Duality in a class of vector Köthe-Orlicz spaces

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Abstract: We deal with a complete normed space $E$, a scalar sequence space $\lambda$, and an Orlicz mapping $M$ to introduce and study some properties of the spaces $\lambda M(E)$ of all $E$-valued sequences that are absolutely $(\lambda, M)$-summable. Denote by $\lambda M(E)_r$ the subspace of $\lambda M(E)$ whose elements are AK-sequences. We describe the continuous linear forms on this space in term of $E^\ast$-valued sequences that are absolutely $(\lambda^\ast, N)$-summable, where $N$ is the Orlicz mapping complement of $M$.

Keywords: Duality, vector and scalar sequence spaces, normed spaces, Orlicz space

1 Introduction

The notion of absolutely and weakly $\lambda$-summable sequences in a locally convex space, for $\lambda$ a perfect Köthe scalar sequence space, was first introduced by A. Pietsch [1] to characterize the nuclearity of locally convex spaces.

Since then many authors have been interested to the study of these spaces defined by a combination of a Köthe scalar sequence spaces and a linear vector space. They consider on $\lambda$, not only its normal topology, but general polar topologies. The space of absolutely $\lambda$ summable sequence has been intensively studied by many authors as in [2,3]. Later, an extension to the modular function has been introduced in [4,5]. The authors in [6,7,8,9,10,11,12] were mainly interested in the weakly $\lambda$-summability. In [12], the author involved the Orlicz mapping to define a new class of these spaces.

In this note, we deal with an Orlicz mapping $M$ and a scalar sequence space $\lambda$, supposed to be perfect, to generalize the notion of the absolute $\lambda$-summability by defining $\lambda M(E)$, the space of all absolute $(\lambda, M)$-summable ones in a complete normed space $E$.

Notice that, for $M(t) = t$, the space $\lambda M(E)$ is nothing else but $\lambda \{ E \}$ of all $E-$ valued sequences that are $\lambda$-summable studied in [2,3].

In this paper, we study some properties of $\lambda M(E)$, such as the description of the topological dual.

2 Preliminaries

Throughout this paper, if $F$ is a normed space then we denote by $F^\ast$, $B_{F^\ast}$ and $\| \cdot \|_{F^\ast}$, respectively, the topological dual, the closed unit ball and the norm of $F$.

Let the symbol $\omega$ stand for the linear space of all complex sequences with respect to the standard unit vector of order $n$ in $\omega$. A linear subspace $\lambda$ of $\omega$ is said to be normal, whenever $\alpha$ and $\beta$ are in $\omega$, and $\alpha \leq \beta$ and $\beta \in \lambda$ then $\beta \in \lambda$.

If $\lambda$ is a sequence space, its $\alpha$-dual will be denoted $\lambda^\ast$ and defined as

$$
\lambda^\ast = \left\{ (\beta_n) \in \omega : \sum_{n=1}^{\infty} |\alpha_n| |\beta_n| < \infty, \forall (\alpha_n) \in \lambda \right\}.
$$

It is easy to check the inclusion $\lambda \subset \lambda^{**} = (\lambda^\ast)^\ast$. We say that $\lambda$ is perfect whenever the equality $\lambda = \lambda^{**}$ holds. Everywhere it occurs in this note, $\lambda$ means a complete and perfect normed sequence space such that

(a) $\| \cdot \|_{\lambda}$ is solid, that is, whatever $\gamma$ and $\delta$ in $\lambda$, if $\gamma \leq \delta$ then $\| \gamma \|_{\lambda} \leq \| \delta \|_{\lambda}$.

(b) Every $(\beta_n)$ in $\lambda$ is the limit of the sequence $(\beta_1, \ldots, \beta_n, 0, \ldots)$, $n \in \mathbb{N}$, the finite sections of $\beta$ with respect to the norm in $\| \cdot \|_{\lambda}$. In other words, the space $\lambda$ satisfies the AK-property.

These two conditions make the continuous dual of $\lambda$ coincide with $\lambda^\ast$. By using Hahn-Banach Theorem, the

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standard norm $\| \cdot \|_{\lambda^*}$ of $\lambda^*$ can then be given as

$$\| \gamma \|_{\lambda^*} = \sup \left\{ \sum_{n=1}^{\infty} |b_n|, \delta = (\delta_n)_{n} \in \lambda, \| \delta \|_{\lambda} \leq 1 \right\}.$$ 

Moreover, it will be needed to assumed that the dual space $(\lambda^*, \| \cdot \|_{\lambda^*})$ of $\lambda^*$ satisfies also the is also AK-property. In that case, $\lambda$ will be a reflexive complete normed space.

An Orlicz mapping is a non-decreasing, non-negative, convex and continuous, function $M$ defined for every $t \geq 0$, with the properties that $M(0) = 0, M(x) > 0$ for $x > 0$ and $\lim M(x) = \infty$.

It is possible to represent an Orlicz mapping $M$ in the integral form

$$M(x) = \int_{0}^{x} m(t) dt,$$

where $m$ is positive, continuous at the right for every $t > 0$, and $m(0) = 0$. Let $n$ be defined by for $t \geq 0$,

$$n(t) = \sup \{ u : m(u) \leq t, \forall f \geq 0 \}.$$

So, $n$ satisfies the same conditions as $m$. Let $N$ be defined by

$$N(u) = \int_{0}^{u} n(x) dx.$$

Then $N$ is also an Orlicz mapping. We say that $N$ complements $M$ and $M$ complements $N$. They satisfy

$$ts \leq M(t) + N(s), \text{ for } t, s \geq 0. \tag{1}$$

For an Orlicz mapping $M$, define the space $\ell_M$ by

$$\ell_M = \left\{ (\alpha_n)_{n} \in \omega : \exists \sigma > 0, \sum_{n=1}^{\infty} M \left( \frac{\alpha_n}{\sigma} \right) < \infty \right\}.$$

Since $M$ is non-decreasing, it is easy to verify that the space $\ell_M$ is normal. Moreover, the quantity

$$\| (\beta_n)_{n} \|_{\ell_M} = \inf \left\{ \sigma > 0, \sum_{n=1}^{\infty} M \left( \frac{\beta_n}{\sigma} \right) \leq 1 \right\},$$

is a solid norm on $\ell_M$ for which $\ell_M$ is a complete normed space.

For $M(t) = t^p$, and $1 \leq p < \infty$, the space $\ell_M$ coincides with the classical complete normed spaces $\ell_p$.

Because of its convexity, $M$ satisfies always the inequality $M(tx) \leq tM(x)$, for every $0 \leq t \leq 1$.

We will assume that there is $L > 0$, verifying $M(2x) \leq LM(x)$, for all $x \in [0, \infty)$. This condition on $M$ is known as the condition $A_2$.

Particularly, from this condition, one derives that $\ell_M$ and $\ell_M$ are $\alpha$–dual each other (Corollary 4.2 of [5]) and are then perfect reflexive normed spaces.

### 3 The vector sequence space $\lambda_M \{ E \}$

For a complete normed space $E$, $\omega(E)$ will denote the vector space of all $E$–valued sequences, and by $\lambda_M \{ E \}$ we mean the subset of $\omega(E)$ constituted by all sequences in $E$ that are absolutely $(\lambda, M)$-summable. By this we mean

$$\lambda_M \{ E \} = \{ (x_n)_{n} \in \omega(E) : \forall (\alpha_n)_{n} \in \lambda^*, (\| \alpha_n x_n \|_{E})_{n} \in \ell_M \}.$$

We have

**Theorem 1.** With respect to the standard component operations, $\lambda_M \{ E \}$, is a linear space, and the quantity

$$\| \gamma \|_{\lambda_M \{ E \}} = \sup_{\alpha \in \lambda} \inf_{\sigma > 0} \left\{ \sigma > 0 : \sum_{n=1}^{\infty} M(\| \alpha_n x_n \| / \sigma) \leq 1 \right\}$$

is a norm on $\lambda_M \{ E \}$.

**Proof.** It follows quickly from the subadditivity of the norm of $E$ and the fact that $\ell_M$ is normal that

$$\ell_M \{ E \} = \{ (x_n)_{n} \in \omega(E) : (\| \alpha_n x_n \|_{E})_{n} \in \ell_M \}$$

is a linear subspace of $\omega(E)$. Now, for all $(\alpha_n)_{n} \in \lambda^*$, define $\varphi_{\alpha} : \omega(E) \to \omega(E)$ by $\varphi_{\alpha}(x) = (\alpha_n a(x_n))$.

Clearly, $\varphi_{\alpha}$ is a linear mapping, and

$$\lambda_M \{ E \} = \bigcap_{\alpha \in \lambda^*} \varphi_{\alpha}^{-1}(\ell_M \{ E \}).$$

Then, $\lambda_M \{ E \}$ is a linear space.

Next, we shall show that the quantity in (2) is finite. To this purpose, let $x = (x_n)_{n}$ be in $\lambda_M \{ E \}$ fixed, and consider the operator $f_x : \lambda^* \to \ell_M$ such that $f_x(\gamma) = (\| \gamma_n x_n \|)_{n}$. An application of the closed graph theorem yields the continuity of $f_x$. It follows that

$$\| \gamma \|_{\lambda_M \{ E \}} = \sup_{\gamma \in \lambda_M \{ E \}} \inf_{\sigma > 0} \left\{ \sigma > 0 : \sum_{n=1}^{\infty} M(\| \gamma_n x_n \| / \sigma) \leq 1 \right\}$$

which gives the required property. The other conditions of the norm are easily checked. \(\blacksquare\)

Now, we prove that the projections are continuous.

**Lemma 1.** If $i_1$ is a natural number, let $P_i$ be the the projection from $\lambda_M \{ E \}$ to $E$, given as

$$x = (x_n) \in \lambda_M \{ E \}, \text{ then } P_i(x) = x_i.$$  

Then, $P_i$ is a linear and continuous mapping.
Proof. Let \( i \in \mathbb{N} \) and \((\gamma_n)_n \in B_{1,1}\) such that \( \gamma > 0 \). Let \( K = 1/(\gamma \| e_1 \|_M) \). For all \( x = (x_n) \in \lambda M \{ E \} \), since the norm \( \| \cdot \|_M \) is solid and \( \| e_1 \|_M \) is finite, we have

\[
\gamma \| x_n \|_M \leq \| e_1 \|_M \sqrt{\sum \| x_n \|^2} = \| x_n \|_M = (x_n)_n \in \lambda M \{ E \}.
\]

Thus,

\[
\forall x = (x_n) \in \lambda M \{ E \}, \quad \| x_n \|_M \leq K \| x_n \|_M \leq \| x_n \|_M.
\]

from which one derives the continuity of \( P \).

Theorem 2. The space \( \lambda M \{ E \} \) so normed is a complete normed space for which \( E \) and \( \lambda \) are closed linear subspaces.

Proof. We will show first that, if \( \alpha = (\alpha_k)_k \in \lambda \) and \( t \in E \), then \( (\alpha_k t)_k \in \lambda M \{ E \} \).

Consider \( 0 \neq \alpha = (\alpha_k)_k \in \lambda, \beta = (\beta_k)_k \in \lambda^* \) and \( t \in E \). Let \( \sigma = \sum k \| \alpha_k \beta_k t \| \) and \( \eta_k = \| \alpha_k \beta_k t \| / \| \sigma \| \), for every \( k \). Then,

\[
\sum \| \alpha_k \beta_k t \| / \| \sigma \| = \sum \| M(\eta_k) \| \leq \sum \| \eta_k M(1) \| = M(1) < \infty.
\]

So, \( (\alpha_k t)_k \in \lambda M \{ E \} \). Now, let us show that for all \( \alpha = (\alpha_k)_k \in \lambda \) and \( t \in E \),

\[
\| (\alpha_k t)_k \|_\lambda M \{ E \} \leq (1 + M(1)) \| \alpha \|_\lambda \| t \|_E.
\]

The assertion (2) is trivial when \( t = 0 \). Assume that \( t \neq 0 \). Let \( \sigma_0 = (1 + M(1)) \| t \|_E \| \alpha \|_\lambda \). If \( \beta = (\beta_n)_n \in \lambda^* \) with \( \| \beta \|_{\lambda^*} \leq 1 \), since \( M \) is convex,

\[
\sum_{n=1}^\infty \left( \left| \frac{\alpha_n \beta_n \| t \|}{\sigma_0} \right| \right) \leq \sum_{n=1}^\infty \left| \frac{\alpha_n \beta_n \| t \|}{\sigma_0} \right| = M(1) \| \alpha \|_\lambda \| t \|_E.
\]

Thus,

\[
\| (\alpha_n t)_n \|_{\lambda M \{ E \}} = \| (\alpha_n t)_n \|_M \leq \| \sigma_0 = (1 + M(1)) \| \alpha \|_\lambda \| t \|_E.
\]

For a nonzero \( \tau = (\tau_n)_n \) fixed in \( \lambda \), the well defined mapping \( f_\tau \) from \( E \) to \( \lambda M \{ E \} \) with \( f_\tau(t) = (\tau_n t)_n \) is linear and one to one; its continuity holds by (2).

Suppose that \( (t_k)_k \) is any sequence of members of \( E \) that satisfies the convergence of \( (\tau_k t_k)_k \) in \( \lambda M \{ E \} \) to \( y = (x_n)_n \) for any natural number \( m \) such that \( \tau_m \neq 0 \), we conclude from Lemma 1 the convergence of the sequence \( (t_k)_k \) to \( \frac{1}{\tau_m} x_m \). Suppose then that \( (t_k)_k \) tends to \( t_k \to \infty \). Thus, when \( \tau_k \neq 0 \), \( x_n = t \) and \( x_n = 0 \) for \( \tau_n = 0 \). Then, \( y = \tau_t \), which means that \( E \) can be assimilated as a closed subspace of \( \mathcal{M} \{ E \} \). The same argument applies to prove that \( \lambda \) can also be assimilated with a closed subspace of \( \lambda \mathcal{M} \{ E \} \).

Consider a Cauchy sequence \( x^k = (x^k_n), k = 1, 2, \ldots, \in \lambda \mathcal{M} \{ E \} \). Let \( b \) be a natural number. Thanks to the continuity of the mapping \( P \), stated and proved in the lemma 1, the projected sequence \( x^*_b, k = 1, 2, \ldots, \) is, in fact, in \( E ; \) a Cauchy sequence, denote its limit by \( x_b \in E \).

We claim that \( x = (x_b)_b \in \lambda \mathcal{M} \{ E \} \) and that \( (x^k_b) \to x \) as \( k \to \infty \). For a fixed \( \alpha = (\alpha_k)_k \in \lambda^* \), we will verify that the mapping \( \varphi_{\alpha} : y = (y_n)_n \mapsto (\alpha_n \| y_n \|)_n \in \mathcal{M} \) is uniformly continuous. Since the norm \( \| \cdot \|_{\mathcal{M}} \) of \( E \) is solid, for all \( y = (y_n)_n \) and \( z = (z_n)_n \in \lambda \mathcal{M} \{ E \} \), we can write

\[
\| \varphi_{\alpha}(y) - \varphi_{\alpha}(z) \|_M = \| (\alpha_n \| y_n \|)_n - (\alpha_n \| z_n \|)_n \|_M.
\]

So,
\[
\varphi_{\alpha}(x^k_b) = \{ \alpha_k \| x^k_n \| \}_{n=1}^\infty.
\]

Since \( \mathcal{M} \) is a complete normed space, this sequence converges to a limit that we denote by \( \beta = (\beta_n)_n \in \mathcal{M} \). Let \( b \) be a natural number. Then

\[
\alpha_b \| x^b_n \| = \alpha_b \lim_{b \to \infty} x^b_n = \beta_n.
\]

By what we proved that \( x \in \lambda \mathcal{M} \{ E \} \).

Again, a more difficult task is to prove the convergence of \( \{ x^k_n \}_{k=1}^\infty \) to \( x \). Consider a positive real number \( \epsilon \).

We can select a natural number \( N \) for which, if \( \alpha = (\alpha_n)_n \) is laying in \( B_{1,1} \) and \( p \) and \( q \) are natural numbers greater than \( N \), there exists \( 0 < \sigma < \epsilon \) that satisfies

\[
\sup_{K \geq N} \sum_{n=1}^K \left( \frac{\| \alpha_n \|_{\lambda M \{ E \}} \| x^k_n - x^q_n \|}{\sigma} \right) = \sum_{n=1}^K \left( \frac{\| \alpha_n \|_{\lambda M \{ E \}} \| x^k_n - x^q_n \|}{\sigma} \right) \leq 1.
\]

Thanks to the is continuity of \( M \), letting \( p \to \infty \), we find

\[
\sum_{n=1}^K \left( \frac{\| \alpha_n \|_{\lambda M \{ E \}} \| x^k_n - x^q_n \|}{\epsilon} \right) \leq 1 \quad \text{for every natural number } K \quad \text{greater than } N.\]

One can therefore conclude that

\[
\| x^p \|_E = \sup_{\alpha \in B_{1,1}} \inf \left( \left\{ \sigma > 0 : \sum_{n=1}^K \left( \frac{\| \alpha_n \|_{\lambda M \{ E \}} \| x^k_n - x^q_n \|}{\sigma} \right) \leq 1 \right\} \right) \leq \epsilon,
\]

whenever \( p \) is greater than \( N \). The proof is over.

4 On the continuous dual of \( \lambda \mathcal{M} \{ E \} \)

For \( x = (x_n)_n \in \varnothing \{ E \} \), let \( \{ x^k_n \}_{k=1}^\infty \) denote the sequence of the finite sections of \( x \). That is

\[
x^k = (x_1, x_2, \ldots, x_k, 0, \ldots) = \sum_{n=1}^k x_n e_n.
\]
It is immediately seen that \( \lambda_M \{E\} \) contains the finite sections of all its elements. In other words, if \( y = (y_n) \in \lambda_M \{E\} \), then \( \{y^{(k)}\}_{k=1}^{\infty} \subseteq \lambda_M \{E\} \). Using the \( \Sigma \) notation for \( y^{(k)} \), we see that if \( y \) is an AK-sequence, that is \( \{y^{(k)}\}_{k=1}^{\infty} \) converges to \( y \), in \( \lambda_M \{E\} \), then

\[
y = \lim_{k \to \infty} y^{(k)} = \sum_{n=1}^{\infty} y_n e_n.
\]  

(3)

Let \( \lambda_M \{E\}_r \) denote the subspace of elements of \( \lambda_M \{E\} \) satisfying the equation (3). The vector sequence space \( \lambda_M \{E\}_r \) is said to have the AK-property, if it coincides with \( \lambda_M \{E\}_r \).

The following result relates topologically these two spaces.

**Theorem 3.** \( \lambda_M \{E\}_r \) is a closed subspace of \( \lambda_M \{E\} \).

**Proof.** Since the norm \( \| \cdot \|_M \) of \( \ell_M \) is solid, the definition of the norm \( \| \cdot \|_{\lambda_M \{E\}_r} \) of \( \lambda_M \{E\}_r \) reveals that it is monotonic; in particular, if \( y = (y_n) \in \lambda_M \{E\}_r \) then \( \|y\| \leq \|y\|_{\lambda_M \{E\}_r} \leq \|y\|_{\lambda_M \{E\}} \). Consider an element \( y \in \lambda_M \{E\}_r \) which is lying in the closure \( \overline{\lambda_M \{E\}_r} \) of \( \lambda_M \{E\}_r \) and a positive number \( \delta \). One has \( z \in \lambda_M \{E\}_r \), and \( K \in \mathbb{N} \) for which \( \|y - z\|_{\lambda_M \{E\}_r} < \delta \) and \( \|z\|_{\lambda_M \{E\}_r} < \delta \) if \( k \geq K \). So, since |||| \( \| \cdot \|_{\lambda_M \{E\}_r} \) is monotonic,

\[
\|y^{(k)} - y\|_{\lambda_M \{E\}_r} \leq \|y^{(k)} - z^{(k)}\|_{\lambda_M \{E\}} + \|z - y\|_{\lambda_M \{E\}_r} < 2\|y - z\|_{\lambda_M \{E\}_r} + \delta < \delta.
\]

if \( k \geq K \). This means that \( y \in \lambda_M \{E\}_r \) and \( \lambda_M \{E\}_r \) is indeed closed in \( \lambda_M \{E\} \).

**Theorem 4.** Suppose that \( \varphi \) is a mapping which is linear and continuous on \( \lambda_M \{E\} \). Define, for every natural number \( n \), the mapping \( x_n^* \) on \( E \) by setting \( x_n^* (t) = \varphi (\epsilon_n t) \). Then, \( (x_n^*)_n \in \lambda_M^* \{E^*\} \). In other words, \( (x_n^*)_n \) is absolutely \( (\lambda^*, \cdot^*) \)-summable in the dual space \( E^* \) of \( E \).

**Proof.** The continuity of \( \varphi \) provides a positive constant \( K \) with the property that

\[
\|\varphi (y)\| \leq K \|y\|_{\lambda_M \{E\}_r}, \text{ whenever } y = (y_n) \in \lambda_M \{E\}_r.
\]

Now, for a natural number \( n \) and a vector \( z \) in \( E \), the inequality (2) yields

\[
\|x_n^* (z)\| = \|\varphi (\epsilon_n z)\| \leq K \|\epsilon_n z\|_{\lambda_M \{E\}_r} \leq K (M(1) + 1) \|\epsilon_n\|_\alpha \|z\|_E.
\]  

(4)

By the inequality (4), one has \( (x_n^*)_n \in o_\lambda (E^*) \).

The proof of \( (x_n^*)_n \in \lambda^* (E^*, \cdot^*) \) is the only thing that is remaining. To do so, consider \( \alpha = (\alpha_n) \in \lambda \). We will prove that \( (\alpha_n \|x_n^*\|)_{n \in \mathbb{N}} \in \ell_\mathbb{N} \). Let \( \gamma = (\gamma_n) \in \ell_M \).

Since \( \|\gamma_n \alpha_n x_n^*\| = \sup \{\|\gamma_n \alpha_n t\| : t \in B_E\} \), if \( \delta > 0 \), one can find, for every natural number \( n \), a vector \( t_n \) in the closed unit ball \( B_E \) of \( E \) satisfying

\[
\|\gamma_n \alpha_n x_n^*\| \leq \|\alpha_n \gamma_n x_n^*\| + \delta \leq \|\alpha_n \gamma_n x_n^*\| + \frac{\delta}{2^n}.
\]

Let \( (\epsilon_n) \in \alpha \) be such that \( \varphi (\gamma_n \alpha_n x_n^* \epsilon_n) = \gamma_n \alpha_n x_n^* \epsilon_n \). For every \( k \in \mathbb{N} \), we have,

\[
\|\sum_{n=1}^{k} \gamma_n \alpha_n x_n^* \epsilon_n\|_E \leq \sum_{n=1}^{k} \|\gamma_n \alpha_n x_n^* \epsilon_n\|_E + \delta = \|\sum_{n=1}^{k} \gamma_n \alpha_n x_n^* \epsilon_n\|_E + \delta
\]

\[
\leq K \|\sum_{n=1}^{k} \sum_{n=1}^{\infty} \gamma_n \alpha_n x_n^* \epsilon_n\|_E + \delta.
\]

Let \( (\beta_n) \in \ell_\mathbb{N} \). Then,

\[
\|\sum_{n=1}^{k} \gamma_n \alpha_n x_n^* \epsilon_n\|_E \leq \|\alpha\|_\gamma \|\gamma\|_E.
\]

Thus, \( \|\sum_{n=1}^{\infty} \gamma_n \alpha_n x_n^* \epsilon_n\|_E \leq \|\alpha\|_\gamma \|\gamma\|_E \), which proves that the series \( \sum_{n=1}^{\infty} \gamma_n \alpha_n x_n^* \epsilon_n \) converges. So, \( (x_n^*)_n \in \lambda_M^* \{E^*\} \).

For the \( \alpha^* \)-duality, we prove what follows.

**Lemma 2.** Denote by \( (\lambda_M \{E\})^* \) the \( \alpha^* \)-dual of \( \lambda_M \{E\} \):

\[\lambda_M \{E\}^* = \{ (\alpha_n) \in \ell^* : \sum_{n=1}^{\infty} |\alpha_n| (x_n) < \infty, \forall (x_n) \in \lambda_M \{E\}\} .\]

Then one has the double inclusion \( \lambda_M \{E\}_r^* \subseteq \lambda_M \{E\}^* \subseteq \lambda_M^* \{E^*\} \).

**Proof.** We first show the inclusion \( \lambda_M \{E\}^* \subseteq \lambda_M \{E\}_r^* \).

As in the proof of the theorem 4, let \( \varphi \) be in \( (\lambda_M \{E\})^* \). For every natural number \( n \) and vector \( z \in E \), define \( b_n (z) = \varphi (z \epsilon_n) \). By the continuity of \( \varphi \) one has \( \rho > 0 \) such that

\[
\|\gamma\|_E \leq \rho \|\gamma\|_{\lambda_M \{E\}}, \text{ whenever } y = (y_n) \in \lambda_M \{E\}_r.
\]

In particular, we get, by (2),

\[b_n (z) = |\varphi (z \epsilon_n)| \leq \rho \|\epsilon_n\|_{\lambda_M \{E\}} \leq \rho (M(1) + 1) \|\epsilon_n\|_\alpha \|z\|_E,
\]

for all \( n \in \mathbb{N} \) and \( z \in E \). This means that \( (b_n) \in o_\alpha (E^*) \).

Now, we are ready to prove that \( (b_n) \in (\lambda_M \{E\}_r)^* \). Let \( x = (x_n) \in \lambda_M \{E\}_r \). By the equation (3),
\[ x = \lim_{k \to \infty} x^{(k)} = \sum_{n=1}^{\infty} x_n e_n. \] Since \( \varphi \) is continuous on \( \lambda_M \{ E \} \), we can write
\[
\varphi(x) = \varphi(\lim_{k \to \infty} x^{(k)}) = \lim_{k \to \infty} \varphi(x^{(k)})
\]
\[ = \lim_{k \to \infty} \sum_{n=1}^{k} \varphi(x_n e_n) = \sum_{n=1}^{\infty} \varphi(x_n e_n)
\]
\[ = \sum_{n=1}^{\infty} b_n(x_n). \] (6)

Then, the series \( \sum_{n=1}^{\infty} a_n(x_n) \) converges. Actually, it converges absolutely. In fact, let \( (e_n)_n \) a sequence of real numbers such that
\[ |a_n(x_n)| = \varepsilon |a_n(x_n)|, \text{ for every } n \in \mathbb{N}. \]

It is not hard to verify that \( y = (e_n x_n)_n \) belongs to \( \lambda_M \{ E \} \), and that
\[ \sum_{n=1}^{\infty} |a_n(x_n)| = \varphi(y). \]

Now, let \( \alpha = (a_n)_n \in (\lambda_M \{ E \})^{\ast} \). We have to prove that for all \( \alpha \in (\lambda_M \{ E \})^{\ast} \), \( \alpha \in \ell_N \). As in the second part of the proof of (4), since \( \ell_N = \ell^\ast_M \), it is enough to prove that the series \( \sum_{n=1}^{\infty} |\gamma_n a_n| |a_n| \) converges, for all \( (\gamma_n)_n \in \ell_M \). For every \( n \in \mathbb{N} \), since
\[ \| \gamma_n a_n a_n \| = \sup\{|a_n(\gamma_n a_n t)| : t \in B_E\}, \]
there exists \( t_n \in B_E \) such that
\[ \| \gamma_n a_n a_n \| \leq |a_n(\gamma_n a_n t_n)| + \frac{1}{2^n}. \]
But, \( (\gamma_n a_n a_n) \in (\lambda_M \{ E \})^{\ast} \). Indeed, if \( \beta = (b_n)_n \in \lambda^\ast \) then \( (\gamma_n a_n a_n) \in (\lambda_M \{ E \})^{\ast} \) and \( \ell_M \) is normal.

Now, \( \sum_{n=1}^{\infty} |a_n| (\gamma_n a_n a_n) < \infty \) since \( a \in (\lambda_M \{ E \})^{\ast} \) and \( (\gamma_n a_n a_n) \in (\lambda_M \{ E \})^{\ast} \), and then
\[ \sum_{n=1}^{\infty} \| \gamma_n a_n a_n \| \leq \sum_{n=1}^{\infty} |a_n| (\gamma_n a_n a_n) + 1 \text{ is finite too. This completes the proof.} \]

**Theorem 5.** Let \( \psi \) be the correspondence from \( (\lambda_M \{ E \})^{\ast} \) to \( (\lambda^\ast \{ E^\ast, N \}) \) which assigns to every continuous linear form on \( \lambda_M \{ E \} \) the element of \( (\lambda^\ast \{ E^\ast, N \}) \) defined by the sequence \( b = (b_n)_n \) given in the theorem 4. Then, \( \psi \) defines a one- to one continuous mapping, when these two spaces are endowed with their standard respective norms.

**Proof.** If \( \varphi \in (\lambda_M \{ E \})^{\ast} \), the sequence \( a = (a_n)_n \) represents \( \varphi \) as seen in (5). So, \( \psi \) is well defined. Moreover, using (5), one can see that \( \psi \) is linear and one to one. Now, let us prove that \( \psi \) is continuous.

Let \( a = (a_n)_n \) be an element of \( \lambda_N \{ E^\ast \} \), and \( \alpha = (\gamma_n)_n \in B_{\lambda} \). Since \( \ell_N = \ell^\ast_M \), the norm of \( \| (a_n|a_n|) \|_{N} \) is defined by,
\[ \| (a_n|a_n|) \|_{N} = \sup \left\{ \sum_{n=1}^{\infty} \| a_n(\gamma_n a_n a_n) \| : \gamma = (\gamma_n) \in B_{\ell_M} \right\} . \]

For \( \varepsilon > 0 \), a sequence \( (t_n)_n \subset B_{E} \) can be found such that, for every \( n \in \mathbb{N} \),
\[ \gamma_n a_n a_n \| a_n \| \leq a_n(\gamma_n a_n a_n) + \frac{\varepsilon}{2^n}. \]

But, as can be easily seen, \( y = (\gamma_n a_n a_n)_n \in (\lambda_M \{ E \})^{\ast} \), and then, if \( \varphi \) is represented by the sequence \( (a_n)_n \), we have
\[ \sum_{n=1}^{\infty} |\gamma_n a_n a_n | \leq |\varphi(y)| \leq \varepsilon + \| \varphi \| (\lambda_M \{ E \})^{\ast} \| y \| \leq \varepsilon + \| \alpha \| \| y \|. \]

**Conflict of Interest**

The authors declare that there is no conflict regarding the publication of this paper.

**References**


