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Some Properties of Incomplete First Appell Hypergeometric Matrix Functions

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Abstract: The aim of this paper to introduce two incomplete first Appell hypergeometric matrix functions (IFAHMFs) γ_1 and Γ_1 by means of the incomplete Pochhammer matrix symbols. Furthermore, there is a derivation of some results such as integral formula, recursion formula, differentiation formula and finite summation formula of the IFAHMFs γ_1 and Γ_1.

Keywords: Gamma matrix function, incomplete Pochhammer symbols, hypergeometric matrix function, Bessel matrix function.

1 Introduction

In 2012, Srivastava *et al.* [1] introduced new incomplete Pochhammer symbols and discussed many related applications. Recently, Bansal *et al.* [2] established certain incomplete ℵ- functions and investigated some properties of them. Several properties of the incomplete multivariable hypergeometric functions have been investigated in the recent papers [3,4,5,6,7,8].

The matrix theory is appearing in the field of mathematical, physical and engineering. In recent years, many researchers have introduced and investigated several kind of special matrix functions [9,10,11,12,13]. Matrix analogue of the two variable Appell hypergeometric functions are defined in [14,15,16]. The incomplete multivariable hypergeometric matrix functions have been studied by many authors (see, e.g., [17,18,19]). Recursion formula, infinite summation formula for the Srivastava's triple hypergeometric matrix functions $H_{\mathscr{A}}$, $H_{\mathscr{B}}$ and $H_{\mathscr{C}}$ are presented in [20]. Verma *et al.* [21] have obtained some results of the Kampé de Feŕiet hypergeometric matrix function.

Throughout in this paper, let $\mathbb{C}^{s\times s}$ be the complex space of complex matrices of common order s. For any matrix $E \in \mathbb{C}^{s\times s}$, its spectrum $\nu(E)$ is the family of eigenvalues of E. Suppose that $f_1(z)$ and $f_2(z)$ are holomorphic functions in Θ an open set of the complex plane and $E \in \mathbb{C}^{s\times s}$ with $\nu(E) \subset \Theta$, then by means of the properties of the matrix functional calculus [22], we get $f_1(E)f_2(E) = f_2(E)f_1(E)$. Moreover, let F be a matrix in $\mathbb{C}^{s\times s}$ for which $\nu(F) \subset \Theta$, then $f_1(E)f_2(F) = f_2(F)f_1(E)$. A matrix $E \in \mathbb{C}^{s\times s}$ is called positive stable (In short, PS) if $Re(\tau) > 0$ for all $\tau \in \sigma(E)$.

The Gamma matrix function $\Gamma(E)$ is given by [23]

$$\Gamma(E) = \int_0^\infty e^{-t}t^{E-I}dt;\ t^{E-I} = \exp((E-I)\ln t), \tag{1}$$

where E is a PS matrix in $\mathbb{C}^{s\times s}$.
In addition, if $E+dI$ is invertible for each integer $d \geq 0$, hence the reciprocal gamma function [23] is stated as:

$$\Gamma^{-1}(E) = (E)_d\, \Gamma^{-1}(E+dI).$$

Here, $(E)_d$ denotes the shifted factorial matrix function for $E \in \mathbb{C}^{s\times s}$ stated as ([24]):

$$(E)_d = \begin{cases} E(E+I)\cdots(E+(d-1)I), & d \geq 1 \\ I, & d = 0. \end{cases} \tag{2}$$

I denotes the identity matrix in $\mathbb{C}^{s\times s}$. If the matrix $E \in \mathbb{C}^{s\times s}$ is PS and $d \geq 1$, so by [23], one has $\Gamma(E) = \lim_{d\to\infty}(d-1)!(E)_d^{-1}d^E$.

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The Gauss hypergeometric matrix function [24] is stated as

$$ {}_2F_1(E,F;G;z_1) = \sum_{d=0}^{\infty} \frac{(E)_d (F)_d (G)_d^{-1}}{d!} z_1^d, \qquad (3) $$

for matrices E, F and G in $\mathbb{C}^{s\times s}$ so that $G+dI$ is invertible for each integer $d \geq 0$ and $|z_1| \leq 1$.

The incomplete gamma matrix functions $\gamma(E,x)$ and $\Gamma(E,x)$ are respectively given as (see [17])

$$ \gamma(E,x) = \int_0^x e^{-t} t^{E-I} dt \qquad (4) $$

and

$$ \Gamma(E,x) = \int_x^{\infty} e^{-t} t^{E-I} dt. \qquad (5) $$

The next decomposition identity

$$ \gamma(E,x) + \Gamma(E,x) = \Gamma(E), \qquad (6) $$

is fulfilled. The incomplete Pochhammer matrix symbols $(E;x)_d$ and $[E;x]_d$ are defined by (see [17])

$$ (E;x)_d = \gamma(E+dI,x)\Gamma^{-1}(E) \qquad (7) $$

and

$$ [E;x]_d = \Gamma(E+dI,x)\Gamma^{-1}(E), \qquad (8) $$

where E and x denote the PS matrix and positive real number, respectively. By using (6), we get the following decomposition formula:

$$ (E;x)_d + [E;x]_d = (E)_d, \qquad (9) $$

where $(E)_d$ is the Pochhammer matrix symbol introduced in (2).
The incomplete Gauss hypergeometric matrix functions are given as (see [17])

$$ {}_2\gamma_1\left[(E;x),F;G;z_1\right] = \sum_{m=0}^{\infty} (E;x)_m (F)_m (G)_m^{-1} \frac{z_1^m}{m!} \qquad (10) $$

and

$$ {}_2\Gamma_1\left[[E;x],F;G;z_1\right] = \sum_{m=0}^{\infty} [E;x]_n (F)_n (G)_n^{-1} \frac{z_1^m}{m!}, \qquad (11) $$

where E, F and G are matrices in $\mathbb{C}^{s\times s}$ such that $G+k_1I$ is invertible for each integer $k_1 \geq 0$.
Furthermore, the integral representation of the incomplete Gauss hypergeometric matrix function ${}_2\Gamma_1$ is stated as:

$$ {}_2\Gamma_1\left[[E;x],F;G;z_1\right] = \left(\int_0^1 {}_1\Gamma_0\left[[E;x];-;z_1 t\right] t^{F-I}(1-t)^{G-F-I} dt\right) \times \Gamma^{-1}(F)\Gamma^{-1}(G-F)\Gamma(G),\ |z_1|<1, \qquad (12) $$

where G, F and $G-F$ are PS, $GF = FG$, and ${}_1\Gamma_0\left[[E;x];-;z_1 t\right]$ is the incomplete Gauss hypergeometric matrix function of one numerator.

The Bessel matrix function (see, e.g.,[25,26,27]) is stated as:

$$ J_E(z) = \sum_{m\geq 0} \frac{(-1)^m \Gamma^{-1}((m+1)I+E)}{m!} \left(\frac{z_1}{2}\right)^{2mI+E}, \qquad (13) $$

where k_1I+E is invertible for all integers $k_1 \geq 0$. Also, the modified Bessel matrix functions are defined as follows (see[27]):

$$ \begin{aligned} I_E &= e^{\frac{-Ei\pi}{2}} J_E(z_1 e^{\frac{i\pi}{2}});\ -\pi < arg(z_1) < \frac{\pi}{2}, \\ I_E &= e^{\frac{Ei\pi}{2}} J_E(z_1 e^{\frac{-i\pi}{2}});\ -\frac{\pi}{2} < arg(z_1) < \pi. \end{aligned} \qquad (14) $$

2 Main Results

This section deals with the IFAHMFs γ_1 and Γ_1 as follows:

$$ \gamma_1\left[(E;x),F,F';G;z_1,w_1\right] = \sum_{m_1,m_2\geq 0} \frac{(E;x)_{m_1+m_2}(F)_{m_1}(F')_{m_2}(G)_{m_1+m_2}^{-1}}{m_1!m_2!} z_1^{m_1} w_1^{m_2}, \qquad (15) $$

$$ \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] = \sum_{m_1,m_2\geq 0} \frac{[E;x]_{m_1+m_2}(F)_{m_1}(F')_{m_2}(G)_{m_1+m_2}^{-1}}{m_1!m_2!} z_1^{m_1} w_1^{m_2}, \qquad (16) $$

where E, F, F', G are PS matrices in $\mathbb{C}^{s\times s}$ such that $G+k_1I$ is invertible for every integer $k_1 \geq 0$ and z_1, w_1 are complex variables.
From (9), we get the following decomposition formula

$$ \gamma_1\left[(E;x),F,F';G;z_1,w_1\right] + \Gamma_1\left[[E;x],F,F';G;z_1,w_1\right] = F_1\left[E,F,F';G;z_1,w_1\right], \qquad (17) $$

where $F_1\left[E,F,F';G;z_1,w_1\right]$ is the first Appell hypergeometric matrix function [16].

Remark. If we set $z_1 = 0$ or $w_1 = 0$ in (15) and (16), we obtain the classical incomplete families of Gauss hypergeometric matrix functions [17].

By means of the properties of $\gamma_1\left[(E;x),F,F';G;z_1,w_1\right]$, we can determine the properties of $\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]$ using the decomposition formula (17).

Theorem 1. *Let E, F, F' and G be matrices in $\mathbb{C}^{s\times s}$ such that $FG = GF$, $FF' = F'F$ and $F'G = GF'$. Then the following function:*

$$\mathscr{S} = \mathscr{S}(z_1, w_1) = \gamma_1\left[(E;x), F, F'; G; z_1, w_1\right] + \Gamma_1\left[[E;x], F, F'; G; z_1, w_1\right]$$

meets the system of partial differential equations:

$$z_1(1-z_1)\frac{\partial^2\mathscr{T}}{\partial z_1^2} + (1-z_1)w_1\frac{\partial^2\mathscr{T}}{\partial z_1\partial w_1} - z_1(E+I)\frac{\partial\mathscr{T}}{\partial z_1} - z_1\frac{\partial\mathscr{T}}{\partial z_1}F - w_1\frac{\partial\mathscr{T}}{\partial w_1}F + \frac{\partial\mathscr{T}}{\partial z_1}G - E\mathscr{T}F = O, \tag{18}$$

$$w_1(1-w_1)\frac{\partial^2\mathscr{T}}{\partial w_1^2} + (1-w_1)z_1\frac{\partial^2\mathscr{T}}{\partial z_1\partial w_1} - w_1(E+I)\frac{\partial\mathscr{T}}{\partial w_1} - w_1\frac{\partial\mathscr{T}}{\partial w_1}F' - z_1\frac{\partial\mathscr{T}}{\partial z_1}F' + \frac{\partial\mathscr{T}}{\partial w_1}G - E\mathscr{T}F' = O. \tag{19}$$

Proof. The relation (17) succeeds into the following proof conjoined with $F_1\left[E, F, F'; G; z_1, w_1\right]$ which adequately fulfil the matrix differential equations given in [14,15].

Theorem 2. *Let E, F, F' and G be non commuting matrices in $\mathbb{C}^{s\times s}$ so that E and G are PS, then we have the following integral representation:*

$$\Gamma_1\left[[E;x], F, F'; G; z_1, w_1\right] = \Gamma^{-1}(E)\left[\int_x^\infty e^{-t}t^{E-I}\Phi_2(F, F'; G; z_1t, w_1t)dt\right], \tag{20}$$

where Φ_2 is Humbert's hypergeometric matrix function given by (see [28])

$$\Phi_2(F, F'; G; z_1, w_1) = \sum_{m_1,m_2\geq 0}\frac{(F)_{m_1}(F')_{m_2}(G)^{-1}_{m_1+m_2}}{m_1!m_2!}z_1^{m_1}w_1^{m_2}. \tag{21}$$

Proof. By substituting $[E;x]_{m_1+m_2}$ in (5) and (8) by its integral representation in (16), we have

$$\begin{aligned}\Gamma_1\left[[E;x], F, F'; G; z_1, w_1\right] &= \Gamma^{-1}(E)\sum_{m_1,m_2\geq 0}\left(\int_x^\infty e^{-t}t^{E+(m_1+m_2-1)I}dt\right)\\ &\quad\times(F)_{m_1}(F')_{m_2}(G)^{-1}_{m_1+m_2}\frac{z^{m_1}w^{m_2}}{m_1!m_2!},\\ &= \Gamma^{-1}(E)\sum_{m_1,m_2\geq 0}\left(\int_x^\infty e^{-t}t^{E-I}(F)_{m_1}(F')_{m_2}(G)^{-1}_{m_1+m_2}\right.\\ &\quad\left.\times\frac{(z_1t)^{m_1}(w_1t)^{m_2}}{m_1!m_2!}dt\right).\end{aligned} \tag{22}$$

Hence, the proof is completed.

Theorem 3. *For matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that $FG = GF$, $FF' = F'F$ and $F'G = GF'$, and F, F', G are PS, we have the following integral representation:*

$$\begin{aligned}&\Gamma_1\left[[E;x], F, F'; G; z_1, w_1\right]\\ &= \left[\int_0^\infty\int_0^\infty e^{-t_1-t_2}\,{}_1\Gamma_1\left[[E;x]; G; z_1t_1 + w_1t_2\right]t_1^{F-I}t_2^{F'-I}dt_1dt_2\right]\\ &\times\Gamma^{-1}(F)\Gamma^{-1}(F').\end{aligned} \tag{23}$$

Proof. By using the integral representation of the Pochhammer matrix symbols $(F)_m$, $(F')_n$ in the definition of (16), we get

$$\begin{aligned}&\Gamma_1\left[[E;x], F, F'; G; z_1, w_1\right]\\ &= \sum_{m_1,m_2\geq 0}\left[\int_0^\infty\int_0^\infty e^{-t_1-t_2}[E;x]_{m_1+m_2}\right.\\ &\left.\times t_1^{F-I}t_2^{F'-I}(G)^{-1}_{m_1+m_2}\frac{(z_1t)^{m_1}(w_1t)^{m_2}}{m_1!m_2!}dt_1dt_2\right]\Gamma^{-1}(F)\Gamma^{-1}(F').\end{aligned} \tag{24}$$

With the help of the summation formula [29]

$$\sum_{M\geq 0}f(M)\frac{(z+w)^M}{M!} = \sum_{m_1,m_2\geq 0}f(m_1+m_2)\frac{z^{m_1}w^{m_2}}{m_1!m_2!}, \tag{25}$$

we get (23).

Theorem 4. *For matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that $FG = GF$, $FF' = F'F$ and $F'G = GF'$, and E, F, F', G are PS, the following integral representation holds true:*

$$\begin{aligned}&\Gamma_1\left[[E;x], F, F'; G; z_1, w_1\right]\\ &= \Gamma^{-1}(E)\left[\int_0^\infty\int_0^\infty\int_x^\infty e^{-s-t_1-t_2}s^{E-I}\right.\\ &\quad\left.\times t_1^{F-I}t_2^{F'-I}{}_0F_1(-; G; z_1t_1s + w_1t_2s)dt_1dt_2ds\right]\\ &\quad\Gamma^{-1}(F)\Gamma^{-1}(F').\end{aligned} \tag{26}$$

Proof. By substituting $[E;x]_{m+n}$ in (5) and (8) by its integral representation in (23), we are led to the desired result (26).

Corollary 1. *We have*

$$\begin{aligned}&\Gamma_1\left[[E;x], F, F'; G+I; -z_1, -w_1\right]\\ &= \Gamma^{-1}(E)\left[\int_0^\infty\int_0^\infty\int_x^\infty e^{-s-t_1-t_2}s^{E-\frac{G}{2}-I}t_1^{F-I}t_2^{F'-I}\right.\\ &\quad\left.\times(z_1t_1 + w_1t_2)^{-\frac{G}{2}}J_G(2\sqrt{z_1t_1s + w_1t_2s})dt_1dt_2ds\right]\\ &\quad\Gamma^{-1}(F)\Gamma^{-1}(F')\Gamma(G+I)\end{aligned} \tag{27}$$

$$\begin{aligned}&\Gamma_1\left[[E;x], F, F'; G+I; z_1, w_1\right]\\ &= \Gamma^{-1}(E)\left[\int_0^\infty\int_0^\infty\int_x^\infty e^{-s-t_1-t_2}s^{E-\frac{G}{2}-I}t_1^{F-I}t_2^{F'-I}\right.\\ &\quad\left.\times(z_1t_1 + w_1t_2)^{-\frac{G}{2}}I_G(2\sqrt{z_1t_1s + w_1t_2s})dt_1dt_2ds\right]\\ &\quad\Gamma^{-1}(F)\Gamma^{-1}(F')\Gamma(G+I),\end{aligned} \tag{28}$$

Theorem 5. *For non commuting matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that E and G are PS, we have the following recursion relation:*

$$\begin{aligned}&\Gamma_1\left[[E;x],F+sI,F';G;z_1,w_1\right]\\&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad+z_1E\left[\sum_{k=1}^{n}\Gamma_1\left[[E+I;x],F+kI,F';G+I;z_1,w_1\right]\right]G^{-1}.\end{aligned}\tag{29}$$

Also, if $F-kI$ is invertible for every integer $k\leq n$ where n is a non-negative integer, then

$$\begin{aligned}&\Gamma_1\left[[E;x],F-sI,F';G;z_1,w_1\right]\\&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad-z_1E\left[\sum_{k=0}^{n-1}\Gamma_1\left[[E+I;x],F-kI,F';G;z_1,w_1\right]\right]G^{-1}.\end{aligned}\tag{30}$$

Proof. By using (20) and the following formula:

$$(F+I)_m=F^{-1}(F)_m(F+mI),$$

we have

$$\begin{aligned}\Gamma_1\left[[E;x],F+I,F';G;z_1,w_1\right]&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad+z_1E\left[\Gamma_1[(E+I;x],F+I,F';G+I;z_1,w_1]\right]G^{-1}.\end{aligned}\tag{31}$$

Now, applying (31) to the matrix function Γ_1 with the matrix parameter $F+2I$, we find that

$$\begin{aligned}\Gamma_1\left[[E;x],F+2I,F';G;z_1,w_1\right]&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad+z_1E\left[\sum_{k=1}^{2}\Gamma_1\left[[E+I;x],F+kI,F';G+I;z_1,w_1\right]\right]G^{-1}.\end{aligned}\tag{32}$$

Recursion formula (29) follows by repeating n-times the process of result (31).
Again, replace F with $F-I$ in (31) to get

$$\begin{aligned}\Gamma_1\left[[E;x],F-I,F';G;z_1,w_1\right]&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad-z_1E\left[\Gamma_1\left[[E+I;x],F,F';G+I;z_1,w_1\right]\right]G^{-1}.\end{aligned}\tag{33}$$

Iteratively, we obtain (30).

By using the relations (31) and (33), we have another form of recursion formulas for Γ_1.

Theorem 6. *For non commuting matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that E and G are PS, we have the following recursion relation:*

$$\begin{aligned}&\Gamma_1\left[[E;x],F+nI,F';G;z_1,w_1\right]\\&=\sum_{k_1\leq n}\binom{n}{k_1}(E)_{k_1}z_1^{k_1}\\&\quad\times\left[\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right]\right](G)_{k_1}^{-1}.\end{aligned}\tag{34}$$

Also, if $F-kI$ is invertible for every integer $k\leq n$ (where n is a non-negative integer), then

$$\begin{aligned}&\Gamma_1\left[[E;x],F-nI,F';G;z_1,w_1\right]\\&=\sum_{k_1\leq n}\binom{n}{k_1}(E)_{k_1}(-z_1)^{k_1}\\&\quad\times\left[\Gamma_1[[E+k_1I;x],F,F';G+k_1I;z_1,w_1]\right](G)_{k_1}^{-1}.\end{aligned}\tag{35}$$

Proof. To prove the result (34), it suffices to apply the induction on $n\in\mathbb{N}$. For $n=1$, (34) holds. Suppose (34) is true for $n=t$, i.e.,

$$\begin{aligned}&\Gamma_1\left[[E;x],F+tI,F';G;z_1,w_1\right]=\\&\sum_{k_1\leq t}\binom{t}{k_1}(E)_{k_1}z_1^{k_1}\left[\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right]\right](G)_{k_1}^{-1}.\end{aligned}\tag{36}$$

Replacing F with $F+I$ in (36) and using (31), we get

$$\begin{aligned}&\Gamma_1\left[[E;x],F+(t+1)I,F';G;z_1,w_1\right]=\\&\sum_{k_1\leq t}\binom{t}{k_1}(E)_{k_1}z_1^{k_1}\left[\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right]\right.\\&\quad+z_1(E+k_1I)\Gamma_1\left[[E+(k_1+1)I;x],F+(k_1+1)I,F';G+(k_1+1)I;z_1,w_1\right]\\&\quad\left.(G+k_1I)^{-1}\right]\times(G)_{k_1}^{-1}.\end{aligned}\tag{37}$$

After some simplification, (37) takes the form

$$\begin{aligned}&\Gamma_1\left[[E;x],F+(t+1)I,F';G;z_1,w_1\right]=\\&\sum_{k_1\leq t}\binom{t}{k_1}(E)_{k_1}z_1^{k_1}\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right](G)_{k_1}^{-1}\\&\quad+\sum_{k_1\leq t+1}\binom{t}{k_1-1}(G)_{k_1}z_1^{k_1}\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right](G)_{k_1}^{-1}.\end{aligned}\tag{38}$$

By applying Pascal's formulas (38), we obtain

$$\begin{aligned}&\Gamma_1\left[[E;x],F+(t+1)I,F';G;z_1,w_1\right]\\&=\sum_{k_1\leq t+1}\binom{t+1}{k_1}(E)_{k_1}z_1^{k_1}\Gamma_1\left[[E+k_1I;x],F+k_1I,F';G+k_1I;z_1,w_1\right](G)_{k_1}^{-1}.\end{aligned}\tag{39}$$

We get the desired formula (34) for $n=t+1$. Hence, through induction, the relation (34) stands true for all values of n. A similar argument will establish the formula (35).

The recursion formulas for $\Gamma_1\left[(E;x),F,F'\pm nI;G;z_1,w_1\right]$ are obtained by replacing $F\leftrightarrow F'$ and $z_1\leftrightarrow w_1$ in Theorems 5 – 6, respectively.

Theorem 7. *Given the matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ so that $EF=FE$, $F'G=GF'$, and E, G are PS, then we have the following recursion relation:*

$$\begin{aligned}&\Gamma_1\left[(E;x),F,F';G-mI;z_1,w_1\right]\\&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad+z_1EF\left[\sum_{l=1}^{m}\Gamma_1\left[[E+I;x],F+I,F';G+(2-l)I;z_1,w_1\right]\right.\\&\quad\left.\times(G-lI)^{-1}(G-(l-1)I)^{-1}\right]\\&\quad+w_1E\left[\sum_{l=1}^{m}\Gamma_1\left[[E+I;x],F,F'+I;G+(2-l)I;z_1,w_1\right]\right.\\&\quad\left.\times(G-lI)^{-1}(G-(l-1)I)^{-1}\right]F'.\end{aligned}\tag{40}$$

Proof. Applying the integral formula (20) of Γ_1 and the following transformation:

$$(G-I)^{-1}_{n_1+n_2}=(G)^{-1}_{n_1+n_2}\left[I+n_1(G-I)^{-1}+n_2(G-I)^{-1}\right],$$

we obtain the contiguous matrix relation

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G-I;z_1,w_1\right]\\&=\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\\&\quad+z_1EF\left[\Gamma_1\left[[E+I;x],F+I,F';G+I;z_1,w_1\right](G-I)^{-1}(G)^{-1}\right]\\&\quad+w_1E\left[\Gamma_1\left[[E+I;x],F,F'+I;G+I;z_1,w_1\right](G-I)^{-1}(G)^{-1}\right]F'.\end{aligned}\tag{41}$$

Replacing G with $G-I$ in (41), we arrive at

$$\begin{aligned}&\Gamma_1\left[[E;x],F,F';G-2I;z_1,w_1\right]\\&=\Gamma_1\left[[E;x],F,F';z_1,w_1\right]\\&\quad+z_1EF\left[\sum_{l=1}^{2}\Gamma_1\left[[E+I;x],F+I,F';G+(2-l)I;z_1,w_1\right]\right.\\&\quad\left.\times(G-lI)^{-1}(G-(l-1)I)^{-1}\right]\\&\quad+w_1E\left[\sum_{l=1}^{2}\Gamma_1\left[[E+I;x],F,F'+I;G+(2-l)I;z_1,w_1\right]\right.\\&\quad\left.\times(G-lI)^{-1}(G-(l-1))^{-1}\right]F'.\end{aligned}\tag{42}$$

Repeating this relation s-times on $\Gamma_1\left[[E;x],F,F';G-mI;z_1,w_1\right]$, we get (40).

Theorem 8. *Given the matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ so that E and G are PS, then we have the following derivative formulas:*

$$\begin{aligned}&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]\\&=(E)_{k_1}\left[\Gamma_1\left[[E+k_1I;x],F,F'+k_1I;G+k_1I;z_1,w_1\right]\right](F')_{k_1}(G)^{-1}_{k_1},F'G=GF';\end{aligned}\tag{43}$$

$$\begin{aligned}&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]w_1^{F'+(k_1-1)I}\right]\\&=\left[\Gamma_1\left[[E;x],F,F'+k_1I;G;z_1,w_1\right]\right]w_1^{F'-I}(F')_{k_1},F'G=GF';\end{aligned}\tag{44}$$

$$\begin{aligned}&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1w_1,w_1\right]w_1^{G-I}\right]\\&=\left[\Gamma_1\left[[E;x],F,F';G-k_1I;z_1w_1,w_1\right]\right](-1)^{k_1}(I-G)_{k_1}w_1^{G-(k_1+1)I},\end{aligned}\tag{45}$$

where $D_{w_1}f=\frac{df}{dw_1}$ and $G-I$ is an invertible matrix for (45).

Proof. By differentiating (20) with respect to w, we get

$$\begin{aligned}&\frac{d}{dw_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]=E\Gamma^{-1}(E+I)\\&\times\left[\int_x^\infty e^{-t}t^{(E+I)-I}\Phi_2(E,E'+I;G+I;z_1t,w_1t)dt\right]F'G^{-1}.\end{aligned}\tag{46}$$

From the relations (20) and (46), we find that

$$\begin{aligned}&\frac{d}{dw_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]\\&=E\left[\Gamma_1\left[[E+I;x],F,F'+I;G+I;z_1,w_1\right]\right]F'G^{-1}.\end{aligned}\tag{47}$$

Hence, (43) is true for $k_1=1$. The significant formula comes by the principle of induction on k_1. Thus, we obtain (43). Formulas (44) and (45) can be established in a similar way.

Theorem 9. *For matrices E, F, F' and G in $\mathbb{C}^{s\times s}$ such that $F'G=GF'$ and E, G are PS, the following summation formula holds true:*

$$\begin{aligned}&\sum_{l=0}^{k_1}\binom{k_1}{l}(E)_lw_1^l\Gamma_1\left[[E+lI;x],F,F'+lI;G+lI;z_1,w_1\right](G)^{-1}_l\\&=\Gamma_1\left[[E;x],F,F'+lI;G;z_1,w_1\right].\end{aligned}\tag{48}$$

Proof. From definition of incomplete matrix function Γ_1 and the generalized Leibnitz formula for differentiation of a product of two functions, we have

$$\begin{aligned}&D^{k_1}_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]w_1^{F'+(k_1-1)I}\right]\\&=\sum_{l=0}^{k_1}\binom{k_1}{l}D^l_{w_1}\left[\Gamma_1\left[[E;x],F,F';G;z_1,w_1\right]\right]D^{k_1-l}_{w_1}\left[w_1^{F'+(k_1-1)I}\right]\\&=\sum_{l=0}^{k_1}\binom{k_1}{l}(E)_l\left[\Gamma_1\left[[E+lI;x],F,F'+lI;G+lI;z_1,w_1\right]\right]\\&\qquad(F')_l(G)^{-1}_lw_1^{F'+(l-1)I}.\end{aligned}\tag{49}$$

We used (43) and some simplification in the second equality. From (44) and (49), we get (48).

*Remark.*The first Appell hypergeometric matrix function F_1 will be obtained if we assume $x = 0$ in the IFAHMF Γ_1. Hence, taking $x = 0$, the obtained formulas for Γ_1 convert to the formulas for the Appell hypergeometric matrix function F_1.

3 Conclusion

In this paper, we studied the IFAHMFs Γ_1 and γ_1. We obtained some integral formula, recursion formula, differentiation formula and finite summation formula of the IFAHMFs Γ_1 and γ_1. The particular case of our results coincides with the results obtained in [4] when taking matrices from $\mathbb{C}^{1\times 1}$.

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