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On Generalized Harmonically ψ-MT-Convex Functions via Local Fractional Integrals and some Applications

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Abstract: In this work, we introduce a new class of harmonically convex functions, namely, generalized harmonically ψ-MT-convex functions established on fractal set techniques, for establishing inequalities of Hermite-Hadamard type and certain related variants with respect to the Raina's function. With the help of an auxiliary identity associated with Raina's function, by generalized Holder inequality and generalized power mean, generalized midpoint type, Ostrowski type, and trapezoid type inequalities via local fractional integral for generalized harmonically ψ-MT-convex functions are given. The introduced technique gives the results by establishing some special values for the parameters or applying restrictive suppositions and is entirely practicable for regaining the existing inequalities in the related literature.

Keywords: Harmonically-convex function, MT-Convex function, Raina‘s function, Fractal set, Generalized Harmonically ψ-MT-convex function, Hermite-Hadamard type inequality and Local fractional integral.

1 Introduction

Through this paper, $\tau\colon \Xi \to \Re$ is said to be convex on $\Xi \subset \Re$, if the following inequality holds,

$$\tau(\top \rtimes_1 + (1-\top)\rtimes_2) \leqslant \top\tau(\rtimes_1) + (1-\top)\tau(\rtimes_2). \quad (1)$$

$\forall\ \rtimes_1, \rtimes_2 \in \Xi$ and $\top \in [0,1]$.
Reader can see ([1]-[9]) for more generalized classes of convex functions. We will restrict our attention only on Hermite-Hadamard-type inequality (simply H-H type inequality) [10] [11], which is given as,
Suppose $\tau\colon \Xi \subseteq \Re \to \Re$ is a convex function, then

$$\tau\left(\frac{\rtimes_1 + \rtimes_2}{2}\right) \leqslant \frac{1}{\rtimes_2 - \rtimes_1}\int_{\rtimes_1}^{\rtimes_2} \tau(\rtimes)d\rtimes \leqslant \frac{\tau(\rtimes_1)+\tau(\rtimes_2)}{2}. \quad (2)$$

with $\rtimes_1 < \rtimes_2$ and $\rtimes_1, \rtimes_2 \in \Xi$.
Recall, the concept of Harmonically convex function,

Definition 1.*[12] Suppose $\tau\colon \Xi \subseteq \Re \to \Re$ is a real valued function, then τ is said to be harmonically convex function if following holds*

$$\tau\left(\frac{\rtimes_1\rtimes_2}{\top \rtimes_1 + (1-\top)\rtimes_2}\right) \leq (1-\top)\tau(\rtimes_1) + \top\tau(\rtimes_2) \quad (3)$$

for all $\rtimes_1, \rtimes_2 \in \Xi$ and $\top \in [0,1]$.

In [12], Iscan gave H-H inequality for harmonically convex function

$$\tau\left(\frac{2\rtimes_1\rtimes_2}{\rtimes_1 + \rtimes_2}\right) \leq \frac{\rtimes_1\rtimes_2}{\rtimes_2 - \rtimes_1}\int_{\rtimes_1}^{\rtimes_2} \frac{\tau(\rtimes)}{\rtimes^2} d\rtimes \leqslant \frac{\tau(\rtimes_1)+\tau(\rtimes_2)}{2}. \quad (4)$$

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Definition 2.[13] *A function $\tau: \Xi \subseteq \Re \to \Re$ is declared to be MT-convex on Ξ, if following inequality holds:*

$$\tau(\top \rtimes_1 + (1-\top)\rtimes_2) \leqslant \frac{\sqrt{\top}}{2\sqrt{1-\top}}\tau(\rtimes_1) + \frac{\sqrt{1-\top}}{2\sqrt{\top}}\tau(\rtimes_2) \quad (5)$$

$\forall \rtimes_1, \rtimes_2 \in \Xi$ *and* $\top \in (0,1)$.

Definition 3.[14] *Suppose $\tau: \Xi \subseteq \Re \to \Re$ is a real valued function, then τ is said to be harmonically MT-convex function if following holds*

$$\tau\left(\frac{\rtimes_1 \rtimes_2}{\top \rtimes_1 + (1-\top)\rtimes_2}\right) \leq \frac{\sqrt{1-\top}}{2\sqrt{\top}}\tau(\rtimes_1) + \frac{\sqrt{\top}}{2\sqrt{1-\top}}\tau(\rtimes_2) \quad (6)$$

for all $\rtimes_1, \rtimes_2 \in \Xi$ *and* $\top \in (0,1)$.

In [15], M.A. Noor gave H-H inequality for harmonically MT-convex function

$$\tau\left(\frac{2\rtimes_1 \rtimes_2}{\rtimes_1 + \rtimes_2}\right) \leq \frac{\rtimes_1 \rtimes_2}{\rtimes_2 - \rtimes_1}\int_{\rtimes_1}^{\rtimes_2} \frac{\tau(\rtimes)}{\rtimes^2} d\rtimes \leqslant \pi \frac{\tau(\rtimes_1) + \tau(\rtimes_2)}{4}. \quad (7)$$

Merging the idea of fractional differentiation and fractal derivative from [15] , [20]-[24] for $\leftthreetimes$-type set $\Re^{\leftthreetimes}$ of real line numbers where, $0 < \leftthreetimes \leq 1$ also characterized by two binary operations, defined by ([26] Proposition 2) for $\rtimes_1^{\leftthreetimes}, \rtimes_2^{\leftthreetimes} \in \Re^{\leftthreetimes}$

$$\rtimes_1^{\leftthreetimes} + \rtimes_1^{\leftthreetimes} = (\rtimes_1 + \rtimes_2)^{\leftthreetimes}$$

and

$$\rtimes_1^{\leftthreetimes} \times \rtimes_2^{\leftthreetimes} = (\rtimes_1 \times \rtimes_2)^{\leftthreetimes}$$

Then, $(\Re^{\leftthreetimes}, +), (\Re^{\leftthreetimes} \backslash 0^{\leftthreetimes}, \times)$ are abelian groups and $(\Re^{\leftthreetimes}, +, \times)$ is field with $0^{\leftthreetimes}$ being additive identity and $[-\rtimes_1^{\leftthreetimes} = (-\rtimes_1)^{\leftthreetimes}]$ being additive inverse of $\rtimes_1^{\leftthreetimes}$ uniquely for $(\Re^{\leftthreetimes}, +)$.
also, $(1^{\leftthreetimes})$ being multiplicative identity and $\left[\left(\frac{1}{\rtimes_1}\right)^{\leftthreetimes} = \frac{1^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}} \neq \frac{1}{\rtimes_1^{\leftthreetimes}}\right]$ being multiplicative inverse of $\rtimes_1^{\leftthreetimes}$ uniquely for $(\Re^{\leftthreetimes} \backslash 0^{\leftthreetimes}, \times)$.
Also, $(\Re^{\leftthreetimes}, +, \times, <)$ is ordered field like $(\Re, +, \times, <)$ as,

$$\rtimes_1^{\leftthreetimes} < \rtimes_2^{\leftthreetimes} \in \Re^{\leftthreetimes} \Leftrightarrow \rtimes_1 < \rtimes_2 \in \Re$$

Definition 4.[15],[20] *A non-differentiable mapping $\tau: \Re \to \Re^{\leftthreetimes}$ is said to be local fractional continuous at $\rtimes_{10}$, if for any $\xi > 0$, there exists $n > 0$, satisfying that*

$$|\tau(\rtimes_1) - \tau(\rtimes_{10})| < \xi^{\leftthreetimes} \quad whenever \quad |\rtimes_1 - \rtimes_{10}| < n \quad (8)$$

if τ is local fractional continuous at $(\rtimes_1, \rtimes_2)$ then we denote it by $\tau(\rtimes) \in C_{\leftthreetimes}(\rtimes_1, \rtimes_2)$

Definition 5.[15],[20] *The local fractional derivative of $\tau: \Re \to \Re^{\leftthreetimes}$ of order $\leftthreetimes$ at $\rtimes_0 = \rtimes_{10}$ is defined by*

$$\tau^{(\leftthreetimes)}(\rtimes_0) = {}_{\rtimes_0}D_{\rtimes} = \left.\frac{d^{\leftthreetimes} s(\rtimes)}{d\rtimes^{\leftthreetimes}}\right|_{\rtimes=\rtimes_0} = \lim_{\rtimes \to \rtimes_0} \frac{\Delta^{\leftthreetimes}(\tau(\rtimes) - \tau(\rtimes_0))}{(\rtimes - \rtimes_0)^{\leftthreetimes}} \quad (9)$$

if τ is local fractional differentiable at $(\rtimes_1, \rtimes_2)$ then we denote it by $\tau(\rtimes) \in D_{\leftthreetimes}(\rtimes_1, \rtimes_2)$

Definition 6.[15],[20] *Let $\Delta = (\rtimes = \rtimes_0, \rtimes_1, \rtimes_2, ..., \rtimes_N = \rtimes_2), (N \in \aleph)$ and let $\tau(\rtimes) \in C_{\leftthreetimes}[\rtimes_1, \rtimes_2]$ be a partition of $[\rtimes_1, \rtimes_2]$ which satisfies $\rtimes_0 < \rtimes_1 < \rtimes_2 < ... < \rtimes_N$: Then, the local fractional integral of τ on $[\rtimes_1, \rtimes_2]$ of order $\leftthreetimes$ is defined as follows*

$${}_{\rtimes_1}\eth_{\rtimes_2}^{\leftthreetimes} = \frac{1}{\Gamma(1+\leftthreetimes)}\int_{\rtimes_1}^{\rtimes_2} \tau(\rtimes)(d\rtimes)^{\leftthreetimes} = \frac{1}{\Gamma(1+\leftthreetimes)} \lim_{\curlyvee_n \to 0} \Sigma_{j=0}^{N-1} \tau(\rtimes_{1j})(\Delta \rtimes_{1j}) \quad (10)$$

where, $\curlyvee n = max(\Delta\rtimes_0, \Delta\rtimes_1, \Delta\rtimes_2, ..., \Delta\rtimes_N)$ and $\Delta\rtimes_j = \rtimes_{j+1} - \rtimes_j$,
$j = 0, 1, 2, ..., N-1$.
if τ is local fractional integrable at $\rtimes \in [\rtimes_1, \rtimes_2]$ then we denote it by $\tau(\rtimes) \in \eth_{\rtimes}^{\leftthreetimes}[\rtimes_1, \rtimes_2]$
For more details one can also read Lemma 6 and Lemma 7 from [15] and Lemma 8 from [29]

Definition 7.[27] *Let $\Xi \subset (0, \infty)$ be an interval and let then $\tau: \Xi \to \Re^{\leftthreetimes}$*
$(0 < \leftthreetimes \leq 1)$ is said to be generalized harmonically convex function if the inequality holds:

$$\tau\left(\frac{\rtimes_1 \rtimes_2}{\top \rtimes_1 + (1-\top)\rtimes_2}\right) \leq (1-\top)^{\leftthreetimes}\tau(\rtimes_1) + \top^{\leftthreetimes}\tau(\rtimes_2) \quad (11)$$

for all $\rtimes_1, \rtimes_2 \in \Xi$ *and* $\top \in [0,1]$.

In [27], Sun gave H-H inequality for generalized harmonically convex function

$$\frac{1}{\Gamma(1+\leftthreetimes)}\tau\left(\frac{2\rtimes_1\rtimes_2}{\rtimes_1 + \rtimes_2}\right) \leq \left(\frac{\rtimes_1\rtimes_2}{\rtimes_2 - \rtimes_1}\right)^{\leftthreetimes} {}_{\rtimes_1}\eth_{\rtimes_2}^{\leftthreetimes} \frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}} d\rtimes \leqslant [\tau(\rtimes_1) + \tau(\rtimes_2)]\frac{\Gamma(1+\leftthreetimes)}{\Gamma(1+2\leftthreetimes)}. \quad (12)$$

Merging the concepts of 6 and 11 we introduce,

Definition 8.*Suppose $\tau: \Xi \subseteq \Re \to \Re$ is a real valued function, then τ is said to be generalized harmonically*

MT-convex function if following holds

$$\tau\left(\frac{\rtimes_1\rtimes_2}{\top\rtimes_1+(1-\top)\rtimes_2}\right) \leq \left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\curlywedge}\tau(\rtimes_1)+\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\curlywedge}\tau(\rtimes_2) \quad (13)$$

for all $\rtimes_1,\rtimes_2\in\Xi$ *and* $\top\in(0,1)$.

Definition 9.*[28] Let* $\Upsilon=\Upsilon(\ell)_{\ell=0}^{\infty}$ *be a bounded sequence of real numbers, and* $\varpi^{\Upsilon}_{\iota,\chi}(.)\iota,\chi>0$ *be Raina's function defined as*

$$\varpi^{\Upsilon}_{\iota,\chi}(\top)=\varpi^{\Upsilon(0),\Upsilon(1),\dots}_{\iota,\chi}(\top)=\Sigma_{m=0}^{\infty}\frac{\Upsilon(\ell)}{\Gamma(\iota\ell+\Upsilon)}\top^{\ell} \quad ;|\top|<\Re \ \ \iota,\chi>0 \quad (14)$$

Definition 10.*[28] Let* $\Upsilon=\Upsilon(\ell)_{\ell=0}^{\infty}$ *be a bounded sequence of real numbers, and* $\iota,\chi>0$. *A nonempty set* $\Xi_{\varpi}\subset(0,\infty)$ *is said to be generalized harmonically* Ψ*-convex set, if*

$$\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\in H_{\varpi} \quad (15)$$

where, $\rtimes_1,\rtimes_2\in\Xi_{\varpi}$ *and* $\top\in[0,1]$

Merging the concepts of 13 and 15, we introduce,

Definition 11.*Let* $\Upsilon=\Upsilon(\ell)_{\ell=0}^{\infty}$ *be a bounded sequence of real numbers, and* $\iota,\chi>0$. *A mapping* $\tau:\Xi\to\Re^{\curlywedge}(0<\curlywedge\leq1)$ *is said to be generalized harmonically* Ψ*-MT-convex function if the inequality holds:*

$$\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right) \leq \left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\curlywedge}\tau(\rtimes_1)+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\curlywedge}\tau(\rtimes_2) \quad (16)$$

where, $\rtimes_1,\rtimes_2\in\Xi_{\varpi}$ *and* $\top\in(0,1)$

*Remark.*Choosing $\curlywedge=1$,we get harmonically Ψ-MT-convex function.

$$\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right) \leq \left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)\tau(\rtimes_1)+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)\tau(\rtimes_2) \quad (17)$$

where, $\rtimes_1,\rtimes_2\in\Xi_{\varpi}$ and $\top\in(0,1)$

*Remark.*Choosing $\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)=(\rtimes_2-\rtimes_1)$,we get generalized harmonically MT-convex function.

*Remark.*Choosing $\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)=(\rtimes_2-\rtimes_1)$ and $\curlywedge=1$,we get harmonically MT-convex function.

2 Hermite-Hadamard type inequalities

This paper is organized as follows, initially in this section we will present our main results including H-H type inequalities via fractional integral operators. However the last section is devoted for applications of our results.

Theorem 1.*Suppose,* $\iota,\chi>0,\Upsilon=\Upsilon(\ell)_{\ell=0}^{\infty}$ *be a bounded sequence of real numbers, and* $\tau:\Xi_{\varpi}=[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]\subset\Re/0\to\Re^{\curlywedge}(0<\curlywedge\leq1)$ *be generalized harmonically* Ψ*-MT-convex function, where* $\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)\in\Xi_{\varpi}$ *also* $\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)>0$ *such that* $\tau(\rtimes)\in D_{\curlywedge}[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]$ *and* $\tau^{\curlywedge}(\rtimes)\in C_{\curlywedge}[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]$ *then, following inequality holds;*

$$\frac{1}{\Gamma(1+\curlywedge)}\tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right) \leq \frac{\rtimes_1^{\curlywedge}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}\ {}_{\rtimes_1}\partial^{\curlywedge}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(n)}{n^{2\curlywedge}} \leq \Gamma(1+\curlywedge)B_{(\frac{1}{2},\frac{1}{2})}\frac{[\tau(\rtimes_1)+\tau(\rtimes_2)]}{2^{\curlywedge}} \quad (18)$$

where,

$$B_{n_1,n_2}=\frac{1}{\Gamma(1+\curlywedge)}\int_0^1\top^{\curlywedge(n_1-1)}(1-\top)^{\curlywedge(n_2-1)}(d\top)^{\curlywedge} \quad ;\ (n_1,n_2>0) \quad (19)$$

*Proof.*Since τ is generalized harmonically Ψ-MT-convex function,

$$\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right) \leq \left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\curlywedge}\tau(\rtimes_1)+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\curlywedge}\tau(\rtimes_2) \quad (20)$$

when $\top=\frac{1}{2}$

$$\tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)\leq\frac{\tau(\rtimes_1)+\tau(\rtimes_2)}{2^{\curlywedge}}$$

Taking,

$$\rtimes_1=\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+(1-\top)\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}$$

and

$$\rtimes_2=\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}$$

Then, by local fractional integral over (0,1) we get,

$$\frac{2^{\leftthreetimes}}{\Gamma(1+\leftthreetimes)}\int_0^1 \tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)(d\top)^{\leftthreetimes}$$
$$\leq \int_0^1\left[\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+(1-\top)\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)\right.$$
$$\left.+\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)\right](d\top)^{\leftthreetimes}$$

$$=\frac{2^{\leftthreetimes}\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\frac{1}{\Gamma(1+\leftthreetimes)}$$
$$\times\int_{\rtimes_1}^{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(n)}{\rtimes^{2\leftthreetimes}}(d\rtimes)^{\leftthreetimes}$$
$$=\frac{2^{\leftthreetimes}\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\;{}_{\rtimes_1}\eth^{\leftthreetimes}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}}$$

Also,

$$\frac{2^{\leftthreetimes}}{\Gamma(1+\leftthreetimes)}\int_0^1 \tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)(d\top)^{\leftthreetimes}$$
$$=\frac{2^{\leftthreetimes}}{\Gamma(1+\leftthreetimes)}\tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)(d\top)^{\leftthreetimes}$$

So,

$$\frac{1}{\Gamma(1+\leftthreetimes)}\tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)$$
$$\leq\frac{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\;{}_{\rtimes_1}\eth^{\leftthreetimes}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}} \tag{21}$$

For other inequality, by 16

$$\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+(1-\top)\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)$$
$$+\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)$$
$$\leq\left[\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}+\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}\right][\tau(\rtimes_1)+\tau(\rtimes_2)]$$

local fractional integration over $(0,1)$ and 19 yields,

$$\frac{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\;{}_{\rtimes_1}\eth^{\leftthreetimes}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}}$$
$$\leq\Gamma(1+\leftthreetimes)B_{(\frac{1}{2},\frac{1}{2})}\frac{[\tau(\rtimes_1)+\tau(\rtimes_2)]}{2^{\leftthreetimes}} \tag{22}$$

both 21 and 22 give the required inequality.

Corollary 1. *Choosing* $\leftthreetimes=1$, *we get*

$$\tau\left(\frac{2\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)$$
$$\leq\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\int_{\rtimes_1}^{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^2}(d\rtimes)$$
$$\leq\frac{\pi}{2}[\tau(\rtimes_1)+\tau(\rtimes_2)] \tag{23}$$

Corollary 2. *Choosing* $\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)=(\rtimes_2-\rtimes_1)$ *we get,*

$$\frac{1}{\Gamma(1+\leftthreetimes)}\tau\left(\frac{2\rtimes_1\rtimes_2}{\rtimes_1+\rtimes_2}\right)\leq\frac{\rtimes_1^{\leftthreetimes}\rtimes_2^{\leftthreetimes}}{(\rtimes_2-\rtimes_1)^{\leftthreetimes}}\;{}_{\rtimes_1}\eth^{\leftthreetimes}_{\rtimes_2}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}}$$
$$\leq\Gamma(1+\leftthreetimes)B_{(\frac{1}{2},\frac{1}{2})}\frac{[\tau(\rtimes_1)+\tau(\rtimes_2)]}{2^{\leftthreetimes}} \tag{24}$$

Corollary 3. *Choosing* $\leftthreetimes=1$ *and* $\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)=(\rtimes_2-\rtimes_1)$ *,we get*

$$\tau\left(\frac{2\rtimes_1\rtimes_2}{\rtimes_1+\rtimes_2}\right)\leq\frac{\rtimes_1\rtimes_2}{(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\frac{\tau(n)}{n^2}$$
$$\leq[\tau(\rtimes_1)+\tau(\rtimes_2)]\frac{\pi}{2} \tag{25}$$

3 Hermite-Hadamard type inequalities via fractional integral operators

In this section, by using following Lemma one can extend to some new H-H type inequalities for generalized harmonically Ψ-MT-convex function.

Lemma 1. *([26] Lemma 16) Suppose,* $\iota,\chi>0,\Upsilon=\Upsilon(\ell)_{\ell=0}^{\infty}$ *be a bounded sequence of real numbers, and* $\tau:\Xi^{o}_{\varpi}=[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]\subset\Re/0\to\Re^{\leftthreetimes}(0<\leftthreetimes\leq 1)$ *(*Ξ^{o}_{ϖ} *is interior of* Ξ_{ϖ}*) be generalized harmonically* Ψ*-convex function, where* $\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)\in\Xi_{\varpi}$ *also* $\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)>0$ *such that* $\tau(\rtimes)\in D_{\leftthreetimes}[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]$ *and* $\tau^{\leftthreetimes}(\rtimes)\in C_{\leftthreetimes}[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]$ *then, following* *inequality holds;*

$$
\begin{aligned}
&\Xi(\rtimes_1,\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1);\hbar,\leftthreetimes)\\
&=\frac{(\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)^{\leftthreetimes})}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\left[\frac{1}{\Gamma(1+\leftthreetimes)}\right.\\
&\times\int_0^{1-\hbar}\top^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2\leftthreetimes}\\
&\times\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)(d\top)^{\leftthreetimes}\\
&+\frac{1}{\Gamma(1+\leftthreetimes)}\\
&\times\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2\leftthreetimes}\\
&\left.\times\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)(d\top)^{\leftthreetimes}\right] \qquad (26)
\end{aligned}
$$

where,

$$
\begin{aligned}
&\Xi(\rtimes_1,\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1);\hbar,\leftthreetimes)\\
&=(1-\hbar)^{\leftthreetimes}\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\\
&+(\hbar)^{\leftthreetimes}\tau\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\\
&-\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{\leftthreetimes}\\
&\Gamma(1+\leftthreetimes)\,{}_{\rtimes_1}\partial^{\leftthreetimes}_{\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}}
\end{aligned}
$$

$\forall\,\hbar\in[0,1]$ *and* $\leftthreetimes\in(0,1]$.

Theorem 2. *Letting the assumptions of Lemma1 are satisfied. If* $|\tau^{\leftthreetimes}|^q$ *is a generalized harmonically* Ψ*-MT-convex function on* Ξ_{ϖ}. *for* $p,q>1,\ p^{-1}+q^{-1}=1$. *Then following holds,*

$$
\begin{aligned}
&|\Xi(\rtimes_1,\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1);\hbar,\leftthreetimes)|\\
&\le\frac{(\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\left[\frac{\Gamma(1+\leftthreetimes)}{\Gamma(1+2\leftthreetimes)}\right]^{1-\frac{1}{q}}\\
&\times\left[\left((1-\hbar)^{2(q-1)\leftthreetimes}\left[V_1|\tau(\rtimes_1)|^q+V_2|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left((\hbar)^{2(q-1)\leftthreetimes}\left[V_3|\tau(\rtimes_1)|^q+V_4|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}\right] \qquad (27)
\end{aligned}
$$

where,

$$
\begin{aligned}
V_1=&\frac{1}{\Gamma(1+\leftthreetimes)}\\
&\times\int_0^{1-\hbar}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes}\\
&\times\left(\frac{\top^{\frac{3}{2}}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
V_2=&\frac{1}{\Gamma(1+\leftthreetimes)}\\
&\times\int_0^{1-\hbar}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes}\\
&\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
V_3=&\frac{1}{\Gamma(1+\leftthreetimes)}\\
&\times\int_{1-\hbar}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes}\\
&\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
V_4=&\frac{1}{\Gamma(1+\leftthreetimes)}\\
&\times\int_{1-\hbar}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes}\\
&\times\left(\frac{(1-\top)^{\frac{3}{2}}}{2\sqrt{\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (31)
\end{aligned}
$$

Proof. From Lemma1, generalized power mean inequality and 16

$$
\begin{aligned}
&|\Xi(\rtimes_1,\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1);\hbar,\leftthreetimes)|\\
&=\frac{(\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\left[\frac{1}{\Gamma(1+\leftthreetimes)}\right.\\
&\times\int_0^{1-\hbar}\top^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2\leftthreetimes}\\
&\times\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\right|(d\top)^{\leftthreetimes}\\
&+\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes}\\
&\times\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2\leftthreetimes}\\
&\left.\times\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\right|(d\top)^{\leftthreetimes}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}} \\
&\times\Bigg[\left(\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\top^{\leftthreetimes}(d\top)^{\leftthreetimes}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\top^{\leftthreetimes} \\
&\times\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)\right|^{q}(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes}(d\top)^{\leftthreetimes}\Bigg)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes} \\
&\times\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)\right|^{q}(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}} \\
&\times\Bigg[\left(\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\top^{\leftthreetimes}(d\top)^{\leftthreetimes}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\top^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\Bigg[\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^{q} \\
&+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^{q}\Bigg](d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes}(d\top)^{\leftthreetimes}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes} \\
&\times\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\Bigg[\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^{q} \\
&+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^{q}\Bigg](d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}} \\
&\times\Bigg[\left(\frac{\Gamma(1+\leftthreetimes)}{\Gamma(1+2\leftthreetimes)}(1-\hbar)^{2\leftthreetimes}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times\int_0^{1-\hbar}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\Bigg[\left(\frac{\top^{\frac{3}{2}}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^{q} \\
&+\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^{q}\Bigg](d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(1+\leftthreetimes)}{\Gamma(1+2\leftthreetimes)}(\hbar)^{2\leftthreetimes}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\Bigg[\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^{q} \\
&+\left(\frac{(1-\top)^{\frac{3}{2}}}{2\sqrt{\top}}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^{q}\Bigg](d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\left[\frac{\Gamma(1+\leftthreetimes)}{\Gamma(1+2\leftthreetimes)}\right]^{1-\frac{1}{q}} \\
&\times\Bigg(\Bigg[(1-\hbar)^{2(q-1)\leftthreetimes} \\
&\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\left(\frac{\top^{\frac{3}{2}}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^{q}(d\top)^{\leftthreetimes}\Bigg) \\
&+\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^{q}(d\top)^{\leftthreetimes}\Bigg)\Bigg]^{\frac{1}{q}} \\
&+\Bigg[(\hbar)^{2(q-1)\leftthreetimes}\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times\int_{1-\hbar}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^{q}(d\top)^{\leftthreetimes}\Bigg) \\
&+\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes} \\
&\times\left(\frac{(1-\top)^{\frac{3}{2}}}{2\sqrt{\top}}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^{q}(d\top)^{\leftthreetimes}\Bigg]^{\frac{1}{q}}\Bigg)
\end{aligned}
$$

$$\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}{\rtimes_1^{\curlywedge}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}\left[\frac{\Gamma(1+\curlywedge)}{\Gamma(1+2\curlywedge)}\right]^{1-\frac{1}{q}}$$
$$\times\left[\left((1-\hbar)^{2(q-1)\curlywedge}\left[V_1|\tau(\rtimes_1)|^q+V_2|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left((\hbar)^{2(q-1)\curlywedge}\left[V_3|\tau(\rtimes_1)|^q+V_4|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}\right]$$

from 28, 29, 30 and 31 we get the required inequality.

Corollary 4. *Choosing $\hbar=0$ we get, from 27*

$$|\tau(\rtimes_2)-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\curlywedge}$$
$$\times\Gamma(1+\curlywedge)_{\rtimes_1}\partial^{\curlywedge}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\curlywedge}}|$$
$$\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}{\rtimes_1^{\curlywedge}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}\left[\frac{\Gamma(1+\curlywedge)}{\Gamma(1+2\curlywedge)}\right]^{1-\frac{1}{q}}$$
$$\times\left(\left[V_5|\tau(\rtimes_1)|^q+V_6|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}$$

where

$$V_5=\frac{1}{\Gamma(1+\curlywedge)}\int_0^1\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\curlywedge}\times\left(\frac{\top^{\frac{3}{2}}}{2\sqrt{1-\top}}\right)^{\curlywedge}(d\top)^{\curlywedge}\quad(32)$$

$$V_6=\frac{1}{\Gamma(1+\curlywedge)}\int_0^1\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\curlywedge}\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\curlywedge}(d\top)^{\curlywedge}\quad(33)$$

Corollary 5. *Choosing $\hbar=1$ we get, from 27*

$$|\tau(\rtimes_1)-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\curlywedge}$$
$$\times\Gamma(1+\curlywedge)_{\rtimes_1}\partial^{\curlywedge}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\curlywedge}}|$$
$$\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}{\rtimes_1^{\curlywedge}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}\left[\frac{\Gamma(1+\curlywedge)}{\Gamma(1+2\curlywedge)}\right]^{1-\frac{1}{q}}$$
$$\times\left(\left[V_7|\tau(\rtimes_1)|^q+V_8|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}$$

$$V_7=\frac{1}{\Gamma(1+\curlywedge)}\int_0^1\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\curlywedge}\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\curlywedge}(d\top)^{\curlywedge}\quad(34)$$

$$V_8=\frac{1}{\Gamma(1+\curlywedge)}\int_0^1\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\curlywedge}\times\left(\frac{(1-\top)^{\frac{3}{2}}}{2\sqrt{\top}}\right)^{\curlywedge}(d\top)^{\curlywedge}\quad(35)$$

Corollary 6. *Taking mean of Corollary 4 and Corollary5 we get, from 32, 33, 34, 35*

$$|\frac{\tau(\rtimes_1)+\tau(\rtimes_2)}{2^{\curlywedge}}-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\curlywedge}$$
$$\times\Gamma(1+\curlywedge)_{\rtimes_1}\partial^{\curlywedge}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\curlywedge}}|$$
$$\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}{\rtimes_1^{\curlywedge}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}\left[\frac{\Gamma(1+\curlywedge)}{\Gamma(1+2\curlywedge)}\right]^{1-\frac{1}{q}}$$
$$\times\left(\left[V_5|\tau(\rtimes_1)|^q+V_6|\tau(\rtimes_2)|^q\right]^{\frac{1}{q}}+\left[V_7|\tau(\rtimes_1)|^q+V_8|\tau(\rtimes_2)|^q\right]^{\frac{1}{q}}\right)$$

Corollary 7. *Choosing $\hbar=\frac{1}{2}$ we get from 27*

$$|s\left(\frac{2\rtimes_1\rtimes_2}{\rtimes_1+\rtimes_2}\right)-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\curlywedge}$$
$$\times\Gamma(1+\curlywedge)_{\rtimes_1}\partial^{\curlywedge}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\curlywedge}}|$$
$$\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}{\rtimes_1^{\curlywedge}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\curlywedge}}$$
$$\times\left[\frac{\Gamma(1+\curlywedge)}{\Gamma(1+2\curlywedge)}\right]^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{2(\frac{q-1}{q})\curlywedge}$$
$$\times\left(\left[V_9|\tau(\rtimes_1)|^q+V_{10}|\tau(\rtimes_2)|^q\right]^{\frac{1}{q}}+\left[V_{11}|\tau(\rtimes_1)|^q+V_{12}|\tau(\rtimes_2)|^q\right]^{\frac{1}{q}}\right)$$

where,

$$V_9=\frac{1}{\Gamma(1+\curlywedge)}\int_0^{\frac{1}{2}}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\curlywedge}\times\left(\frac{\top^{\frac{3}{2}}}{2\sqrt{1-\top}}\right)^{\curlywedge}(d\top)^{\curlywedge}$$

$$V_{10}=\frac{1}{\Gamma(1+\curlywedge)}\int_0^{\frac{1}{2}}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2q\curlywedge}\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\curlywedge}(d\top)^{\curlywedge}$$

$$V_{11}=\frac{1}{\Gamma(1+\leftthreetimes)}\int_{\frac{1}{2}}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes}\times\left(\frac{\sqrt{\top(1-\top)}}{2}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes}$$

$$V_{12}=\frac{1}{\Gamma(1+\leftthreetimes)}\int_{\frac{1}{2}}^{1}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2q\leftthreetimes}\times\left(\frac{(1-\top)^{\frac{3}{2}}}{2\sqrt{\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes}$$

Theorem 3. *Letting the assumptions of Lemma 1 are satisfied. If $|\tau^{\leftthreetimes}|^q$ is a generalized harmonically Ψ-MT-convex function on Ξ_{ϖ}. for $p,q>1,\ p^{-1}+q^{-1}=1$. Then following holds,*

$$|\Xi(\rtimes_1,\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1);\hbar,\leftthreetimes)|\leq\frac{(\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\Bigg[\left(U_1\right)^{\frac{1}{p}}\times\left(\Big[A(\hbar,\leftthreetimes)|\tau(\rtimes_1)|^q+B(\hbar,\leftthreetimes)|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}+\left(U_2\right)^{\frac{1}{p}}\times\left(\Big[C(\hbar,\leftthreetimes)|\tau(\rtimes_1)|^q+D(\hbar,\leftthreetimes)|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}\Bigg]\quad(36)$$

where,

$$U_1=\frac{1}{\Gamma(1+\leftthreetimes)}\times\int_0^{1-\hbar}\top^{p\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes}\quad(37)$$

$$A(\hbar,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes}\quad(38)$$

$$B(\hbar,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes}\quad(39)$$

$$U_2=\frac{1}{\Gamma(1+\leftthreetimes)}\times\int_{1-\hbar}^{1}(1-\top)^{p\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes}\quad(40)$$

$$C(\hbar,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes}\quad(41)$$

$$D(\hbar,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes}\quad(42)$$

Proof. From Lemma 1, generalized Holder‘s inequality and 16

$$\begin{aligned}&|\Xi(\rtimes_1,\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1);\hbar,\leftthreetimes)|\\&=\frac{(\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\times\Bigg[\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^{1-\hbar}\top^{\leftthreetimes}\times\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2\leftthreetimes}\\&\quad\times\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\right|(d\top)^{\leftthreetimes}+\frac{1}{\Gamma(1+\leftthreetimes)}\int_{1-\hbar}^{1}(1-\top)^{\leftthreetimes}\\&\quad\times\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2\leftthreetimes}\times\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\right|(d\top)^{\leftthreetimes}\Bigg]\end{aligned}$$

$$\begin{aligned}&\leq\frac{(\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\Bigg[\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\times\int_0^{1-\hbar}\top^{p\leftthreetimes}\\&\quad\times\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{p}}\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\\&\quad\times\int_0^{1-\hbar}\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\right|^q(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}}+\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\times\int_{1-\hbar}^{1}(1-\top)^{p\leftthreetimes}\\&\quad\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{p}}\times\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)}\\&\quad\times\int_{1-\hbar}^{1}\left|\tau^{\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi_{\iota,\chi}^{\Upsilon}(\rtimes_2-\rtimes_1)}\right)\right|^q(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{q}}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\Bigg[\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times \int_0^{1-\hbar}\top^{p\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{p}} \\
&\times\left(\frac{1}{\Gamma(1+\leftthreetimes)}\right. \\
&\times \int_0^{1-\hbar}\left[\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^q\right. \\
&\left.\left.+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^q\right](d\top)^{\leftthreetimes}\right)^{\frac{1}{q}} \\
&+\Bigg(\frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times \int_{1-\hbar}^1 (1-\top)^{p\leftthreetimes} \\
&\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes}\Bigg)^{\frac{1}{p}} \\
&\times\left(\frac{1}{\Gamma(1+\leftthreetimes)}\right. \\
&\times \int_{1-\hbar}^1\left[\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}|\tau(\rtimes_1)|^q\right. \\
&\left.\left.+\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}|\tau(\rtimes_2)|^q\right](d\top)^{\leftthreetimes}\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\Bigg[\Big(U_1\Big)^{\frac{1}{p}} \\
&\times\left(\left[A(\hbar,\leftthreetimes)|\tau(\rtimes_1)|^q+B(\hbar,\leftthreetimes)|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}} \\
&+\Big(U_2\Big)^{\frac{1}{p}}\times\Big(\Big[C(\hbar,\leftthreetimes)|\tau(\rtimes_1)|^q \\
&+D(\hbar,\leftthreetimes)|\tau(\rtimes_2)|^q\Big]\Big)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

from 37, 38, 39, 40, 41, 42 we get the required inequality.

Corollary 8. *Choosing* $\hbar=0$ *we get, from 36*

$$
\begin{aligned}
&|\tau(\rtimes_2)-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\leftthreetimes}\Gamma(1+\leftthreetimes) \\
&\times {}_{\rtimes_1}\partial^{\leftthreetimes}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}}| \\
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\Bigg[\Big(U_3\Big)^{\frac{1}{p}} \\
&\times\left(\left[A(0,\leftthreetimes)|\tau(\rtimes_1)|^q+B(0,\leftthreetimes)|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

where,

$$
\begin{aligned}
U_3 &= \frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times \int_0^1 \top^{pa}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2pa}(d\top)^{\leftthreetimes} \qquad (43)
\end{aligned}
$$

$$A(0,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^1\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (44)$$

$$B(0,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^1\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (45)$$

Corollary 9. *Choosing* $\hbar=1$ *we get, from 36*

$$
\begin{aligned}
&|\tau(\rtimes_1)-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\leftthreetimes}\Gamma(1+\leftthreetimes) \\
&\times {}_{\rtimes_1}\partial^{\leftthreetimes}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\leftthreetimes}}| \\
&\leq \frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}{\rtimes_1^{\leftthreetimes}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\leftthreetimes}}\Bigg[\Big(U_4\Big)^{\frac{1}{p}} \\
&\times\left(\left[C(1,\leftthreetimes)|\tau(\rtimes_1)|^q+D(1,\leftthreetimes)|\tau(\rtimes_2)|^q\right]\right)^{\frac{1}{q}}\Bigg]
\end{aligned}
$$

where,

$$
\begin{aligned}
U_4 &= \frac{1}{\Gamma(1+\leftthreetimes)} \\
&\times \int_0^1 (1-\top)^{p\leftthreetimes}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2p\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (46)
\end{aligned}
$$

$$C(1,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^1\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (47)$$

$$D(1,\leftthreetimes)=\frac{1}{\Gamma(1+\leftthreetimes)}\int_0^1\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\leftthreetimes}(d\top)^{\leftthreetimes} \qquad (48)$$

Corollary 10.*Taking mean of Corollary 8 and Corollary9 we get, from 43, 44, 45, 46, 47, 48.*

$$\Big|\frac{\tau(\rtimes_1)+\tau(\rtimes_2)}{2^{\lambda}}-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\lambda}\times\Gamma(1+\lambda)\times{}_{\rtimes_1}\partial^{\lambda}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\lambda}}\Big|$$
$$\leq\frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\lambda}}{2^{\lambda}\rtimes_1^{\lambda}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\lambda}}\Bigg[\Big(U_3\Big)^{\frac{1}{p}}\times\left(\Big[A(0,\lambda)|\tau(\rtimes_1)|^q+B(0,\lambda)|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}+\Big(U_4\Big)^{\frac{1}{p}}\left(\Big[C(1,\lambda)|\tau(\rtimes_1)|^q+D(1,\lambda)|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}\Bigg]$$

Corollary 11.*Choosing $\hbar=\frac{1}{2}$ we get from 36*

$$\Big|\tau\left(\frac{2\rtimes_1\rtimes_2}{\rtimes_1+\rtimes_2}\right)-\left(\frac{\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}\right)^{\lambda}\times\Gamma(1+\lambda)_{\rtimes_1}\partial^{\lambda}_{\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\lambda}}\Big|$$
$$\leq\frac{(\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\lambda}}{\rtimes_1^{\lambda}(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))^{\lambda}}\Bigg[\Big(U_5\Big)^{\frac{1}{p}}\times\left(\Big[A\Big(\frac{1}{2},\lambda\Big)|\tau(\rtimes_1)|^q+B\Big(\frac{1}{2},\lambda\Big)|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}+\Big(U_6\Big)^{\frac{1}{p}}\times\left(\Big[C\Big(\frac{1}{2},\lambda\Big)|\tau(\rtimes_1)|^q+D\Big(\frac{1}{2},\lambda\Big)|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}\Bigg]$$

where,

$$U_5=\frac{1}{\Gamma(1+\lambda)}\times\int_0^{\frac{1}{2}}\top^{p\lambda}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2p\lambda}(d\top)^{\lambda}$$

$$A\Big(\frac{1}{2},\lambda\Big)=\frac{1}{\Gamma(1+\lambda)}\int_0^{\frac{1}{2}}\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\lambda}(d\top)^{\lambda}$$
$$B\Big(\frac{1}{2},\lambda\Big)=\frac{1}{\Gamma(1+\lambda)}\int_0^{\frac{1}{2}}\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\lambda}(d\top)^{\lambda}$$

$$U_6=\frac{1}{\Gamma(1+\lambda)}\times\int_{\frac{1}{2}}^{1}(1-\top)^{p\lambda}\left(\frac{\rtimes_1(\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1))}{\rtimes_1+\top\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)}\right)^{2p\lambda}(d\top)^{\lambda}$$

$$C\Big(\frac{1}{2},\lambda\Big)=\frac{1}{\Gamma(1+\lambda)}\int_{\frac{1}{2}}^{1}\left(\frac{\sqrt{\top}}{2\sqrt{1-\top}}\right)^{\lambda}(d\top)^{\lambda}$$
$$D\Big(\frac{1}{2},\lambda\Big)=\frac{1}{\Gamma(1+\lambda)}\int_{\frac{1}{2}}^{1}\left(\frac{\sqrt{1-\top}}{2\sqrt{\top}}\right)^{\lambda}(d\top)^{\lambda}$$

4 Applications

Definition 12.*In Definition 9 if $\Upsilon=(1,1,1,\ldots)$ with $c=\curlyvee,(\Re(\curlyvee)>0)$, $\Upsilon=1$ and $\top\in\check{C}$,(complex) Then we have,*

$$\complement_{\curlyvee_1}(\top)=\Sigma_{\ell=0}^{\infty}\frac{1}{\Gamma(1+\curlyvee_1\ell)}\top^{\ell}\tag{49}$$

Theorem 4.*Let $\tau:\Xi_{\varpi}=[\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)]\subset\Re\{0\}\to\Re^{\lambda}(0<\lambda\leq 1)$ be generalized harmonically Ψ-MT-convex function, where $\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)\in\Xi_{\varpi}$, $\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)>0$ such that $\tau\in D_{\lambda}[\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)]$ and $\tau^{\lambda}\in C_{\lambda}[\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)]$. then from 19, the following hold;*

$$\frac{1}{\Gamma(1+\lambda)}\tau\left(\frac{2\rtimes_1(\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1))}{2\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)}\right)\leq\frac{\rtimes_1^{\lambda}(\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1))^{\lambda}}{(\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1))^{\lambda}}\,{}_{\rtimes_1}\partial^{\lambda}_{\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)}\frac{\tau(\rtimes)}{\rtimes^{2\lambda}}\leq\Gamma(1+\lambda)B_{(\frac{1}{2},\frac{1}{2})}\frac{[\tau(\rtimes_1)+\tau(\rtimes_2)]}{2^{\lambda}}\tag{50}$$

Theorem 5.*Suppose, $\iota,\chi>0,\Upsilon=\Upsilon(\ell)_{\ell=0}^{\infty}$ be a bounded sequence of real numbers, and $\tau^{o}_{\varpi}=[\rtimes_1,\rtimes_1+\varpi^{\Upsilon}_{\iota,\chi}(\rtimes_2-\rtimes_1)]\subset\Re/0\to\Re^{a}(0<\lambda\leq 1)(\Xi^{o}_{\varpi}$ is the interior of$\Xi_{\varpi})$ be generalized harmonically Ψ-MT-convex function, where $\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)\in\Xi_{\varpi}$ also $\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)>0$ such that $\tau(\rtimes)\in D_{\lambda}[\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)]$ and $\tau^{\lambda}(\rtimes)\in C_{\lambda}[\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1)]$ then, following inequality holds;*

$$|\Xi(\rtimes_1,\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1);\hbar,\lambda)|\leq\frac{(\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1))^{\lambda}}{\rtimes_1^{\lambda}(\rtimes_1+\complement_{\curlyvee_1}(\rtimes_2-\rtimes_1))^{\lambda}}\left[\frac{\Gamma(1+\lambda)}{\Gamma(1+2\lambda)}\right]^{1-\frac{1}{q}}\times\Bigg[\left((1-\hbar)^{2(q-1)\lambda}\Big[W_1|\tau(\rtimes_1)|^q+W_2|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}+\left((\hbar)^{2(q-1)\lambda}\Big[W_3|\tau(\rtimes_1)|^q+W_4|\tau(\rtimes_2)|^q\Big]\right)^{\frac{1}{q}}\Bigg]$$

where,

$$W_1 = \frac{1}{\Gamma(1+\leftthreetimes)} \int_0^{1-\hbar} \left(\frac{\rtimes_1(\rtimes_1 + \mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1))}{\rtimes_1 + \top\mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1)} \right)^{2q\leftthreetimes} \times \left(\frac{\top^{\frac{3}{2}}}{2\sqrt{1-\top}} \right)^{\leftthreetimes} (d\top)^{\leftthreetimes}$$

$$W_2 = \frac{1}{\Gamma(1+\leftthreetimes)} \int_0^{1-\hbar} \left(\frac{\rtimes_1(\rtimes_1 + \mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1))}{\rtimes_1 + \top\mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1)} \right)^{2q\leftthreetimes} \times \left(\frac{\sqrt{\top(1-\top)}}{2} \right)^{\leftthreetimes} (d\top)^{\leftthreetimes}$$

$$W_3 = \frac{1}{\Gamma(1+\leftthreetimes)} \int_{1-\hbar}^{1} \left(\frac{\rtimes_1(\rtimes_1 + \mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1))}{\rtimes_1 + \top\mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1)} \right)^{2q\leftthreetimes} \times \left(\frac{\sqrt{\top(1-\top)}}{2} \right)^{\leftthreetimes} (d\top)^{\leftthreetimes}$$

$$W_4 = \frac{1}{\Gamma(1+\leftthreetimes)} \int_{1-\hbar}^{1} \left(\frac{\rtimes_1(\rtimes_1 + \mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1))}{\rtimes_1 + \top\mathsf{C}_{\Upsilon_1}(\rtimes_2 - \rtimes_1)} \right)^{2q\leftthreetimes} \times \left(\frac{(1-\top)^{\frac{3}{2}}}{2\sqrt{\top}} \right)^{\leftthreetimes} (d\top)^{\leftthreetimes}$$

5 Conclusions

In this study, we discussed some inequalities of Hermite-Hadamard type and certain related variants with respect to the Raina's function for a new class of harmonically convex functions, namely, generalized harmonically ψ-MT-convex functions established on fractal set techniques With the help of an auxiliary identity associated with Raina's function, by generalized Holder inequality and generalized power mean, generalized midpoint type, Ostrowski type, and trapezoid type inequalities via local fractional integral for generalized harmonically ψ-MT-convex functions. Further, the presented technique yields results by establishing some special values for the parameters or applying limiting suppositions, and it is completely practicable for restoring the existing inequalities in the associated literature. In the future, reviewers may discover many novel inequities from a variety of applied and pure disciplines based on our findings. They can also use our technique to establish applications to special ways for diverse generalized harmonically ψ-MT-convex functions.

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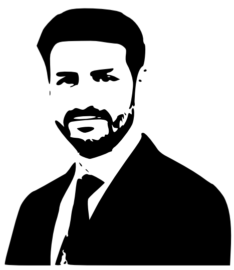

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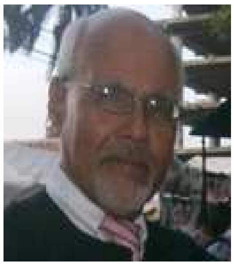

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