e^{m\tau}E^*\\&-(\rho+d)e^{m\tau}\frac{E\mathcal{I}^*}{\mathcal{I}(t)}+\frac{\rho+d}{\rho}e^{m\tau}\Big(1-\frac{\mathcal{I}^*}{\mathcal{I}(t)}\Big)\\&\times[-(\gamma+\delta+d)\mathcal{I}(t)-\yen(\mathcal{I}(t))]\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\ln\frac{\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\\=&\Big(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)[d\mathcal{S}^*-d\mathcal{S}(t)]\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\left(2-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}+\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\right.\\&\left.-\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}-\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}\right)\\&-\frac{\rho+d}{\rho}e^{m\tau}\Big(1-\frac{\mathcal{I}^*}{\mathcal{I}(t)}\Big)\\&\times\left(\frac{\rho E^*-\yen(\mathcal{I}^*)}{\mathcal{I}^*}\mathcal{I}(t)+\yen(I(t))\right)\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\ln\frac{\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}.\end{aligned}$$

Adding and subtracting some terms, it follows that

$$\begin{aligned}D^{\alpha}P_1(t)=&\Big(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)[d\mathcal{S}^*-d\mathcal{S}(t)]+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\\&\times\Bigg[\Big(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}+\ln\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)\\&+\Big(1-\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\\&+\ln\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\Big)\\&+\Big(1-\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}+\ln\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}\Big)\\&+\Big(1-\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}+\ln\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big)\Bigg]\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\left[\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}-2+\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\right]\\&-(\rho+d)e^{m\tau}\Big(1-\frac{\mathcal{I}^*}{\mathcal{I}(t)}\Big)\frac{E^*\mathcal{I}(t)}{\mathcal{I}^*}\\&-\frac{\rho+d}{\rho}e^{m\tau}\Big(1-\frac{\mathcal{I}^*}{\mathcal{I}(t)}\Big)\Big(-\frac{\yen(\mathcal{I}^*)\mathcal{I}(t)}{\mathcal{I}^*}+\yen(\mathcal{I}(t))\Big)\\&-\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\Bigg[\ln\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}+\ln\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}\\&+\ln\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}+\ln\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Bigg]\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\ln\frac{\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}.\end{aligned}$$

Since,

$$\begin{aligned}&\ln\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}+\ln\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}+\ln\frac{E^*J(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\\&+\ln\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}=\ln\frac{\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))},\end{aligned}$$

then,

$$\begin{aligned}D^{\alpha}P_1(t)=&d\mathcal{S}^*\Big(1-\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)\Big(1-\frac{\mathcal{S}(t)}{\mathcal{S}^*}\Big)\\&+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\Bigg[Z\Big(\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)+Z\Big(\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}\Big)\\&+Z\Big(\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\Big)\\&+Z\Big(\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big)\Bigg]+\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\\&\times\left[\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}-1-\frac{\mathcal{I}}{\mathcal{I}^*}+\frac{\mathcal{I}\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\right]\\&-\frac{\rho+d}{\rho}e^{m\tau}\yen(\mathcal{I}^*)\Big(1-\frac{\mathcal{I}^*}{\mathcal{I}(t)}\Big)\Big(\frac{\yen(\mathcal{I}(t))}{\yen(\mathcal{I}^*)}-\frac{\mathcal{I}(t)}{\mathcal{I}^*}\Big),\end{aligned}$$

it follows that,

$$D^\alpha P_1(t) = d\mathcal{S}^*\Big(1 - \frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)\Big(1 - \frac{\mathcal{S}(t)}{\mathcal{S}^*}\Big)$$
$$+ \mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)\Bigg[Z\Big(\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big) + Z\Big(\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}\Big)$$
$$+ Z\Big(\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\Big)$$
$$+ Z\Big(\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big)\Bigg] + \mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)$$
$$\times\Big(\frac{\mathcal{I}(t)}{\mathcal{I}^*} - \frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)\Big(\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))} - 1\Big)$$
$$+ \frac{\rho+d}{\rho}e^{m\tau}¥(\mathcal{I}^*)\Big(\frac{\mathcal{I}^*}{\mathcal{I}(t)} - 1\Big)\Big(\frac{¥(\mathcal{I}(t))}{¥(\mathcal{I}^*)} - \frac{\mathcal{I}(t)}{\mathcal{I}^*}\Big).$$

The monotonicity of $\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))$ with respect to $\mathcal{S}$ yields

$$\Big(1 - \frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)\Big(1 - \frac{\mathcal{S}(t)}{\mathcal{S}^*}\Big) \le 0.$$

Also, by the characteristics of the function $Z(\upsilon)$, we get

$$Z\Big(\frac{\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big) \le 0, \quad Z\Big(\frac{E(t)\mathcal{I}^*}{E^*\mathcal{I}(t)}\Big) \le 0,$$
$$Z\Big(\frac{E^*\mathcal{J}(\mathcal{S}(t-\tau),\mathcal{I}(t-\tau))}{E(t)\mathcal{J}(\mathcal{S}^*,\mathcal{I}^*)}\Big) \le 0,$$
$$Z\Big(\frac{\mathcal{I}(t)\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{I}^*\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}\Big) \le 0.$$

Finally, from conditions (C_5) and (C_6), we have

$$\Big(\frac{\mathcal{I}(t)}{\mathcal{I}^*} - \frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}\Big)\Big(\frac{\mathcal{J}(\mathcal{S}(t),\mathcal{I}^*)}{\mathcal{J}(\mathcal{S}(t),\mathcal{I}(t))} - 1\Big) \le 0,$$
$$\Big(\frac{\mathcal{I}^*}{\mathcal{I}(t)} - 1\Big)\Big(\frac{¥(\mathcal{I}(t))}{¥(\mathcal{I}^*)} - \frac{\mathcal{I}(t)}{\mathcal{I}^*}\Big) \le 0.$$

Hence, we see that $D^\alpha P_1(t) \le 0$ for all $\mathcal{S}(t), E(t), \mathcal{I}(t) > 0$ and $D^\alpha P_1(t) = 0$ holds only for $\mathcal{S} = \mathcal{S}^*$, $E = E^*$ and $\mathcal{I} = \mathcal{I}^*$. It is simple to demonstrate that the largest invariant set of $D^\alpha P_1(t) = 0$ is the singleton $\mathcal{E}_1$. The Lyapunov Lasalle theorem [29] states that $\mathcal{E}_1$ is asymptotic stable globally for any $\tau > 0$. This is the end of the proof.

5 Example and numerical simulation

This section contains numerical simulations that document the theoretical analyzes presented in the previous sections. MATLAB was used to do numerical calculations to solve the delay differential equation of fractional-order using the modified Adams-Bashforth-Moulton predictor-corrector method [30]. A special case of model (1) is presented in the following example.

$$\begin{aligned} D^\alpha\mathcal{S}(t) =& \varDelta - \tfrac{b\mathcal{S}(t)\mathcal{I}(t)}{1+a_1\mathcal{S}(t)+a_2\mathcal{I}(t)+a_3\mathcal{S}(t)\mathcal{I}(t)} - d\mathcal{S}(t),\\ D^\alpha E(t) =& e^{-m\tau}\tfrac{b\mathcal{S}(t-\tau)\mathcal{I}(t-\tau)}{1+a_1\mathcal{S}(t-\tau)+a_2\mathcal{I}(t-\tau)+a_3\mathcal{S}(t-\tau)\mathcal{I}(t-\tau)}\\ &-(\rho+d)E(t),\\ D^\alpha\mathcal{I}(t) =& \rho E(t) - (\gamma+\delta+d)\mathcal{I}(t) - \tfrac{r\mathcal{I}^2(t)}{1+c\mathcal{I}(t)},\\ D^\alpha R(t) =& \gamma\mathcal{I}(t) + \tfrac{r\mathcal{I}^2(t)}{1+c\mathcal{I}(t)} - dR(t). \end{aligned} \quad (13)$$

Table 1: Parameters values of system (13).

Parameter	Value	Parameter	Value
$\varDelta$	0.74	ρ	0.04
d	0.05	γ	0.25
b	*varied*	δ	0.01
r	0.3	a_1	0.03
c	0.02	a_2	0.02
m	*varied*	a_3	0.01

For this example, we use the specific nonlinear functional response $\mathcal{J}(\mathcal{S},\mathcal{I}) = \frac{b\mathcal{S}\mathcal{I}}{1+a_1\mathcal{S}+a_2\mathcal{I}+a_3\mathcal{S}\mathcal{I}}$ as an incidence rate and $¥(\mathcal{I}) = \frac{r\mathcal{I}^2(t)}{1+c\mathcal{I}(t)}$ as a treatment rate. The parameter b represents the maximum transmission rate between S and I, r is the disease treatment rate, c is the limitation rate in treatment availability, and a_1, a_2 and a_3 represent a measure of inhibition, and all are positive constants.

We are now checking the conditions (C_1) - (C_4):

(C_1) Obviously,

$$\mathcal{J}(\mathcal{S},\mathcal{I}) > 0,\ \mathcal{J}(\mathcal{S},0) = \mathcal{J}(0,\mathcal{I}) = 0 \quad for\ any\ \ \mathcal{S},\mathcal{I} > 0.$$

$$(C_2)\ \ \mathcal{J}'_{\mathcal{S}}(\mathcal{S},\mathcal{I}) = \frac{b\mathcal{I}(1+a_2\mathcal{I})}{(1+a_1\mathcal{S}+a_2\mathcal{I}+a_3\mathcal{S}\mathcal{I})^2} > 0,$$
$$\mathcal{J}'_{\mathcal{I}}(\mathcal{S},\mathcal{I}) = \frac{b\mathcal{S}(1+a_1\mathcal{S})}{(1+a_1\mathcal{S}+a_2\mathcal{I}+a_3\mathcal{S}\mathcal{I})^2} > 0,$$
$$\mathcal{J}'_{\mathcal{S}}(\mathcal{S},0) = 0, \quad \mathcal{J}'_{\mathcal{I}}(\mathcal{S},0) = \frac{b\mathcal{S}}{(1+a_1\mathcal{S})} > 0,$$
$$for\ any\ \ \mathcal{S},\mathcal{I} > 0.$$

$$(C_3)\ \ \Big(\frac{\mathcal{J}(\mathcal{S},\mathcal{I})}{\mathcal{I}}\Big)'_{\mathcal{I}} = \Big(\frac{b\mathcal{S}}{1+a_1\mathcal{S}+a_2\mathcal{I}+a_3\mathcal{S}\mathcal{I}}\Big)'_{\mathcal{I}} = \frac{-b\mathcal{S}(a_2+a_3\mathcal{S})}{(1+a_1\mathcal{S}+a_2\mathcal{I}+a_3\mathcal{S}\mathcal{I})^2} < 0,$$
$$for\ any\ \ \mathcal{S},\mathcal{I} > 0.$$

$$(C_4)\ \ ¥(\mathcal{I}) = \frac{r\mathcal{I}^2(t)}{1+c\mathcal{I}(t)} \ge 0, \quad ¥(0) = 0,$$
$$¥'(\mathcal{I})\frac{r\mathcal{I}(2+c\mathcal{I})}{(1+c\mathcal{I})^2} \ge 0, \ \ for\ any\ \ \mathcal{I} \ge 0,$$

The basic reproduction number of this example has the following form:

$$\mathcal{R}_0 = \frac{\rho b e^{-m\tau}\mathcal{S}^0}{(\rho+d)(d+\delta+\gamma)(1+a_1\mathcal{S}^0)}.$$

The parameter values listed in Table 1 are used to perform numerical simulations of system (13) using the three initial values shown below:

IV1: $\mathcal{S}(0) = 18,\ E(0) = 5.5,\ \mathcal{I}(0) = 0.8,\ R(0) = 2.5$,
IV2: $\mathcal{S}(0) = 10,\ E(0) = 2.5,\ \mathcal{I}(0) = 0.3,\ R(0) = 1.5$,
and
IV3: $\mathcal{S}(0) = 4,\ E(0) = 0.8,\ \mathcal{I}(0) = 0.07,\ R(0) = 0.5$.

5.1 Stability of equilibrium points

Based on Table 1, and taking $\alpha = 0.97$, $\tau = 0.5$, $m = 0.01$ and b varies as follows:

Case(i) Stability of $\mathcal{E}_0$: Selecting $b = 0.047$ yields $\mathcal{R}_0 = 0.6872 < 1$, and according to Lemma 2, the only infection-free equilibrium $\mathcal{E}_0 = (14.8, 0, 0, 0)$ exists for system (13). Figure 1 demonstrates that the trajectories starting with IV1-IV3 asymptotically converge to the equilibrium $\mathcal{E}_0 = (14.8, 0, 0, 0)$. Thus, $\mathcal{E}_0$ is asymptotically stable globally, and therefore, the numerical outcomes confirm the theoretic conclusions of Theorem 1.

Case(ii) Stability of $\mathcal{E}_1$: The choice of $b = 0.087$ result in $\mathcal{R}_0 = 1.272 > 1$. From Lemma 2, it is evident that the endemic equilibrium $\mathcal{E}_1$ for system (13) exists with $\mathcal{E}_1 = (12.7034, 1.1643, 0.1333, 0.7746)$. Figure 2 shows that $\mathcal{E}_1$ is asymptotically stable globally and this agrees Theorem 2.

5.2 Effect of the fractional-order operator α on solution trajectories of system (13)

Figures 3 and 4 depict the influence of fractional-order operator on the solution trajectories of system (13) for Case(i) and Case(ii) in the previous Subsection 5.1. The fractional-order dampens the oscillation behaviour and extends the stability range. We notice that as the derivative order is decreased, the phase portrait expands. The fractional derivative indicates the model long memory, which obviously appear the beginning stages.

5.3 Effect of the time delay τ on solution trajectories of system (13)

In this subsection, we look into how time delay affects the system dynamics. We select the values $\alpha = 0.97$, $b = 0.087$ and $m = 0.1$ with the values listed in Table 1. We consider different values of τ and the initial value IV2. We note that the existence of time delay reduces the basic reproduction number $\mathcal{R}_0$. As seen in Figure 5, with an increase in τ, the number of susceptible people increases, whereas the number of infected, exposed and recovered people decreases.

5.4 Effect of the disease treatment rate r on system (13)

Using Table 1 and choosing $\alpha = 0.97$, $b = 0.087$, $m = 0.1$ and $\tau = 0.5$. We take different values of the disease treatment rate r as well as the initial value, IV4:$\mathcal{S}(0) = 12, E(0) = 1.5, \mathcal{I}(0) = 0.15, R(0) = 0.8$. Figure 6 illustrates how raising the treatment parameter r can decrease the number of infected and exposed people while raising the number of susceptible people.

6 Conclusion

This study investigated the delayed $\mathcal{SE}\mathcal{I}R$ epidemiological model in its fractional-order version that uses a general function of the infection transmission and a general treatment function that alter with the epidemiological state. The Caputo case of the fractional derivative is considered, which is appropriate for initial-value problems. We have shown that the proposed model contains non-negative and bounded solutions, as well as two equilibrium points, one infection-free and the other endemic. The basic reproduction number $\mathcal{R}_0$, which is dependent on the time delay, was computed using the next generation technique. It governs not only the existence of the equilibrium points, but also the model's dynamic behaviour. Using the Lyapunov approach, we demonstrated that the model's two equilibrium points are asymptotically stable globally for any $\tau > 0$. The equilibrium of infection-free has proven to be stable when $\mathcal{R}_0 < 1$, and for $\mathcal{R}_0 > 1$, the equilibrium of endemic state becomes stable under some mild conditions on the infection transmission and treatment functions. To illustrate our theoretical findings, we used a special case of the underling model as an example and the modified Adams-Bashforth-Moulton technique to simulate it. In the numerical portion, we also looked at the effect of the disease treatment parameter r on the model dynamics and the benefits of applying the time delay and fractional-order to a differential equations. Our study shows that the addition of a time delay and fractional-order to a differential equations significantly enhances the system dynamics and raises the complexity of the observable behaviour. Moreover, we found that as r increased, the number of $\mathcal{I}(t)$ and $E(t)$ decreased while the number of $\mathcal{S}(t)$ increased. Finally, we believe that our model is a generalization of several previously published epidemiological models.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no competing interests.

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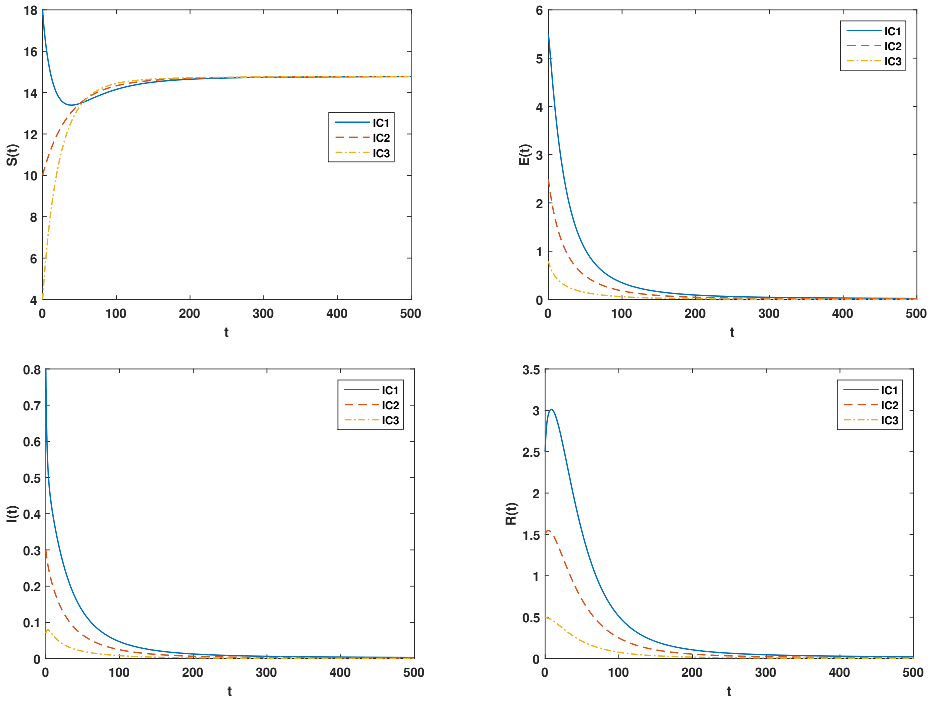

Fig. 1: The solution trajectories behaviour of system (13) corresponding to Table 1 when $\alpha = 0.97$, $\tau = 0.5$, $b = 0.047$ and $R_0 = 0.6598 < 1$ with the initial values IV1-IV3: Pointing to asymptotically stability of $\mathcal{E}_0 = (14.8, 0, 0, 0)$.

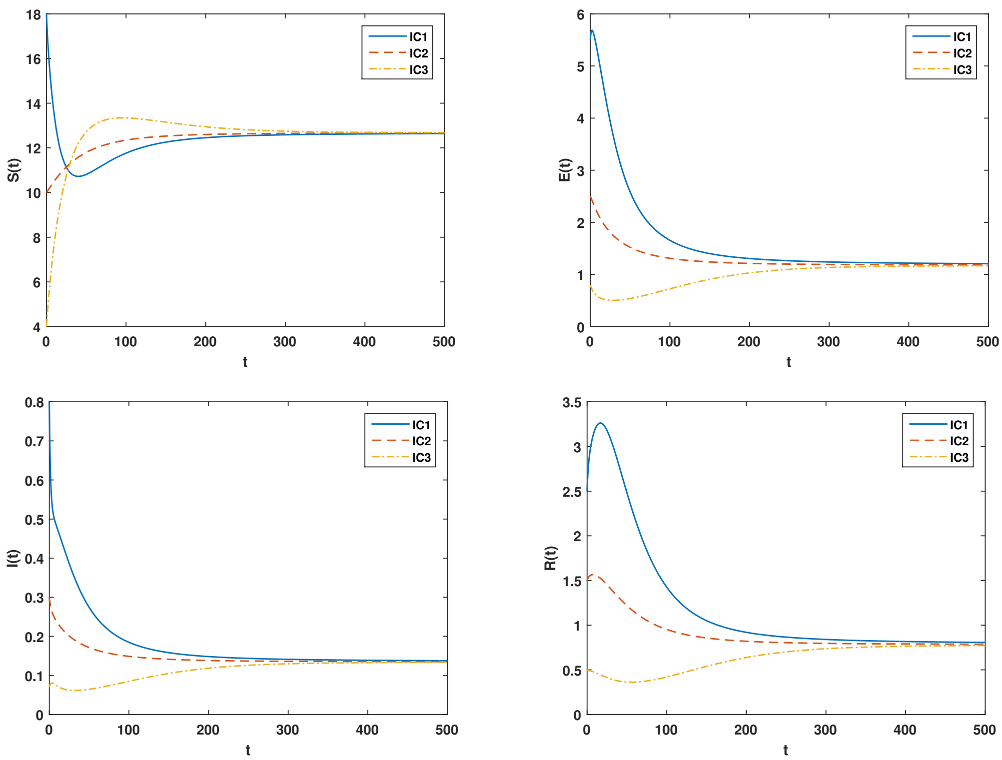

Fig. 2: The solution trajectories behaviour of system (13) corresponding to Table 1 when $\alpha = 0.97$, $\tau = 0.5$, $b = 0.087$ and $R_0 = 1.272 > 1$ with the initial values IV1-IV3: Pointing to asymptotically stability of $\mathcal{E}_1 = (12.7034, 1.1643, 0.1333, 0.7746)$.

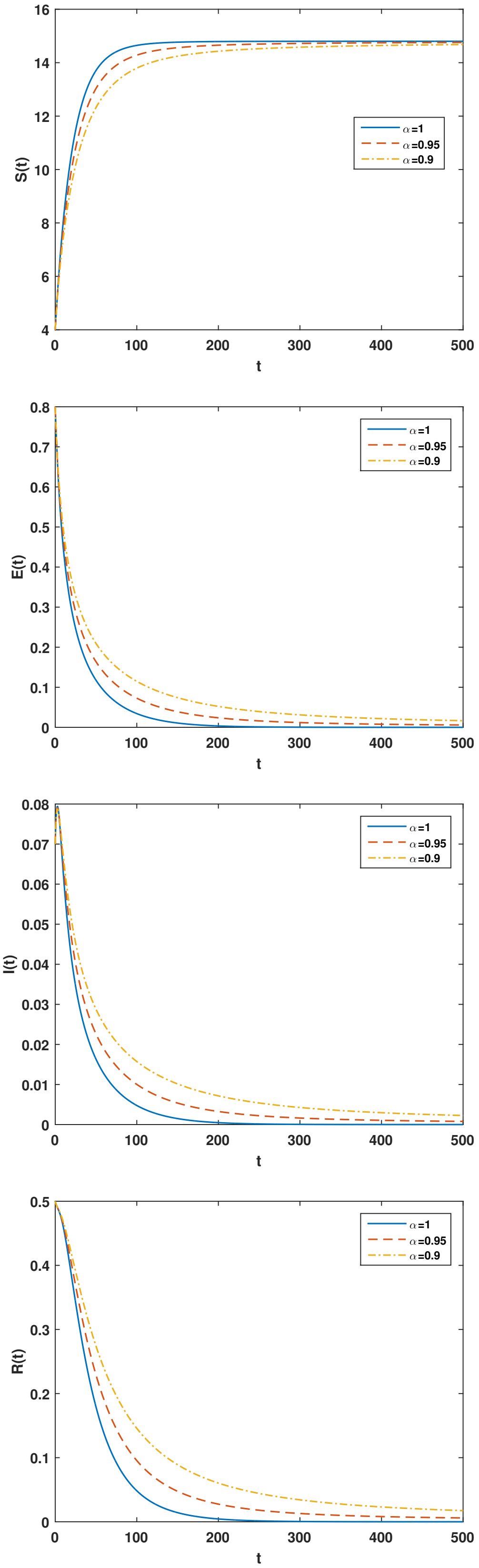

Fig. 3: The stability behaviour of the infection-free equilibrium $\mathcal{E}_0$ with the initial value IV3 and different values of α.

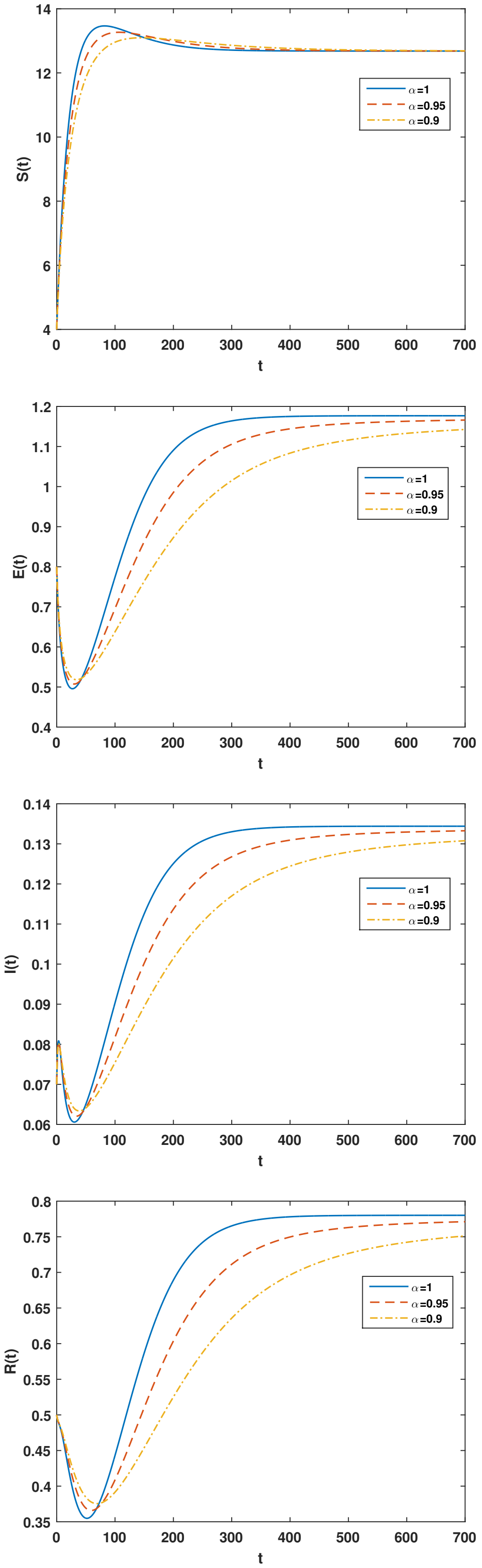

Fig. 4: The stability behaviour of the endemic equilibrium point $\mathcal{E}_1$ with the initial value IV3 and different values of α.

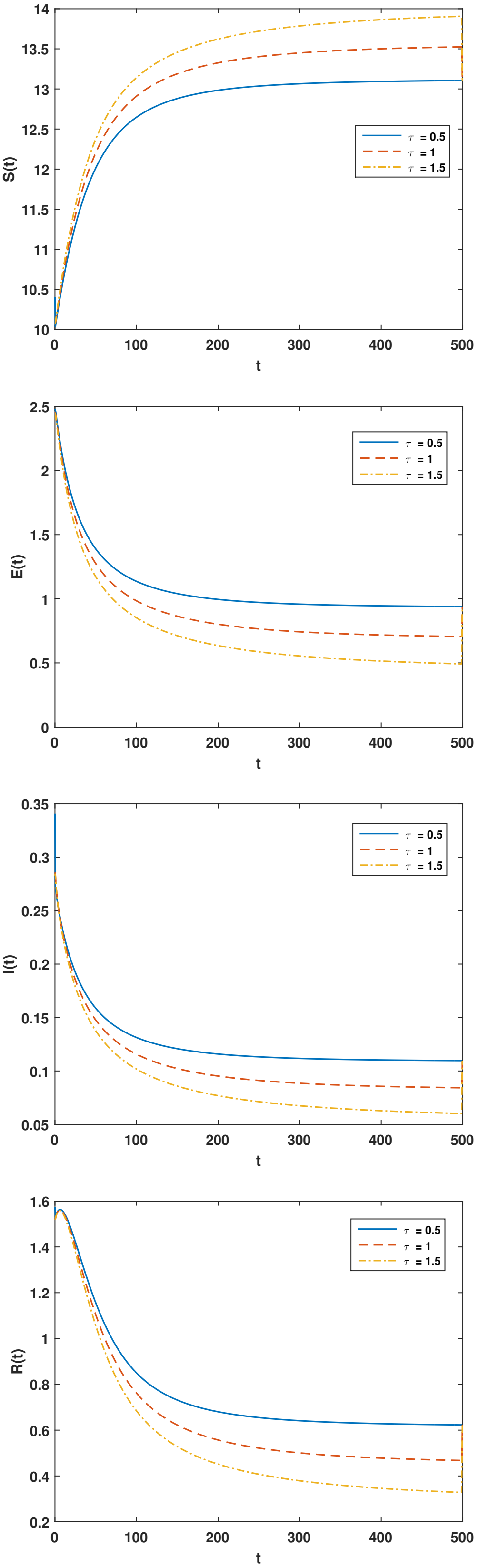

Fig. 5: Solution behaviour of system (13) when τ is different with $\alpha = 0.97$ and $\mathcal{R}_0 > 1$.

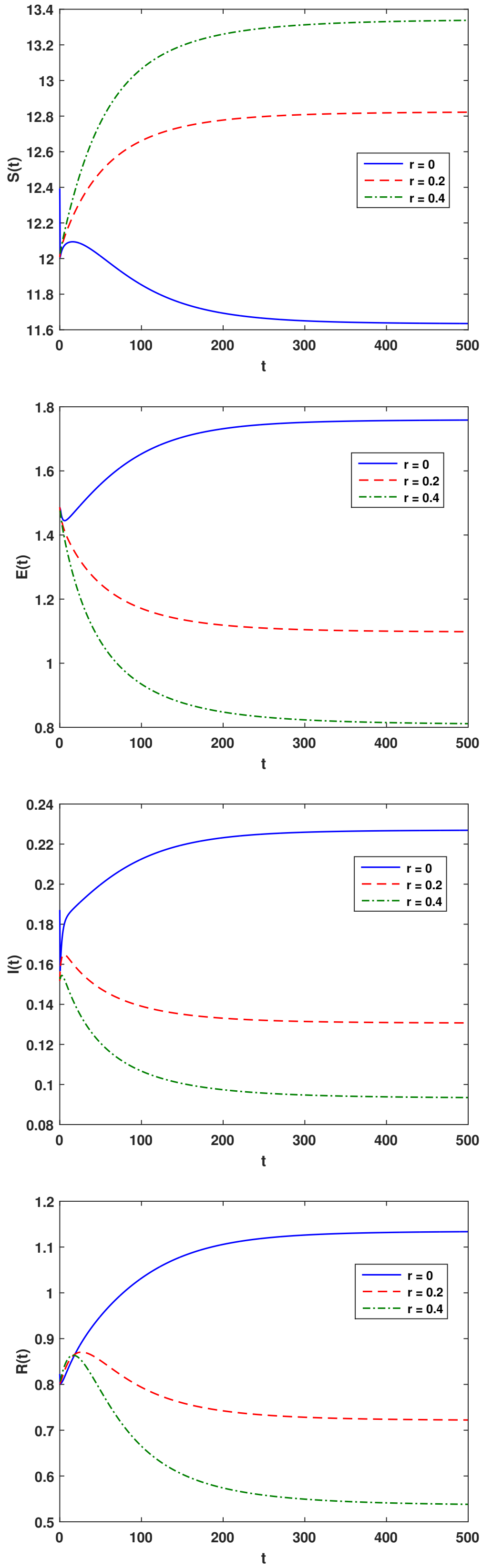

Fig. 6: Solution behaviour of system (13) when r is different with $\alpha = 0.97$ and $\mathcal{R}_0 > 1$.

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