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On the blow up criterion for the 3D nematic liquid crystal flows involving the second eigenvalue of the deformation tensor

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Abstract: In this paper, we study the blow up criterion of the smooth solutions to the three-dimensional incompressible nematic liquid crystal flows in terms of λ_2^+ in the multiplier space $\dot{X}_1$ and ∇d in BMO. It is shown that the solution (u,d) can be extended beyond $t=T$ if

$$\int_0^T \left(\frac{\|\lambda_2^+(\cdot,t)\|^2_{\dot{X}_1}}{\ln(e+\|\nabla u(\cdot,t)\|_{\dot{X}_1})} + \frac{\|\nabla d(\cdot,t)\|^2_{BMO}}{\ln(e+\|\nabla d(\cdot,t)\|_{BMO})} \right) \mathrm{d}t < \infty.$$

Keywords: the 3D nematic liquid crystal flows, regularity criterion, BMO, the multiplier space $\dot{X}_1$.

1 Introduction

In this paper, we consider the initial value problem of incompressible nematic liquid crystals in $\mathbb{R}^3$ (see [1]) :

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\pi = -\nabla\cdot(\nabla d \odot \nabla d), \\ d_t + (u\cdot\nabla)d = \Delta d + |\nabla d|^2 d \\ \nabla\cdot u = 0, \end{cases} \tag{1}$$

with initial data

$$u(x,0) = u_0(x), \;\; d(x,0) = d_0(x), \tag{2}$$

where $u=u(x,t)$ is the velocity field of the flow, $d=d(x,t)$ represents the macroscopic molecular orientation and π is the pressure.

Here the 3×3 matrix $\nabla d \odot \nabla d$ is given by $(\nabla d\odot\nabla d)_{ij} = \partial_i d\cdot\partial_j d$ (for $1\le i,j\le 3$) and hence

$$\nabla\cdot(\nabla d\odot\nabla d) = \nabla\left(\frac{|\nabla d|^2}{2}\right) + \Delta d\nabla d \tag{3}$$

using the formula $a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$, and the fact that $|d|=1$ implies that $d\Delta d = -|\nabla d|^2$, where "$\times$" denotes the vector cross product in $\mathbb{R}^3$.

In [1], Huang and Wang proved that if $u_0\in H^s(\mathbb{R}^3,\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$ and $d_0\in H^{s+1}(\mathbb{R}^3,\mathbb{S}^2)$ for $s\ge 3$, then there is $T_0>0$ which depends only on $\|u_0\|_{H^s}$ and $\|d_0\|_{H^{s+1}}$ such that (1)-(2) has a unique smooth solution (u,d) in $\mathbb{R}^3\times[0,T_0)$ with

$$\begin{cases} u\in C^1([0,T];H^{s-1}(\mathbb{R}^3))\cap C([0,T];H^s(\mathbb{R}^3)), \\ d\in C^1([0,T];H^s(\mathbb{R}^3,\mathbb{S}^2))\cap C([0,T];H^{s+1}(\mathbb{R}^3,\mathbb{S}^2)), \end{cases} \tag{4}$$

for $0<T<T_0$. But, it is still open to prove whether the local solution is global or not.

In a series of works [2,3,4,5,6], Neustupa-Penel obtained regularity criteria via only the middle eigenvalue λ_2 of the strain (deformation) tensor

$$\lambda_2^+ = \max\{0,\lambda_2\}\in L^p(0,T;L^q(\mathbb{R}^3)) \; q>\frac{3}{2}, \; \frac{3}{q}+\frac{2}{p} = 2 \tag{5}$$

while the deformation tensor def u has components

$$(\text{def } u)_{ij} \stackrel{\text{def}}{=} S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1\le i,j\le 3,$$

where for $S\in\mathbb{R}^{3\times 3}$

$$|S| = (\sum_{i,j=1}^{3} S_{ij}^2)^{1/2}.$$

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Recently, Miller [7] obtained another proof of (5). Later, Wu [8,9] extended (5) to the anisotropic Lebesgue spaces in the 3D double-diffusive convection equations.

Inspired by [7,2], we establish a blow-up criterion which impose conditions only on λ_2^+ in terms of the norm in multiplier $\dot{X}_1$ space and ∇d in terms of the BMO-norm. We will prove the following.

Theorem 1. *Assume that $(u_0,d_0)\in H^2(\mathbb{R}^3)\times H^3(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$ in $\mathbb{R}^3$ in the sense of distribution. Let (u,d) be a local smooth solution to (1)-(2) on $[0,T)$. If*

$$\int_0^T\left(\frac{\|\lambda_2^+(\cdot,t)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,t)\|_{\dot{X}_1})}+\frac{\|\nabla d(\cdot,t)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,t)\|_{BMO})}\right)dt<\infty,$$

then (u,d) remains smooth on $[0,T]$. In other words, if the solution blows up at $t=T$, then

$$\int_0^T\left(\frac{\|\lambda_2^+(\cdot,t)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,t)\|_{\dot{X}_1})}+\frac{\|\nabla d(\cdot,t)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,t)\|_{BMO})}\right)dt=\infty,$$

where the space $\dot{X}_1$ is the multiplier space.

2 Preliminaries

2.1 Definitions of BMO and $\dot{X}_r$

Let $e^{t\Delta}$ denotes the standard heat semigroup defined by

$$e^{t\Delta}f=G_t*f,\ \ G_t(x)=(4\pi t)^{-\frac{3}{2}}\exp\left(-\frac{|x|^2}{4t}\right),\ t>0,x\in\mathbb{R}^3,$$

where the symbol $*$ is the convolution. Let $\mathscr{S}=\mathscr{S}(\mathbb{R}^3)$ denotes the Schwartz class. The divergence is denoted also by $\nabla\cdot$, for example, $\nabla\cdot F$ for a tensor $F=(F_{ij})_{i,j=1,2,3}$ is $\left(\sum_j\partial_jF_{ij}\right)_{i=1,2,3}$, where $\partial_j=\frac{\partial}{\partial x_j}$.

A measurable function f is in BMO if

$$\|f\|_{BMO}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}\frac{1}{|B(x,R)|}\int_{B(x,R)}\left|f(y)-f_{B(x,R)}\right|dy,$$

where $f_{B(x,R)}=\frac{1}{|B(x,R)|}\int_{B(x,R)}f(z)dz$. Here $B(x,R)$ denotes the ball in $\mathbb{R}^3$ of radius R and centered at $x\in\mathbb{R}^3$. By the Carleson measure characterization of BMO, a tempered distribution f is in BMO if

$$\|f\|_{BMO}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}\int_0^{R^2}\left|\nabla e^{\Delta t}f(y)\right|^2dtdy\right)^{\frac{1}{2}}<\infty.$$

The Carleson measure characterization due to Strichartz [13] leads us the equivalent norms:

$$\|f\|_{BMO}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}\int_0^{R^2}\left|e^{\Delta t}f(y)\right|^2\frac{dt}{t}dy\right)^{\frac{1}{2}}<\infty.$$

This is equivalent to the standard definition (see [10]).

The multiplier space $\dot{X}_r$ [11,12] for $0\le r<3/2$ is

$$\dot{X}_r=\{f\in L^2_{loc}\ :\ \forall\, g\in\dot{H}^r\ \ fg\in L^2\}$$

where $\dot{H}^r$ is the completion of the space $D(\mathbb{R}^d)$ with respect to the norm

$$\|g\|_{\dot{H}^r}=\|(-\Delta)^{r/2}g\|_{L^2}=\||\xi|^r\hat{g}(\xi)\|_{L^2}.$$

The norm of $\dot{X}_r$ is given by

$$\|f\|_{\dot{X}_r}=\sup_{\|g\|_{\dot{H}^r}\le1}\|fg\|_{L^2}.$$

2.2 Some Lemmas

We need also the following lemmas established in [7].

Lemma 1. *For all $\alpha\in\left]-\frac{3}{2},\frac{3}{2}\right[$ and for all u divergence free in the sense that $\xi\cdot\widehat{u}(\xi)=0$ almost everywhere,*

$$\|S\|_{\dot{H}^\alpha}^2=\|A\|_{\dot{H}^\alpha}^2=\frac{1}{2}\|\omega\|_{\dot{H}^\alpha}^2=\frac{1}{2}\|u\|_{\dot{H}^{\alpha+1}}^2. \qquad (6)$$

Lemma 2. *Suppose $S\in L^2([0,T];H^1(\mathbb{R}^3))\cap L^\infty([0,T];L^2(\mathbb{R}^3))$ be a strong solution of (1) on $\mathbb{R}^3\times(0,T)$ and let $\lambda_1\le\lambda_2\le\lambda_3$ be the eigenvalues of the strain tensor S associated to the classical solution u. Letting*

$$\lambda_2^+=\max\{\lambda_2,0\}.$$

Then, the following inequality holds

$$-\det(S)\le\frac{1}{2}|S|^2\lambda_2^+.$$

2.3 Reformulation of the original equation $(1)_1$

We first define some matrices. The gradient tensor is given by

$$(\nabla\otimes u)_{ij}=\frac{\partial u_j}{\partial x_i},$$

where $i,j=1,2,3$. The vorticity is given by $\omega=\nabla\times u$, which is related to

$$A_{ij}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right),$$

by

$$A=\frac{1}{2}\begin{pmatrix}0&\omega_3&-\omega_2\\-\omega_3&0&\omega_1\\\omega_2&-\omega_1&0\end{pmatrix}. \qquad (7)$$

Then, obviously we have $\nabla\otimes u=S+A$.

Thanks to (3) and $\nabla\times\nabla\left(\frac{|\nabla d|^2}{2}\right)=0$, the evolution equation for vorticity is

$$\partial_t\omega-\Delta\omega+(u\cdot\nabla)\omega=S\omega-\nabla\times(\Delta d\cdot\nabla d). \quad (8)$$

Now we define a differential operator that maps a vector to the symmetric part of its gradient tensor :

$$\nabla_{\text{sym}}(v)_{ij}=\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right).$$

Note that $S=\nabla_{\text{sym}}u=\nabla_{\text{sym}}(-\Delta)^{-1}\nabla\times\omega$. We apply ∇_{sym} to $(1)_1$, we find

$$\partial_t S-\Delta S+\nabla_{\text{sym}}((u\cdot\nabla)u)+\text{Hess}(p+\frac{|\nabla d|^2}{2})=\nabla_{\text{sym}}(\Delta d\cdot\nabla d). \quad (9)$$

We have

$$\begin{aligned}\nabla_{\text{sym}}((u\cdot\nabla)u)_{ij}&=\frac{1}{2}\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3}u_k\frac{\partial u_j}{\partial x_k}\right)+\frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}u_k\frac{\partial u_i}{\partial x_k}\right)\\&=\sum_{k=1}^{3}u_k\frac{\partial}{\partial x_k}\left(\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right)\\&\quad+\frac{1}{2}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}+\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right).\end{aligned}$$

We see from our definitions of S and A that

$$\begin{aligned}S_{ij}^2&=\frac{1}{4}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right)\\&=\frac{1}{4}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}+\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}+\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}+\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right)\end{aligned} \quad (10)$$

and

$$\begin{aligned}A_{ij}^2&=\frac{1}{4}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i}-\frac{\partial u_i}{\partial x_k}\right)\left(\frac{\partial u_j}{\partial x_k}-\frac{\partial u_k}{\partial x_j}\right)\\&=\frac{1}{4}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}-\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}-\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}+\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right)\end{aligned} \quad (11)$$

By combining (10) with (11), we obtain

$$S_{ij}^2+A_{ij}^2=\frac{1}{2}\sum_{k=1}^{3}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}+\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}\right).$$

Therefore

$$\nabla_{\text{sym}}((u\cdot\nabla)u)=(u\cdot\nabla)S+S^2+A^2.$$

We have

$$A^2=\frac{1}{4}(\omega\otimes\omega)-\frac{1}{4}|\omega|^2 I_3.$$

Hence

$$\partial_t S-\Delta S+(u\cdot\nabla)S+S^2+\frac{1}{4}(\omega\otimes\omega)-\frac{1}{4}|\omega|^2 I_3 \quad (12)$$

$$+\text{Hess}(p+\frac{|\nabla d|^2}{2})=\nabla_{\text{sym}}(\Delta d\cdot\nabla d). \quad (13)$$

From (7), $A\omega=0$, therefore

$$S\omega=(S+A)\omega=(\omega\cdot\nabla)u.$$

3 Proof of Theorem 1

Suppose that $[0,T^*)$ is the maximal interval on which the local smooth solution exists. For $T^*<T$, we will show that

$$\limsup_{t\to T^*}\left(\|\Delta u(\cdot,t)\|_{L^2}^2+\|\nabla\Delta d(\cdot,t)\|_{L^2}^2\right)<\infty,$$

under the assumption (22). Thus, $[0,T^*)$ cannot be maximal, which conflicts with the definition of T^*.

Step 1. We mutiply $(1)_1$ by u and integrate by parts, we use (3) and $\nabla\cdot u=0$

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2+\|\nabla u(\cdot,t)\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla d\cdot\Delta d\cdot u dx, \quad (14)$$

We multiply $(1)_2$ by $-\Delta d$ and integrate, we use $|d|=1$, $\Delta(|d|^2)=0$ and $|\nabla d|^2=-d\cdot\Delta d$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla d(\cdot,t)\|_{L^2}^2+\|\Delta d(\cdot,t)\|_{L^2}^2-\int_{\mathbb{R}^3}(u\cdot\nabla)d\cdot\Delta d\mathrm{d}x\\&=-\int_{\mathbb{R}^3}|\nabla d|^2 d\cdot\Delta d\mathrm{d}x\\&=\int_{\mathbb{R}^3}|d\cdot\Delta d|^2\mathrm{d}x\le\int_{\mathbb{R}^3}|\Delta d|^2\mathrm{d}x.\end{aligned} \quad (15)$$

Summing up (14) and (15)

$$\frac{1}{2}\frac{d}{dt}(\|u(\cdot,t)\|_{L^2}^2+\|\nabla d(\cdot,t)\|_{L^2}^2)+\|\nabla u(\cdot,t)\|_{L^2}^2\le 0. \quad (16)$$

Integrating (16) in time gives

$$\begin{aligned}&\|\nabla d(\cdot,t)\|_{L^2}^2+\|u(\cdot,t)\|_{L^2}^2\\&+\int_0^t\|\nabla u(\cdot,\tau)\|_{L^2}^2 d\tau\le\|\nabla d_0\|_{L^2}^2+\|u_0\|_{L^2}^2,\end{aligned} \quad (17)$$

for all $0<t<\infty$.

Taking the L^2 inner product of (8) with ω, we have

$$\begin{aligned}&-\langle\Delta\omega,\omega\rangle+\frac{1}{2}\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2+\langle(u\cdot\nabla)\omega,\omega\rangle\\&=\langle\nabla d\cdot\Delta d,\nabla\times\omega\rangle+\langle S\omega,\omega\rangle.\end{aligned} \quad (18)$$

We integrate by parts, we get

$$\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2+2\|\nabla\omega(\cdot,t)\|_{L^2}^2=2\langle S,\omega\otimes\omega\rangle+2\langle\nabla d\cdot\Delta d,\Delta u\rangle, \quad (19)$$

using $\nabla\times\omega=\nabla\times(\nabla\times u)=-\Delta u+\nabla\nabla\cdot u=-\Delta u$ and the divergence free condition.
From Lemma 1 with $\alpha=0$ and (19), it follows that

$$\frac{d}{dt}\|S(\cdot,t)\|_{L^2}^2+2\|\nabla S(\cdot,t)\|_{L^2}^2=\langle S,\omega\otimes\omega\rangle+\langle\nabla d\cdot\Delta d,\Delta u\rangle,$$

which yields

$$\frac{1}{3}\frac{d}{dt}\|S(\cdot,t)\|_{L^2}^2+\frac{2}{3}\|\nabla S(\cdot,t)\|_{L^2}^2=\frac{1}{3}\langle S,\omega\otimes\omega\rangle+\frac{1}{3}\langle\nabla d\cdot\Delta d,\Delta u\rangle. \quad (20)$$

Now, taking the L^2 inner product of (13) with S, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|S(\cdot,t)\|_{L^2}^2+\langle-\Delta S,S\rangle+\langle S^2,S\rangle+\frac{1}{4}\langle(\omega\otimes\omega),S\rangle$$
$$=-\langle(u\cdot\nabla)S,S\rangle+\frac{1}{4}\left\langle|\omega|^2 I_3,S\right\rangle$$
$$-\left\langle \mathrm{Hess}(p+\frac{|\nabla d|^2}{2}),S\right\rangle+\langle\nabla_{\mathrm{sym}}(\nabla d\cdot\Delta d),S\rangle.$$

By using the following identities

$$\left\langle|\omega|^2 I_3,S\right\rangle$$
$$=\int_{\mathbb{R}^3}|\omega|^2\left(\sum_{i,j=1}^{3}S_{ij}I_{ij}\right)dx=\int_{\mathbb{R}^3}|\omega|^2\,\mathrm{tr}(S)dx=0,$$
$$\langle S^2,S\rangle=\int_{\mathbb{R}^3}\mathrm{tr}(S^3)dx,\quad \langle(u\cdot\nabla)S,S\rangle=0,$$

$$\left\langle \mathrm{Hess}(p+\frac{|\nabla d|^2}{2}),S\right\rangle=\left\langle \widehat{\mathrm{Hess}(p+\frac{|\nabla d|^2}{2})},\widehat{S}\right\rangle$$
$$=-4\pi^2\int\overline{\widehat{p}}(\xi)\left(\sum_{i,j=1}^{3}\xi_i\xi_j\widehat{S}_{ij}(\xi)\right)d\xi$$
$$=0,$$

and

$$\begin{aligned}2\langle\nabla_{sym}(\Delta d\cdot\nabla d),S\rangle&=2\langle\nabla\otimes(\Delta d\cdot\nabla d),S\rangle\\&=\langle\Delta d\cdot\nabla d,-2\mathrm{div}(S)\rangle\\&=\langle\Delta d\cdot\nabla d,-2\mathrm{div}(\nabla_{\mathrm{sym}}u)\rangle\\&=\langle\Delta d\cdot\nabla d,-\Delta u-\nabla\mathrm{div}(u)\rangle\\&=\langle\Delta d\cdot\nabla d,-\Delta u\rangle,\end{aligned}$$

thus

$$2\|\nabla S(\cdot,t)\|_{L^2}^2+\frac{d}{dt}\|S(\cdot,t)\|_{L^2}^2+2\int\mathrm{tr}(S^3)dx$$
$$+\frac{1}{2}\int(\omega\otimes\omega)\cdot Sdx$$
$$=\int_{\mathbb{R}^3}\Delta d\cdot\nabla d\cdot\Delta udx,$$

which is equivalent to

$$\frac{2}{3}\frac{d}{dt}\|S(\cdot,t)\|_{L^2}^2+\frac{4}{3}\|\nabla S(\cdot,t)\|_{L^2}^2$$
$$=-\frac{4}{3}\int_{\mathbb{R}^3}\mathrm{tr}(S^3)dx-\frac{1}{3}\int_{\mathbb{R}^3}(\omega\otimes\omega)\cdot Sdx$$
$$+\frac{2}{3}\int_{\mathbb{R}^3}\Delta d\cdot\nabla d\cdot\Delta udx. \tag{21}$$

From (20) and (21)

$$\frac{d}{dt}\|S(\cdot,t)\|_{L^2}^2+2\|\nabla S(\cdot,t)\|_{L^2}^2$$
$$=-\frac{4}{3}\int_{\mathbb{R}^3}\mathrm{tr}(S^3)dx+\int_{\mathbb{R}^3}\Delta d\cdot\nabla d\cdot\Delta udx. \tag{22}$$

We apply Δ to $(1)_2$, we dote the resulting equation with Δd, we integrate by parts and use the divergence free property, we use

$$\int_{\mathbb{R}^3}(u\cdot\nabla\Delta d)\cdot\Delta d\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla)(|\Delta d|^2)\mathrm{d}x=0,$$

we obtain

$$\|\nabla\Delta d(\cdot,t)\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\Delta d(\cdot,t)\|_{L^2}^2$$
$$=\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)d)\cdot\Delta d\mathrm{d}x$$
$$=-2\int_{\mathbb{R}^3}\nabla u\cdot\nabla\nabla d\cdot\Delta d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta u\cdot\nabla d\cdot\Delta d\mathrm{d}x$$
$$+\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d\mathrm{d}x,$$

By adding (22) and (21), we find

$$\frac{d}{dt}(\|S(\cdot,t)\|_{L^2}^2+\|\Delta d\|_{L^2}^2)+2(\|\nabla S(\cdot,t)\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2)$$
$$=-\frac{4}{3}\int_{\mathbb{R}^3}\mathrm{tr}(S^3)dx+\int_{\mathbb{R}^3}(\nabla d\cdot\Delta d)\cdot\Delta u\mathrm{d}x \tag{23}$$
$$+\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)d)\cdot\Delta d\mathrm{d}x$$
$$=\mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3+\mathscr{I}_4. \tag{24}$$

Moreover, we use $\mathrm{tr}(S)=\nabla\cdot u=0$ and the fact $\lambda_1+\lambda_2+\lambda_3=0$, Lemma 2 and Young's inequality yield

$$\begin{aligned}\mathscr{I}_1&=-\frac{4}{3}\int_{\mathbb{R}^3}\mathrm{tr}(S^3)dx=-\frac{4}{3}\int_{\mathbb{R}^3}(\lambda_1^3+\lambda_2^3+\lambda_3^3)dx\\&=-\frac{4}{3}\int_{\mathbb{R}^3}(\lambda_1^3+\lambda_2^3+(-\lambda_1-\lambda_2)^3)dx\\&=4\int_{\mathbb{R}^3}\lambda_1\lambda_2(\lambda_1+\lambda_2)dx=-4\int_{\mathbb{R}^3}\lambda_1\lambda_2\lambda_3dx\\&=-4\int_{\mathbb{R}^3}\det(S)dx\le 2\int_{\mathbb{R}^3}|S|^2\lambda_2^+dx\\&\le C\left\|\lambda_2^+\right\|_{\dot{X}_1}\|S\|_{L^2}\|\nabla S\|_{L^2}\\&\le\frac{1}{2}\|\nabla S\|_{L^2}^2+C\left\|\lambda_2^+\right\|_{\dot{X}_1}^2\|S\|_{L^2}^2.\end{aligned} \tag{25}$$

with the aid of the following inequality

$$\|\nabla f\|_{L^4}^2\le C\|f\|_{BMO}\|\Delta f\|_{L^2},$$

Owing to Young's inequality, $\mathscr{I}_2$ can be estimated as follows

$$\begin{aligned}\mathscr{I}_2+\mathscr{I}_3&=\int_{\mathbb{R}^3}(\Delta d\cdot\nabla d)\cdot\Delta u\mathrm{d}x-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)d)\cdot\Delta d\mathrm{d}x\\&=-\sum_{i,k=1}^{3}\int_{\mathbb{R}^3}(2\nabla u_i\partial_i\nabla d_k\Delta d_k+u_i\partial_i\Delta d_k\Delta d_k)\mathrm{d}x\\&=-2\sum_{i,k=1}^{3}\int_{\mathbb{R}^3}\nabla u_i\partial_i\nabla d_k\Delta d_k\mathrm{d}x\\&\le C\|\Delta d\|_{L^4}^2\|\nabla u\|_{L^2}\\&\le C\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}\|\nabla d\|_{BMO}\\&\le\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2+C\|\nabla d\|_{BMO}^2\|S\|_{L^2}^2.\end{aligned}$$

By using $|d| = 1$ and $|\nabla d|^2 = -d \cdot \Delta d$, after integration by parts, by using the Hölder inequality, the Young inequality, $\mathscr{I}_4$ can be estimated as

$$\begin{aligned}
\mathscr{I}_4 &\leq \left|\int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d)\cdot \Delta d \mathrm{dx}\right| = \left|\int_{\mathbb{R}^3} \nabla(|\nabla d|^2 d)\cdot \nabla\Delta d \mathrm{dx}\right| \\
&\leq \left|\int_{\mathbb{R}^3} |\nabla d|^2 \nabla d \cdot \nabla\Delta d \mathrm{dx}\right| + \left|\int_{\mathbb{R}^3} d\nabla(|\nabla d|^2)\cdot \nabla\Delta d \mathrm{dx}\right| \\
&\leq \left|\int_{\mathbb{R}^3} |\nabla d|^2 \nabla d(\nabla\Delta d)\mathrm{dx}\right| + \left|\int_{\mathbb{R}^3} (\nabla d)\Delta d \cdot d\nabla\Delta d \mathrm{dx}\right| \\
&\leq C\|d\Delta d\nabla d\|_{L^2}\|\nabla\Delta d\|_{L^2} \\
&\leq C\|\nabla d\|_{L^4}\|\Delta d\|_{L^4}\|\nabla\Delta d\|_{L^2} \\
&\leq \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + C\|\nabla d\|_{L^4}^2\|\Delta d\|_{L^4}^2 \\
&\leq \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + C\|d\|_{L^\infty}\|\Delta d\|_{L^2}\|\nabla d\|_{BMO}\|\nabla\Delta d\|_{L^2} \\
&\leq \frac{1}{4}\|\nabla\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2\|\nabla d\|_{BMO}^2,
\end{aligned}$$

using

$$\|\nabla f\|_{L^4}^2 \leq C\|\Delta f\|_{L^2}\|f\|_{L^\infty}.$$

By substituting the above estimates into (24), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\Delta d\|_{L^2}^2 + \|S(\cdot,t)\|_{L^2}^2) + \|\nabla\Delta d\|_{L^2}^2 + \|\nabla S(\cdot,t)\|_{L^2}^2 \\
&\leq C(\|S\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)(\|\lambda_2^+\|_{\dot{X}_1}^2 + \|\nabla d\|_{BMO}^2). \qquad (26)
\end{aligned}$$

Due to (22), one concludes that for $\varepsilon > 0$, there exists $T^* < T$ such that

$$\int_{T^*}^{T}\left(\frac{\|\lambda_2^+(\cdot,t)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,t)\|_{H^1})} + \frac{\|\nabla d(\cdot,t)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,t)\|_{H^2})}\right)dt \leq \varepsilon. \qquad (27)$$

For any $t \in (T^*,T]$, we denote

$$\mathscr{Z}(t) = \sup_{\tau\in[T^*,t]}\left(\|d(\cdot,\tau)\|_{H^3}^2 + \|u(\cdot,\tau)\|_{H^2}^2\right).$$

It should be noted that the function $\mathscr{Z}(t)$ is nondecreasing. Applying the logarithmic Sobolev inequality, we get that for any $t \in [T^*,T)$,

$$\begin{aligned}
&\|S(\cdot,t)\|_{L^2}^2 + \|\Delta d(\cdot,t)\|_{L^2}^2 + \int_{T^*}^t (\|\nabla S(\cdot,\tau)\|_{L^2}^2 + \|\nabla\Delta d(\cdot,\tau)\|_{L^2}^2)d\tau \\
&\leq (\|S(\cdot,T^*)\|_{L^2}^2 + \|\Delta d(\cdot,T^*)\|_{L^2}^2) \\
&\quad \times \exp\left(C\int_{T^*}^t (\frac{\|\lambda_2^+(\cdot,\tau)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,\tau)\|_{\dot{X}_1})}\ln(e+\|\nabla u(\cdot,\tau)\|_{\dot{X}_1})\right) \\
&\quad \times \exp\left(C\int_{T^*}^t \frac{\|\nabla d(\cdot,\tau)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,\tau)\|_{BMO})}\ln(e+\|\nabla d(\cdot,\tau)\|_{BMO}))d\tau\right) \\
&\leq C(T^*)\exp\left(C\int_{T^*}^t (\frac{\|\lambda_2^+(\cdot,\tau)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,\tau)\|_{\dot{X}_1})}\ln(e+\|u(\cdot,\tau)\|_{H^2})d\tau\right) \\
&\quad \times \exp\left(C\int_{T^*}^t \frac{\|\nabla d(\cdot,\tau)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,\tau)\|_{BMO})}\ln(e+\|\nabla d(\cdot,\tau)\|_{H^2})d\tau\right) \\
&\leq C(T^*)\exp\left(C\int_{T^*}^t (\frac{\|\lambda_2^+(\cdot,\tau)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,\tau)\|_{\dot{X}_1})}\ln(e+\|u(\cdot,\tau)\|_{H^2})d\tau\right) \\
&\quad \times \exp\left(C\int_{T^*}^t \frac{\|\nabla d(\cdot,\tau)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,\tau)\|_{BMO})}\ln(e+\|d(\cdot,\tau)\|_{H^3}))d\tau\right) \\
&\leq C(T^*)\exp\left(C\int_{T^*}^t \left(\frac{\|\lambda_2^+(\cdot,\tau)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,\tau)\|_{\dot{X}_1})} + \frac{\|\nabla d(\cdot,\tau)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,\tau)\|_{BMO})}\right) \times \ln(e+\|u(\cdot,\tau)\|_{H^2}^2 + \|d(\cdot,\tau)\|_{H^3}^2)d\tau\right) \\
&\leq C(T^*)\exp\left(C\int_{T^*}^t \left(\frac{\|\lambda_2^+(\cdot,\tau)\|_{\dot{X}_1}^2}{\ln(e+\|\nabla u(\cdot,\tau)\|_{\dot{X}_1})} + \frac{\|\nabla d(\cdot,\tau)\|_{BMO}^2}{\ln(e+\|\nabla d(\cdot,\tau)\|_{BMO})}\right)d\tau\right) \\
&\quad \times \left\{\sup_{T^*\leq\tau\leq t} \ln(e+\|u(\cdot,\tau)\|_{H^2}^2 + \|d(\cdot,\tau)\|_{H^3}^2)\right\} \\
&\leq C(T^*)\exp\{C\varepsilon\ln(e+\mathscr{Z}(t)\}, \qquad (28)
\end{aligned}$$

where $C(T^*)$ depends on $\|S(\cdot,T^*)\|_{L^2}$ and $\|\Delta d(\cdot,T^*)\|_{L^2}$.

We apply Gronwall's inequality on (28) for $[T^*,t]$,

$$\begin{aligned}
&\|\Delta d(\cdot,t)\|_{L^2}^2 + \|S(\cdot,t)\|_{L^2}^2 \\
&+ \int_{T^*}^t (\|\nabla\Delta d(\cdot,\tau)\|_{L^2}^2 + \|\nabla S(\cdot,\tau)\|_{L^2}^2)d\tau \\
&\leq C(T^*)(e+\mathscr{Z}(t))^{C\varepsilon}.
\end{aligned}$$

Step 2.

We apply Δ in $(1)_1$, then multiply the resulting equation by u, and integrate by parts

$$\begin{aligned}
&\|\nabla\Delta u(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u(\cdot,t)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3}[\Delta\nabla\cdot(\nabla d\odot\nabla d)]\cdot\Delta u dx - \int_{\mathbb{R}^3}\Delta((u\cdot\nabla)u)\cdot\Delta u dx \\
&= -\int_{\mathbb{R}^3}\Delta(\Delta d\cdot\nabla d)\cdot\Delta u dx - \int_{\mathbb{R}^3}\Delta((u\cdot\nabla)u)\cdot\Delta u dx, \qquad (29)
\end{aligned}$$

Take $\nabla\Delta$ in $(1)_2$, then multiply by $\nabla\Delta d$, and integrate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta d(\cdot,t)\|_{L^2}^2 + \left\|\Delta^2 d(\cdot,t)\right\|_{L^2}^2 \\
&= \int \nabla\Delta(|\nabla d|^2 d)\cdot\nabla\Delta d\mathrm{dx} - \int \nabla\Delta((u\cdot\nabla)d)\cdot\nabla\Delta d\mathrm{dx}. \qquad (30)
\end{aligned}$$

Summing up (29) and (30), we get

$$\frac{1}{2}\frac{d}{dt}(\|\Delta u(\cdot,t)\|_{L^2}^2+\|\nabla\Delta d(\cdot,t)\|_{L^2}^2) \quad (31)$$
$$+\left\|\Delta^2 d(\cdot,t)\right\|_{L^2}^2+\|\nabla\Delta u(\cdot,t)\|_{L^2}^2$$
$$=-\int_{\mathbb{R}^3}\Delta(\Delta d\cdot\nabla d)\cdot\Delta u dx-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)u)\cdot\Delta u dx$$
$$+\int_{\mathbb{R}^3}\nabla\Delta(|\nabla d|^2 d)\cdot\nabla\Delta d\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\Delta((u\cdot\nabla)d)\cdot\nabla\Delta d\mathrm{d}x$$
$$=\mathscr{J}_1+\mathscr{J}_2+\mathscr{J}_3+\mathscr{J}_4. \quad (32)$$

Use $\nabla\cdot u=0$ and integrate by parts, we bound J_1 as follows

$$\mathscr{J}_1=-\int_{\mathbb{R}^3}\Delta((u\cdot\nabla)u)\cdot\Delta u dx$$
$$=-2\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_j u_i\partial_i\partial_j u\cdot\Delta u dx-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\Delta u_i\partial_i u\cdot\Delta u dx$$
$$\le C\|\Delta u\|_{L^4}^2\|\nabla u\|_{L^2}$$
$$\le C\|\nabla u\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}\|\nabla\Delta u\|_{L^2}^{\frac74}$$
$$\le\frac16\|\nabla\Delta u\|_{L^2}^2+C\|\nabla u\|_{L^2}^8\|\nabla u\|_{L^2}^2,$$

where the identity $\int_{\mathbb{R}^3}u_i\partial_i\Delta u\cdot\Delta u dx=0$ and the following Gagliardo-Nirenberg inequality are used

$$\left\|\nabla^2 u\right\|_{L^4}\le C\|\nabla u\|_{L^2}^{\frac18}\|\nabla\Delta u\|_{L^2}^{\frac78}.$$

It follows from Young inequality that

$$\mathscr{J}_2=\int_{\mathbb{R}^3}\nabla(\Delta d\cdot\nabla d)\cdot\nabla\Delta u\mathrm{d}x$$
$$\le\|\nabla\Delta u\|_{L^2}\|\nabla(\Delta d\cdot\nabla d)\|_{L^2}$$
$$\le C\left\|\nabla\Delta d\cdot\nabla d+\Delta d\cdot\nabla^2 d\right\|_{L^2}^2+\frac18\|\nabla\Delta u\|_{L^2}^2$$
$$\le C(\|\nabla\Delta d\cdot\nabla d\|_{L^2}^2+\left\|\Delta d\cdot\nabla^2 d\right\|_{L^2}^2)+\frac18\|\nabla\Delta u\|_{L^2}^2$$
$$\le C\|\nabla\Delta d\|_{L^4}^2\|\nabla d\|_{L^4}^2+C\|\Delta d\|_{L^4}^4+\frac18\|\nabla\Delta u\|_{L^2}^2$$
$$\le C\left\|\Delta^2 d\right\|_{L^2}^{\frac74}\|\Delta d\|_{L^2}^{\frac14}\|\Delta d\|_{L^2}\|d\|_{L^\infty}$$
$$+C\|\Delta d\|_{L^2}^{\frac52}\left\|\Delta^2 d\right\|_{L^2}^{\frac32}+\frac18\|\nabla\Delta u\|_{L^2}^2$$
$$\le\frac18\left\|\Delta^2 d\right\|_{L^2}^2+C\|\Delta d\|_{L^2}^{10}+\frac18\|\nabla\Delta u\|_{L^2}^2,$$

where we have applied the following Gagliardo-Nirenberg inequality

$$\|\nabla f\|_{L^{2q}}\le C\|f\|_{L^\infty}^{\frac12}\|\Delta f\|_{L^q}^{\frac12},\quad\text{for }\ q>\frac32.$$

In order to estimate the term $\mathscr{J}_3$, recall that

$$\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\le C\left(\|\nabla f\|_{L^q}\left\|\Lambda^{s-1}g\right\|_{L^r}+\|\Lambda^s f\|_{L^r}\|g\|_{L^q}\right),$$

for $f,g\in W^{k,p}$ with $1<p<\infty$ and $1\le s\le k$ such that

$$\frac1p=\frac1q+\frac1r,\quad 1<q\le\infty,\quad 1<r<\infty,$$

where $[\Lambda^s,f]g=\Lambda^s(fg)-f\Lambda^s(g)$ and $\Lambda=(-\Delta)^{\frac12}$. Using the cancelation property $(u\cdot\nabla)\nabla\Delta d\cdot\nabla\Delta d=0$, one has

$$\mathscr{J}_3=-\int_{\mathbb{R}^3}[\nabla\Delta(u\cdot\nabla d)-u\cdot\nabla\Delta(\nabla d)]\cdot\nabla\Delta d\mathrm{d}x$$
$$\le\|\nabla\Delta(u\cdot\nabla d)-u\cdot\nabla\Delta\nabla d\|_{L^{\frac43}}\|\nabla\Delta d\|_{L^4}$$
$$\le C(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^4}+\|\nabla d\|_{L^4}\|\nabla\Delta u\|_{L^2})\|\nabla\Delta d\|_{L^4}$$
$$\le C\|\nabla u\|_{L^2}\|\Delta d\|_{L^2}^{\frac14}\left\|\Delta^2 d\right\|_{L^2}^{\frac74}$$
$$+C\|d\|_{L^\infty}\|\Delta d\|_{L^2}\left\|\Delta^2 d\right\|_{L^2}^{\frac74}\|\Delta d\|_{L^2}^{\frac14}+\frac16\|\nabla\Delta u\|_{L^2}^2$$
$$\le C\|\nabla u\|_{L^2}^8\|\Delta d\|_{L^2}^2+\frac16\left\|\Delta^2 d\right\|_{L^2}^2+C\|\Delta d\|_{L^2}^{10}+\frac16\|\nabla\Delta u\|_{L^2}^2.$$

$\mathscr{J}_4$ is simply bounded as

$$\mathscr{J}_4=\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\nabla^2\Delta d\mathrm{d}x$$
$$=\int[2\nabla(|\nabla d|^2)\nabla d+\Delta(|\nabla d|^2)d+|\nabla d|^2\Delta d]\cdot\nabla^2\Delta d\mathrm{d}x$$
$$=\int[2\nabla^2 d\nabla d\nabla d+\nabla(2\nabla^2 d\nabla d)d-d\Delta d\Delta d]\cdot\nabla^2\Delta d\mathrm{d}x$$
$$=\int[2(\nabla^2 d\nabla^2 d+\nabla^3 d\nabla d)d-2\nabla^2 d(d\Delta d)-d|\Delta d|^2]\cdot\nabla^2\Delta d\mathrm{d}x$$
$$\le\int[2(\left|\nabla^2 d\right|^2+\left|\nabla^3 d\right||\nabla d|)|d|+2\left|\nabla^2 d\right||\Delta d|+|d||\Delta d|^2]\cdot\nabla^2\Delta d\mathrm{d}x$$
$$\le C(\left\|\left|\nabla^3 d\right||\nabla d|\right\|_{L^2}+\left\|\left|\nabla^2 d\right|^2\right\|_{L^2})\left\|\Delta^2 d\right\|_{L^2}$$
$$\le C(\|\nabla\Delta d\|_{L^4}\|\nabla d\|_{L^4}+\left\|\nabla^2 d\right\|_{L^4}^2)\left\|\Delta^2 d\right\|_{L^2}$$
$$\le C\|\Delta d\|_{L^2}^{\frac18}\left\|\Delta^2 u\right\|_{L^2}^{\frac78}\|d\|_{L^\infty}^{\frac12}\|\Delta d\|_{L^2}^{\frac12}\left\|\Delta^2 d\right\|_{L^2}$$
$$+C\|\Delta d\|_{L^2}^{\frac54}\left\|\Delta^2 d\right\|_{L^2}^{\frac34}\left\|\Delta^2 d\right\|_{L^2}$$
$$\le C\|\Delta d\|_{L^2}^{10}+\frac16\left\|\Delta^2 d\right\|_{L^2}^2.$$

Insert the above estimates into (32) and absorb the dissipative terms,

$$\frac{d}{dt}(\|\Delta u(\cdot,t)\|_{L^2}^2+\|\nabla\Delta d(\cdot,t)\|_{L^2}^2)+\|\nabla\Delta u(\cdot,t)\|_{L^2}^2$$
$$+\left\|\Delta^2 d(\cdot,t)\right\|_{L^2}^2$$
$$\le C(\|\nabla u\|_{L^2}^8+\|\Delta d\|_{L^2}^8)(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2),$$

which together with basic energy (17) yields

$$\frac{d}{dt}(\|u(\cdot,t)\|_{H^2}^2+\|d(\cdot,t)\|_{H^3}^2)+\|u(\cdot,t)\|_{H^3}^2+\|d(\cdot,t)\|_{H^4}^2$$
$$\le C(\|\nabla u\|_{L^2}^8+\|\Delta d\|_{L^2}^8)(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2). \quad (33)$$

Integrating the inequality (33) over (T^*,t), we get

$$\begin{aligned}
&\|u(\cdot,t)\|_{H^2}^2+\|d(\cdot,t)\|_{H^3}^2-\|u(\cdot,T^*)\|_{H^2}^2+\|d(\cdot,T^*)\|_{H^3}^2\\
&\leq C\int_{T^*}^{t}(\|\nabla u(\cdot,\tau)\|_2^8+\|\Delta d(\cdot,\tau)\|_2^8)(\|\nabla u(\cdot,\tau)\|_2^2\\
&\quad+\|\Delta d(\cdot,\tau)\|_2^2)d\tau\\
&\leq C\int_{T^*}^{t}(e+\mathscr{Z}(\tau))^{4C\varepsilon}(\|\nabla u(\cdot,\tau)\|_{L^2}^2+\|\Delta d(\cdot,\tau)\|_{L^2}^2)d\tau\\
&\leq C(e+\mathscr{Z}(t))^{4C\varepsilon}\int_{T^*}^{t}(\|\nabla u(\cdot,\tau)\|_{L^2}^2+\|v(\cdot,\tau)\|_{L^2}^2)d\tau\\
&\leq C(e+\mathscr{Z}(t))^{5C\varepsilon}.
\end{aligned}$$

Hence, it follows that

$$\mathscr{Z}(t)-\mathscr{Z}(T^*)\leq C(e+\mathscr{Z}(\tau))^{5C\varepsilon}.$$

Now we choose ε small enough such that $5C\varepsilon<1$ to deduce that

$$e+\mathscr{Z}(t)\leq C(T^*)<\infty,$$

which implies

$$\begin{aligned}
&\max_{\tau\in[0,T]}\left(\|u(\cdot,\tau)\|_{H^2}^2+\|d(\cdot,\tau)\|_{H^3}^2\right)\\
&\leq C(u_0,d_0,u(T^*),d(T^*),T^*,T)<\infty.
\end{aligned}$$

Therefore, we get the boundedness of $H^2\times H^3$-norm of (u,d) for all $t\in[0,T]$. The local existence results allow us to extend (u,d) past time T. This achieves the proof of Theorem 1.

Conflict of Interest

The authors declare that there is no conflict regarding the publication of this paper.

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