Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170312

Application of Effective Semi-Analytical Algorithms for Neutral and Retarded Volterra Integrodifferential Equations

Mohamed S. M. Bahgat[*] *and Mofida Zaki*

Mathematics Department, Faculty of Science, Minia University, Minia, Egypt

Received: 22 Dec. 2022, Revised: 22 Feb. 2023, Accepted: 23 Apr. 2023
Published online: 1 May 2023

Abstract: This study developed three semi-analytical algorithms to solve nonlinear delay Volterra and delay Fredholm-Volterra integrodifferential equations under initial conditions. These algorithms embrace Laplace Adomian decomposition algorithm (LADA), the modified Laplace Adomian decomposition algorithm (MLADA), and the Laplace variational iteration algorithm (LVIA). Using the suggested approaches, we find the solutions without discretization, or limiting traditions while considering suitable initial conditions. Moreover, solution terms are easily calculable and fast-converging series are generated. The proposed methodologies are tested numerically on three numerical applications to prove their efficacy and dependability as well as to compare their computational efficiency. Based on the numerical results, it is evident that the procedures offered are both effective and correct.

Keywords: Laplace Adomian decomposition algorithm, Integrodifferential equations, Error analysis, Functional differential equation, Laplace transform.

1 Introduction

Volterra delay integrodifferential equations have many applications in science and engineering, for example, their usage in electrodynamics, dynamical systems, mechanics, mathematics, viscoelasticity, oscillating magnetic fields, heat conduction, electromagnetic, biology, and other domains. Functional differential and integrodifferential equations with variable delays are extensively employed in modeling biological phenomena and play an important role in different fields [1, 2, 3, 4]. They also describe various reactor-related chemical and physics operations. Biological processes such as growth, birth, and death can also be described using neutral equations. Additionally, these equations have numerous medical applications. For example, they can be used to mimic sugar size in the blood, cancer chemotherapy, immunity, and epidemiology can all be used to show different characteristics of humans, see in [5]. Abd-Elhameed and Youssri [6] introduced and discussed the second kind Chebyshev quadrature collocation method for solving a mixed Volterra-Fredholm integral equation. The numerical solution of Volterra integrodifferential equations with type delay has received a lot of attention in the last few years, Refs. [7]-[16] and provides an overview of approximation strategies for solving these types of equations. The Laplace decomposition algorithm shows how the Laplace transform can be utilized to estimate the solutions of nonlinear integrodifferential equations by adjusting the decomposition approach. This method divides the equation under inquiry into linear and nonlinear parts and creates a solution in the form of a convergent series with easily computed terms that can be found defined by a recursive relationship and utilizing Adomian polynomials for nonlinear terms. Furthermore, LADA discovers the solution without any constricting assumptions, free from round-off errors and without taking a long time or a lot of computer memory. This method is powerful and effective, and it solves a wide range of linear and nonlinear ordinary and partial differential equations, as well as integral equations (see in Refs. [17]-[21]). Additionally, LADA is able to drastically diminish the numerical calculation while maintaining high accuracy in the numerical solution because it does not produce a sizeable set of linear or nonlinear equations, in contrast to other numerical

[*] Corresponding author e-mail: msmbahgat66@hotmail.com

methods. Though, solution series converge very rapidly in very narrow regions or near boundary points but they converge very slowly in wider and/or outer regions. This is one of the drawbacks of LADA. Furthermore, LADA has the disadvantage that his Adomian polynomials for the nonlinear terms need to be found and evaluated, which is computationally expensive due to the extensive computation required. MLADA [22, 23] is a new modified Laplace Adomian decomposition algorithm for nonlinear equations based on an appropriate initial solution selection. The modified Laplace decomposition approach overcomes the noise oscillation during the iteration procedure with this adjustment. The LVIA [24, 25, 26] is a novel adjustment of the variational iteration method (VIM). This approach was planned by combining the Laplace transform and VIM, and we can derive new variational iteration formulas. The main gain of LVIA is that it comes up with a new concept of Lagrange multipliers from the Laplace transform without tiresome calculations. A key feature of LVIA is its flexibility and ability to solve nonlinear equations accurately and without linearization and polynomials Adomian for nonlinear terms. Furthermore, LVIA is the freedom to choose the initial guess without unknown parameters. LVIA leads to a convergent solution, but the solution requires a lot of time and a lot of computer memory. The flexibility and customization offered by these proposed algorithms made them strong candidates for realistic solutions leading to closed-form exact solutions.

2 The proposed semi-analytical algorithms

This work studies the semi-analytical solution of delay Volterra integrodifferential equation (DVIDE), neutral Volterra delay integrodifferential equation (NVDIDE) and the nonlinear delay Fredholm-Volterra integrodifferential equation (NDF-VIDE). The general form of neutral Volterra delay integrodifferential equation (NVDIDE) is

$$\begin{aligned} v'(t) &= f\left(t,\, v(t),\, v(h(t)),\, \int_0^t K\big(t,\, x,\, v(x),\, v(h(x)),\, v'(h(x))\big)dx\right), t \geq a \\ v(t) &= \phi(t), \;\; t \in [a^*,\, a] \end{aligned} \tag{1}$$

while the general form of retarded Volterra integrodifferential equation is

$$\begin{aligned} v'(t) &= f\left(t,\, v(t),\, v(h(t)),\, \int_0^t K\big(t,\, x,\, v(x),\, v(h(x))\big)dx\right), \quad t \geq a \\ v(t) &= \phi(t), \;\; t \in [a^*,\, a] \end{aligned} \tag{2}$$

and the nonlinear retarded Fredholm Volterra integrodifferential is

$$\begin{aligned} v'(t) &= f\Bigg(t,\, v(t),\, v(h(t)),\, \int_0^t K_1\big(t,\, x,\, v(x),\, v(h(x))\big)dx,\, \int_a^b K_2\big(t,\, x,\, v(x),\, v(h(x))\big)dx\Bigg), t \geq a \\ v(t) &= \phi(t), \;\; t \in [a^*,\, a] \end{aligned}$$

where $f,\ h,\ K,\ K_1,\ K_2$ and ϕ are functions that are given. On the finite interval $[a,b]$, the solution $v(t)$ is desired. We will use the following format to rewrite equation (2):

$$\begin{aligned} v'(t) &= f(t,\, v(t),\, v(h(t)),\, z(t)), \quad a \leq t \leq b \\ z(t) &= \int_a^t K\big(t,\, x,\, v(x),\, v(h(x))\big)dx. \end{aligned} \tag{3}$$

It is assumed that the delay function $h(t)$ is continuous in the interval $[a,\ b]$ and that it satisfies the inequality $a^* \leq h(t) \leq t,\ t \in [a,\ b]$. Let $\phi \in C^r[a^*,\ a]$ and suppose that $f : [a,\ b] \times \mathbb{R}^3 \to \mathbb{R},\ (t,\ u,\ w,\ z) \to f(t,\ u,\ w,\ z)$ is defined and continuous together with its r^{th} derivatives, $r \in N$ in the domain $D : |t - t_0| < \sigma_1,\ D : |v - v_0| < \sigma_2$, satisfying

$$|f^{(q)}(t,\, u,\, w,\, z)| \leq S_1, \quad q = 1(1)r+1, \tag{4}$$

and the Lipschitz conditions

$$\begin{aligned} &\left|f^{(q)}(t,\, u_1,\, w_1,\, z_1) - f^{(q)}(t,\, u_2,\, w_2,\, z_2)\right| \leq \\ &L_1\left\{|u_1 - u_2| + |w_1 - w_2| + |z_1 - z_2|\right\}, \end{aligned} \tag{5}$$

then, $\exists\, \beta$ such that

$$\begin{aligned} |w_1 - w_2| \leq \beta\, |f^{(q)}&(t,\, u_1,\, w_1,\, z_1) \\ &- f^{(q)}(t,\, u_2, w_2,\, z_2)|, \end{aligned} \tag{6}$$

$$\forall (t,\, u,\, w,\, z)\,,\; (t,\, u_1,\, w_1,\, z_1)\,,\; (t,\, u_2,\, w_2,\, z_2) \in \left([a,\, b] \times \mathbb{R}^3\right).$$

Assume that $K : [a,\ b] \times [a,\ b] \times \mathbb{R}^2 \to \mathbb{R}$, is both defined and continuous function and that it satisfies

$$|K(t,\, x,\, u,\, w)| \leq S_2 \tag{7}$$

and the Lipschitz conditions

$$\begin{aligned} &|K(t,\, x,\, u_1,\, w_1) - K(t,\, x,\, u_2,\, w_2)| \leq \\ &\quad L_2\{|u_1 - u_2| + |w_1 - w_2|\} \end{aligned} \tag{8}$$

$\exists\, \beta$ such that

$$|w_1 - w_2| \leq \beta\, |K(t,\, x,\, u_1,\, w_1) - K(t,\, x,\, u_2,\, w_2)| \tag{9}$$

$$\forall (t,\, x,\, u,\, w)\,,\; (t,\, x, u_1,\, w_1)\,,\; (t,\, x,\, u_2,\, w_2) \in \left([a,\, b] \times [a,\, b] \times \mathbb{R}^2\right),$$

where $\beta < min\{\frac{1}{L_1},\ \frac{1}{L_2}\}$. These conditions guarantee the existence of a unique solution of problem (3).

In this paper, we employ three semi-analytical algorithms to solve the following functional integrodifferential equation with variable delays

$$v^{(n)}(\mathrm{t}) + \sum_{i=0}^{n-1}\left[N_{1,i}\left(v(t), v^{(i)}(h_i(t))\right) + R_{1,i}\left(v(t), v^{(i)}(h_i(t))\right)\right] = \mathrm{g}(\mathrm{t}) + \mathrm{I}\left[K, \sum_{i=0}^{n-1} R_{2,i}, \sum_{i=0}^{n-1} N_{2,i}\right], \quad (10)$$

with initial conditions which are given

$$v^{(k)}(0) = f_k(t), \qquad k = 0, 1, 2, \ldots\ldots, n-1, \quad (11)$$

where $v^{(n)}(t) = \frac{d^n v}{dt^n}$ and,

$$\mathrm{I}\left[K, \sum_{i=0}^{n-1} R_{2,i}, \sum_{i=0}^{n-1} N_{2,i}\right] = \int_0^t K(t, x)\left[\sum_{i=0}^{n-1} R_{2,i}\left(v(t), v^{(i)}(h_i(t))\right), \sum_{i=0}^{n-1} N_{2,i}\left(v(t), v^{(i)}(h_i(t))\right)\right], \quad (12)$$

$K(t, x)$ is the kernel of the integral equation, $R_{1,i}$, $R_{2,i}$, $N_{1,i}$ and $N_{2,i}$ are linear and nonlinear functions of $v(t)$ and $v^{(i)}(h_i(t))$ that will be determined, respectively. It also takes into account the first-order integrodifferential equation with variable delays and neutral terms, which has the form.

$$v'(t) + R_1(t)v'(t-\tau(t)) = R_0(t)v(t) + \int_a^t K(t, x)\, v(x)\, dx + g(t), \quad t \geq 0$$
$$v(t) = v_0, \quad (13)$$

where $R_0(t)$, $R_1(t)$, $v(t)$, $g(t)$ and $\tau(t)$ (delay term) are defined as continuous functions. The detailed definitions provided below, as found in [27-28], must be used for this study.

Definition 2.1. Suppose that $u(t)$ is real-valued function of the variable $t > 0$ and s is a real parameter. The Laplace transform of $u(t)$ is defined by

$$£_t[u(t)] = \int_0^\infty e^{-st} u(t)\, dt.$$

Theorem 2.2 . Suppose that $u(t)$, $u'(t)$, $\ldots$, $u^{(n-1)}(t)$ real-valued functions are continuous on $(0, \infty)$, then

$$£_t\left[u^{(n)}(t)\right] = s^n £_t[u(t)] - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - u^{(n-1)}(0).$$

Definition 2.3 . Let $v(t)$ be a continuous differentiable function on $(0, \infty)$ and let τ be a constant delay such that: $v(t) = \varphi(t)$ for $-\tau \leq t < 0$. Then the Laplace transform of delay function is given by :

$$£_t[v(t-\tau)] = e^{-s\tau}(£_t[v(t)] + \Phi(s, \tau)),$$

where $\Phi(s, \tau) = \int_{-\tau}^{0} e^{-st}\varphi(t)\, dt$.

Definition 2.4 Let $g(x)$ have a power series in a neighbourhood of $x = 0$. If polynomials $P(x)$ and $Q(x)$, of degrees m and n respectively, can be found such that $g(z) = \frac{P(x)}{Q(x)} + O(|x|^{m+n+1})$, with $Q(0) = 1$, then $\frac{P(x)}{Q(x)}$ is a Pade$'$ approximant to $g(x)$. When $m = n$, $\frac{P(x)}{Q(x)}$ is called a diagonal Pade$'$ approximant to $g(x)$. The following examples are given for an illustrative purpose, which we need in the Laplace transform,
if $m = n = 1$, Pade$'(e^s) = \frac{2+s}{2-s}$ and Pade$'(e^{-s}) = \frac{2-s}{2+s}$. If $m = n = 2$, Pade$'(e^s) = \frac{s^2+6s+12}{s^2-6s+12}$, and Pade$'(e^{-s}) = \frac{s^2-6s+12}{s^2+6s+12}$.

Corollary 2.5. For any iteration m, define $v_m(t)$, $m = 1, 2, \ldots, n$, as a series. The algorithms will reach the analytical solution of (1) shortly afterward. The following are only a few of the many types of errors:

1. Residual error $(Res^m(t))$ defined by

$$Res^m(t) = \left|\left(v'_m(t) - f\left(t, v_m(t), v_m(h(t)), \int_a^t K(t, x, v_m(x), v_m(h(x)), v'_m(h(x)))dx\right)\right)\right|$$

2. Exact error $(Ext^m(t))$ that is determined by

$$Ext^m(t) = \left|v_{exact}(t) - v_m(t)\right|.$$

3. Consecutive error $(Con^m(t))$, which is defined by

$$Con^m(t) = \left|v_{m+1}(t) - v_m(t)\right|.$$

We apply the our proposed algorithms to solve the problem (10)-(11) in the following subsections.

2.1 Formulation of LADA

In the present section we use LADA to solve functional integrodifferential equation with variable delays of the following form:

$$v^{(n)}(\mathrm{t}) + \sum_{i=0}^{n-1}\left[N_{1,i}\left(v(t), v^{(i)}(h_i(t))\right) + R_{1,i}\left(v(t), v^{(i)}(h_i(t))\right)\right] = \mathrm{g}(\mathrm{t}) + \mathrm{I}\left[K, \sum_{i=0}^{n-1} R_{2,i}, \sum_{i=0}^{n-1} N_{2,i}\right]. \quad (14)$$

Using the initial conditions that are stated

$$v^{(k)}(0)=f_k(t), \qquad k=0,\ 1,\ 2,\ \ldots\ldots.,\ n-1. \tag{15}$$

Following are the steps that should be taken to solve Eq. (14) using LADA:

***Step1*:** Eq. (14), when transformed by the Laplace transform, yields

$$\pounds_t\left[v^{(n)}(t)\right]+\sum_{i=0}^{n-1}\pounds_t\left[N_{1,i}\left(v(t),\ v^{(i)}(h_i(t))\right)+R_{1,i}\left(v(t),\ v^{(i)}(h_i(t))\right)\right]=\pounds_t\left[I\left[K,\ \sum_{i=0}^{n-1}R_{2,i},\ \sum_{i=0}^{n-1}N_{2,i}\right]\right]+\pounds_t\left[g(t)\right].$$

In view of definition 2.1 and the initial conditions (15), we obtain

$$s^n V(s)-\sum_{j=0}^{n-1}s^{n-j-1}v^{(j)}(0)+\sum_{i=0}^{n-1}\pounds_t\left[N_{1,i}\left(v(t),v^{(i)}(h_i(t))\right)+R_{1,i}\left(v(t),v^{(i)}(h_i(t))\right)\right]=G(s)+\pounds_t[I[K,\sum_{i=0}^{n-1}R_{2,i},\ \sum_{i=0}^{n-1}N_{2,i}]], \tag{16}$$

where $V(s)=\pounds_t[v(t)]$ and $G(s)=\pounds_t[g(t)]$.

Step2: LADA describes the solutions by the infinite series of components.

$$v(t)=\sum_{n=0}^{\infty}v_n(t), \tag{17}$$

and the nonlinear terms $N_{1,i}$ and $N_{2,i}$ in Eq. (16) is decomposed as follows:

$$\begin{aligned}N_{1,i}\left(v(t),\ v^{(i)}(h_i(t))\right)&=A_{n,i},\ i=0,\ 1,\ 2,\ldots,n-1,\\ N_{2,i}\left(v(t),\ v^{(i)}(h_i(t))\right)&=B_{n,i},\ \ i=0,\ 1,\ 2,\ldots,n-1,\end{aligned} \tag{18}$$

where $A_{i,n}$ and $B_{i,n}$ are the Adomian polynomials. The Adomian polynomials have the following general formula:

$$\begin{aligned}A_{n,i}&=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_{1,i}\left(\sum_{k=0}^{n}\lambda^k v_k(t),\ \sum_{k=0}^{n}\lambda^k v_k(h_i(t)),\sum_{k=0}^{n}\lambda^k v_k^{(i)}(h_i(t))\right)\right]\Bigg|_{\lambda=0},\\ B_{n,i}&=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_{2,i}\left(\sum_{k=0}^{n}\lambda^k v_k(t),\ \sum_{k=0}^{n}\lambda^k v_k(h_i(t)),\ \sum_{k=0}^{n}\lambda^k v_k^{(i)}(h_i(t))\right)\right]\Bigg|_{\lambda=0}.\end{aligned} \tag{19}$$

Step3: Eq.(16) is transformed into a collection of recursive equations produced by the decomposition analysis method.

$$\begin{aligned}\pounds_t[v_0(t)]&=\sum_{k=0}^{n-1}\frac{f_k(t)}{s^{k+1}}+\frac{1}{s^n}G(s),\\ \pounds_t[v_1(t)]&=\frac{1}{s^n}\pounds_t\left[\sum_{i=0}^{n-1}-(A_{0,i}(v_0)+R_{1,i}(v_0))+I[K,\ \sum_{i=0}^{n-1}R_{2,i}(v_0),\ \sum_{i=0}^{n-1}B_{0,i}(v_0)]\right],\ \cdots\end{aligned} \tag{20}$$

So in general, for $k\geq\ 0$, we obtain

$$\pounds_t[v_k(t)]=\frac{1}{s^n}\pounds_t\left[\sum_{i=0}^{n-1}-(A_{k-1,i}(v_0,\ v_1,\ \cdots,\ v_{k-1})+R_{1,i}(v_{k-1})+I\left[K,\ \sum_{i=0}^{n-1}R_{2,i}v_{k-1},\ \sum_{i=0}^{n-1}B_{k-1,i}(v_0,\ v_1,\ \cdots,\ v_{k-1})\right]\right]. \tag{21}$$

Step4: We can use the inverse Laplace transform to evaluate

$$\begin{aligned}v_0(t)&=\pounds_t^{-1}\left\{\sum_{k=0}^{n-1}\frac{f_k(t)}{s^{k+1}}+\frac{1}{s^n}G(s)\right\}=H(t),\\ v_1(t)&=\pounds_t^{-1}\left\{\frac{1}{s^n}\pounds_t\left[-A_{0,i}(v_0)-R_{1,i}(v_0)+I[K,R_{2,i}(v_0),B_{0,i}(v_0)]\right]\right\},i=0,1,2,\ \ldots,n-1\\ &\vdots\end{aligned} \tag{22}$$

So, the $n-$semi-analytical solution is given by:

$$\phi_n(t)=\sum_{i=0}^{n}v_i(t), \tag{23}$$

and the analytical solution will be $\lim_{n\to\infty}\phi_n(t)$, in some circumstances, the analytical solution in closed form can be obtained.

2.2 Formulation of MLADA

As we know, the LADA suggest that the zeroth component $v_0(t)$ usually defined by function $H(t)$ that arises from the source term $g(t)$ and the prescribed initial conditions. In comparison to the old methods, Wazwaz [20] developed a decomposition method that minimizes the step size in the calculation with good results. A noise oscillation occurs during iteration if Eq. (22) is selected as the initial solution. Based on the assumption that the function $H(t)$ can be divided into two parts, namely $H_1(t)$ and $H_2(t)$, one of which is assigned to the first term, the other to the second. Given the presumption we established,

$$H(t) = H_1(t) + H_2(t). \tag{24}$$

Our formulation of the repetition relations is modified according to Eq. (24). By taking a part of $H(t)$, we identify the zero component $v_0(t)$ and add the rest to $v_1(t)$. Hence, a set of recursive equations are given by

$$\begin{aligned} v_0(t) &= \pounds_t^{-1}\left\{\sum_{k=0}^{n-1}\frac{f_k(t)}{s^{k+1}} + \frac{1}{s^n}G_1(s)\right\},\\ v_1(t) &= \pounds_t^{-1}\left\{\frac{1}{s^n}G_2(s) - \frac{1}{s^n}\pounds_t\left[\sum_{i=0}^{n-1}(A_{0,i}(v_0)+R_{1,i}(v_0)) - I[K,\sum_{i=0}^{n-1}R_{2,i}(v_0),\sum_{i=0}^{n-1}B_{0,i}(v_0)]\right]\right\},\\ v_2(t) &= \pounds_t^{-1}\left\{-\frac{1}{s^n}\pounds_t\left[\sum_{i=0}^{n-1}(A_{1,i}(v_0,v_1)+R_{1,i}(v_1)) + I[K,\sum_{i=0}^{n-1}R_{2,i}(v_1),\sum_{i=0}^{n-1}B_{0,i}(v_0,v_{10})]\right]\right\},\\ &\vdots \end{aligned} \tag{25}$$

So, the $n-$semi-analytical solution is given by:

$$\psi_n(t) = \sum_{i=0}^{n} v_i(t), \tag{26}$$

and the analytical solution will be $\lim_{n\to\infty}\psi_n(t)$.

2.3 Formulation of LVIA

The procedure begins with the application of the Laplace transform to (14), followed by the use of the Laplace transform's differentiation feature to obtain

$$\begin{aligned} &[s^n\pounds_t[v_n(t)] - \sum_{k=0}^{n-1}s^{n-k-1}\frac{\partial^k v(t)}{\partial t^k}\Big|_{t=0}\\ &+\sum_{i=0}^{n-1}\pounds_t\left[N_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right) + R_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right)\right]\\ &= G(s) + \pounds_t[I[K,\sum_{i=0}^{n-1}R_{2,i},\sum_{i=0}^{n-1}N_{2,i}]]. \end{aligned} \tag{27}$$

The basic iterative approach employing the Lagrange multiplier can be advised using the iteration formula of (27), then

$$\begin{aligned} \pounds_t[v_{n+1}(t)] &= \pounds_t[v_n(t)] + \lambda(s)\Bigg[s^n\pounds_t[v_n(t)] - \sum_{k=0}^{n-1}s^{n-k-1}\frac{\partial^k v(t)}{\partial t^k}\Big|_{t=0}\\ &+\sum_{i=0}^{n-1}\pounds_t\left[N_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right) + R_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right)\right] - G(s)\\ &-\pounds_t\left[I\left[K,\sum_{i=0}^{n-1}R_{2,i},\sum_{i=0}^{n-1}N_{2,i}\right]\right]\Bigg]. \end{aligned} \tag{28}$$

Using $\pounds_t\left[N_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right) + R_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right)\right]$ and $\pounds_t[I(K,\, R_{2,i},\, N_{2,i})]$ as restricted terms, a Lagrange multiplier can be calculated as

$$\lambda(s) = \frac{-1}{s^n}, \tag{29}$$

with Eq. (29) and $\pounds_t^{-1}$, Eq. (28) becomes

$$\begin{aligned} v_{n+1}(t) = \pounds_t^{-1}\Bigg\{&\frac{1}{s^n}\sum_{k=0}^{n-1}s^{n-k-1}\frac{\partial^k v(t)}{\partial t^k}\Big|_{t=0}\\ &-\frac{1}{s^n}\sum_{i=0}^{n-1}\pounds_t[N_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right) + R_{1,i}\left(v(t),\, v^{(i)}(h_i(t))\right)]\\ &+\frac{1}{s^n}G(s) + \frac{1}{s^n}\pounds_t\left[I\left[K,\sum_{i=0}^{n-1}R_{2,i},\sum_{i=0}^{n-1}N_{2,i}\right]\right]\Bigg\}, \end{aligned} \tag{30}$$

where $v_0(t))$ is the initial iteration and can be calculated by

$$v_0(t) = \pounds_t^{-1}\left\{\sum_{k=0}^{n-1}\frac{f_k(t)}{s^{k+1}}\right\}. \tag{31}$$

From Eq. (30), Eq. (31) and Eq. (15), we get the solution

$$v_0(t) = \pounds_t^{-1}\left[\frac{1}{s^n}\sum_{k=0}^{n-1} s^{n-k-1}\frac{\partial^k v(t)}{\partial t^k}\Big|_{t=0}\right] = \pounds_t^{-1}\left\{\sum_{k=0}^{n-1}\frac{f_k(t)}{s^{k+1}}\right\},$$

$$v_1(t) = v_0(t) + \pounds_t^{-1}\left\{\frac{1}{s^n}G(s)\right\} - \pounds_t^{-1}\left\{\frac{1}{s^n}\sum_{i=0}^{n-1}\pounds_t\left[N_{1,i}\left(v_0(t),\, v_0^{(i)}(h_i(t))\right) + R_{1,i}\left(v_0(t),\, v_0^{(i)}(h_i(t))\right)\right]\right\} + \pounds_t^{-1}\left\{\frac{1}{s^n}\pounds_t\left[I\left[K,\, \sum_{i=0}^{n-1}R_{2,i},\, \sum_{i=0}^{n-1}N_{2,i}\right]\right]\right\}, \tag{32}$$

$$v_2(t) = v_0(t) + \pounds_t^{-1}\left\{\frac{1}{s^n}G(s)\right\} - \pounds_t^{-1}\left\{\frac{1}{s^n}\sum_{i=0}^{n-1}\pounds_t\left[N_{1,i}\left(v_1(t),\, v_1^{(i)}(h_i(t))\right) + R_{1,i}\left(v_1(t),\, v_1^{(i)}(h_i(t))\right)\right]\right\} + \pounds_t^{-1}\left\{\frac{1}{s^n}\pounds_t\left[I\left[K,\, \sum_{i=0}^{n-1}R_{2,i},\, \sum_{i=0}^{n-1}N_{2,i}\right]\right]\right\},$$

$$\vdots$$

The analytical solution is $v(t) = lim_{n\to\infty} v_n(t)$.

3 Numerical results and discussions

The applicability of LADA, MLADA and LVIA in solving neutral Volterra delay integrodifferential equations and Volterra delay integrodifferential equations of constant type are demonstrated in this section with numerical results for our suggested algorithms. Three specific examples are presented to demonstrate the validity of the results.

Example 4.1: Consider the following nonlinear delay Fredholm Volterra integrodifferential equation [7]

$$v'(t) = g(t) + \int_0^1 (t+x)\ v(x)\,dx + \int_0^t (t-x)\ v^2\left(\frac{x}{2}\right)\,dx. \tag{33}$$

The function $g(t)$ is chosen such that the analytical solution is $v(t) = t^2 - 2,\ t \geq 0$.

Firstly,The LADA will be used to solve this equation. The recursive relations, according to Eqs. (22), are as follows:

$$v_0(\mathrm{t}) = \pounds_t^{-1}\{\frac{1}{s}(v(0) + G(s))\}$$

$$v_{k+1}(\mathrm{t}) = \pounds_t^{-1}\left\{\frac{1}{s}\pounds_t\left[I_1[k_1,\, v_k(t)] + I_2[k_2,\, C_k(v_k(t))]\right]\right\},\quad k\geq 0 \tag{34}$$

where C_k denotes the Adomian polynomials that express the nonlinear terms $v^2(\frac{x}{2})$, k_1 and k_2 denote the kernels of the integrals I_1 and I_2, respectively, and I_1 and I_2 represent integral operators defined as $I_1 = \int_0^1 (t+x)\ v(x)\ dx$ and $I_2 = \int_0^t (t-x)\ \ v^2\left(\frac{x}{2}\right)dx$. By using Eq. (19), the Adomian polynomials for $v^2\left(\frac{x}{2}\right)$, begin with the first few terms.

$$C_0 = v_0^2\left(\frac{x}{2}\right),$$

$$C_1 = 2v_0\left(\frac{x}{2}\right)\ v_1\left(\frac{x}{2}\right), \tag{35}$$

$$C_2 = 2v_0\left(\frac{x}{2}\right)v_2\left(\frac{x}{2}\right) + v_1^2\left(\frac{x}{2}\right)$$

$$\vdots$$

While $G(s) = \pounds_t[g(t)]$
and $g(t) = \frac{3}{4} + \frac{11t}{3} - 2t^2 + \frac{t^4}{12} - \frac{t^6}{480}$.
The decomposition series' first few terms are given**:**

$$v_0(t) = -2 + \frac{3t}{4} + \frac{11t^2}{6} - \frac{2t^3}{3} + \frac{t^5}{60} - \frac{t^7}{3360},$$

$$v_1(t) = -\frac{12781t}{30240} - \frac{94979t^2}{161280} + \frac{2t^3}{3} - \frac{t^4}{16} - \frac{65t^5}{2304} + \frac{13t^6}{2304} + \frac{17t^7}{24192} - \frac{113t^8}{483840} + \frac{169t^9}{11612160} + \frac{157t^{10}}{232243200} - \frac{457t^{11}}{5109350400} - \frac{t^{12}}{619315200} + \frac{17t^{13}}{44281036800} - \frac{t^{15}}{1127153664000} + \frac{t^{17}}{754672730112000}, \tag{36}$$

$$\vdots$$

So, the $n-$ semi-analytical solution is given by:

$$\psi_n(t) = \sum_{i=0}^{n} v_i(t),$$

and the analytical solution will be

$$\lim_{n\to\infty}\psi_n(t).$$

Secondly, in order to solve Example 4.1 using the MLADA. Source function $g(t)$ is divided into two functions, which are $g_1(t)=\frac{3}{4}$ and $g_2(t)=\frac{11t}{3}-2t^2+\frac{t^4}{12}-\frac{t^6}{480}$. According to Eqs. (25), the iteration formulas will be

$$v_0(t)=\pounds_t^{-1}\left\{\frac{1}{s}\left(v(0)+G_1(s)\right)\right\},$$

$$v_1(t)=\pounds_t^{-1}\left\{\frac{1}{s}\left(G_2(s)\right)\right\}+\pounds_t^{-1}\left\{\frac{1}{s}\pounds_t\left[I_1\left[k_1,\ v_0(t)\right]+I_2[k_2,\ C_0\left(v_0(t)\right)]\right]\right\},$$

$$v_2(t)=\pounds_t^{-1}\left\{\frac{1}{s}\pounds_t\left[I_1\left[k_1,\ v_1(t)\right]+I_2[k_2,\ C_1\left(v_0(t),\ v_1(t)\right)]\right]\right\},\ldots$$

Hence,

$$v_0(t)=-2+\frac{3}{4}t,$$

$$v_1(t)=-\frac{3t}{4}+\frac{49t^2}{48}-\frac{t^4}{16}+\frac{3t^5}{1280}, \tag{37}$$

$$\vdots$$

The series form of the solution is then defined by

$$v(t)=\sum_{i=0}^{n}v_i(t),$$

$$v(t)=-2+\frac{49t^2}{48}-\frac{t^4}{16}+\frac{3t^5}{1280}+\cdots \tag{38}$$

Third, we will obtain LVIA solution for Eq. (33). The recursive relation is given by

$$v_0(t)=\pounds_t^{-1}\left\{\frac{1}{s}\left(v(0)\right)\right\},$$

$$v_{k+1}(t)=v_0(t)+\pounds_t^{-1}\left\{\frac{1}{s}\left(G(s)\right)\right\}+\pounds_t^{-1}\left\{\frac{1}{s}\pounds_t\left[I_1\left[k_1,\ v_k(t)\right]+I_2[k_2,\ v_k\left(\frac{t}{2}\right)]\right]\right\}$$

$$v_0(t)=-2,$$

$$v_1(t)=-2-\frac{t}{4}+\frac{5t^2}{6}+\frac{t^5}{60}-\frac{t^7}{3360}, \tag{39}$$

$$\vdots$$

Hence, the analytical solution is

$$v(t)=\lim_{n\to\infty}v_n(t).$$

Numerical results of Example 4.1

We compare our solutions obtained by LADA, MLADA and LVIA with the analytical solution for various n values in Figure 1(a, b, c), this graph illustrates how the suggested algorithms are more straightforward and more accurate. Figures 2-4 (a, b, c) view the graphics of the absolute errors $\text{Ext}^n_{\text{approx}}(t)=\left|v_{\text{Exact}}-v_{approx\ n}\right|$, the consecutive errors $\text{Con}^n(t)=\left|v_{n+1}(t)-v_n(t)\right|$ and the residual errors

$$\text{Res}^n(t)=\left(\left|v'_n(t)-f\left(t,\ v_n(t),\ v_n\left(h(t)\right),\ I\left[k,\ v_n\right]\right)\right)$$

for three analytical methods (LADA, MLADA and LVIA) at different values of $n\,(n=3,\ 4,\ 5)$ respectively. From these figures, it is obviously that semi-analytical solution obtained by LADA, MLADA and LVIA converges rapidly to analytical solution and MLADA's semi-analytical solution is closer to the analytical solution than LADA and LVIA's semi-analytical solutions.

<u>**Example 4.2:**</u> Consider the following linear of second-order Volterra integrodifferential equation including variable delay [15]

$$v''(t)+v\left(\frac{t}{2}\right)-\frac{3}{4}v(t)-\int_0^t xv(x)\,dx=-\frac{11}{4}\sin(t)+t\cos(t)+\sin\left(\frac{t}{2}\right). \quad 0\le t\le 1 \tag{40}$$

With initial conditions $v(0)=0,\ v'(0)=1,$ and the analytical solution is $v(t)=\sin(t)$.

LADA solution for Example 4.2

Firstly, Both sides of Example 4.2 are transformed using the Laplace transform.

$$s^2\pounds_t\left[v(t)\right]-sv(0)-v'(0)+\pounds_t\left[v\left(\frac{t}{2}\right)\right]-\frac{3}{4}\pounds_t\left[v(t)\right]-\pounds_t[I\left[K,\ v(t)\right]]=G(s). \tag{41}$$

Where $K(t,\ x)=x$, $g(t)=t\cos(t)-\frac{11}{4}\sin(t)+\sin\left(\frac{t}{2}\right)$, $G(s)=\pounds_t[g(t)]$ and $I\left[K,\ v(t)\right]=\int_0^t xv(x)\,dx$.
By using initial condition's Example 4.2

$$\pounds_t\left[v(t)\right]=\frac{1}{s^2-\frac{3}{4}}\left(1+G(s)+\pounds_t\left[v\left(\frac{t}{2}\right)\right]+\pounds_t\left[I\left[K,\ v(t-1)\right]\right]\right) \tag{42}$$

Second, Infinite series of components are used by LADM to define the solutions,

$$v(t)=\sum_{n=0}^{\infty}v_n(t).$$

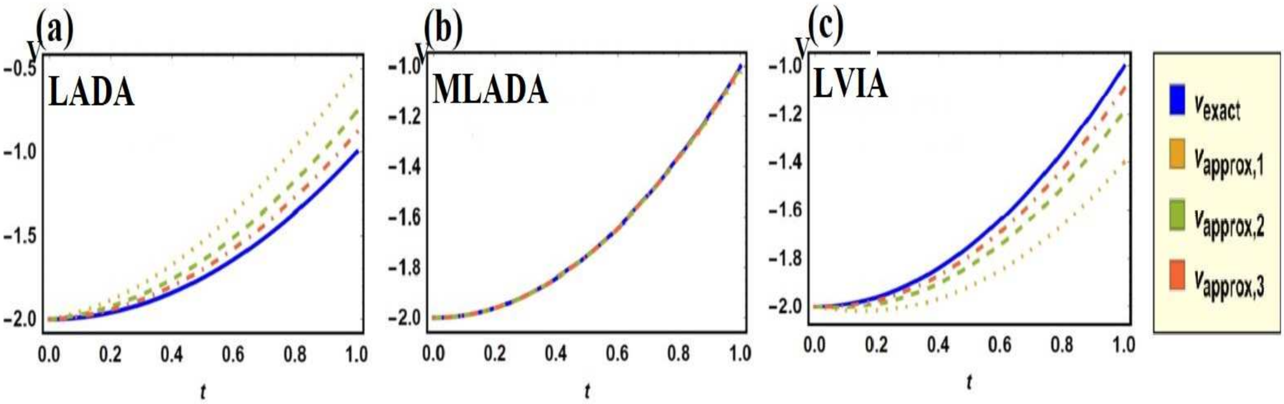

Figure 1. The behaviours of $v_{approx,\ n}(t)$ while employing (a) LADA (b) MLADA (c) LVIA for various values of $n = 1,\ 2,\ 3$ of terms when, $0 \leq t \leq 1$, with the analytical solution for Example 4.1.

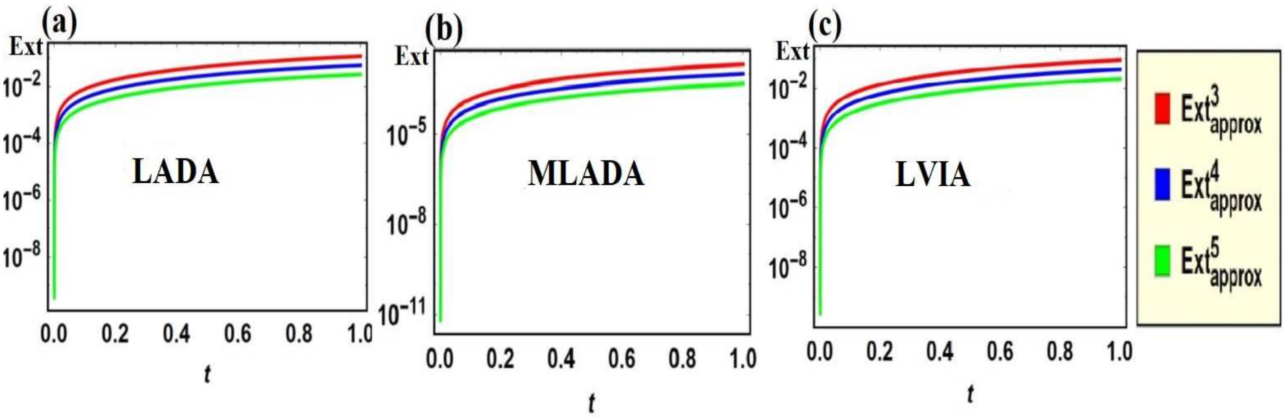

Figure 2. Graphics of the absolute errors $Ext^n_{approx}(t)$ for (a) LADA (b) MLADA (c) LVIA at different values of $n (n = 3,\ 4,\ 5)$ when $0 \leq t \leq 1$, for Example 4.1.

The decomposition analysis approach is applied to (41) to create a set of recursive equations that are provided by

$$\begin{aligned} v_0(\mathrm{t}) &= \pounds_t^{-1}\{\frac{1}{s^2-\frac{3}{4}}(1+G(s))\}, \\ v_{k+1}(\mathrm{t}) &= \pounds_t^{-1}\left\{\frac{1}{s^2-\frac{3}{4}}\left(\pounds_t\left[v_k\left(\frac{t}{2}\right)\right]\right.\right. \\ &\quad \left.\left.+\pounds_t\left[I\left[K,\ v_k(t)\right]\right]\right)\right\}, \\ k &\geq 0 \end{aligned} \tag{43}$$

Hence,

$$\begin{aligned} v_0(t) &= \frac{1}{98}(5\sqrt{3}\,e^{-\frac{\sqrt{3}t}{2}} - 5\sqrt{3}\,e^{\frac{\sqrt{3}t}{2}} \\ &\quad - 56t\cos(t) - 98\sin(\frac{t}{2}) + 218\sin(t), \end{aligned}$$

$$\begin{aligned} v_1(t) &= \frac{1}{561834}[37981\sqrt{3}\,e^{\frac{\sqrt{3}t}{2}} - 37981\sqrt{3}\,e^{-\frac{\sqrt{3}t}{2}} \\ &\quad + 50960\sqrt{3}\,e^{-\frac{\sqrt{3}t}{4}} - 50960\sqrt{3}\,e^{\frac{\sqrt{3}t}{4}} \\ &\quad + 57330\,e^{-\frac{\sqrt{3}t}{2}}t + 57330\,e^{\frac{\sqrt{3}t}{2}}t \\ &\quad + 9555\sqrt{3}\,e^{-\frac{\sqrt{3}t}{2}}t^2 - 1284192t\cos(\frac{1}{2}t) \\ &\quad + 1500408t\cos(t) - 691488\sin(\frac{1}{4}t) \\ &\quad - 9555\sqrt{3}\,e^{\frac{\sqrt{3}t}{2}}t^2 + 4781322\sin(\frac{1}{2}t) \\ &\quad - 2586168\sin(t) + 183456t^2\sin(t), \end{aligned}$$

$$\vdots$$

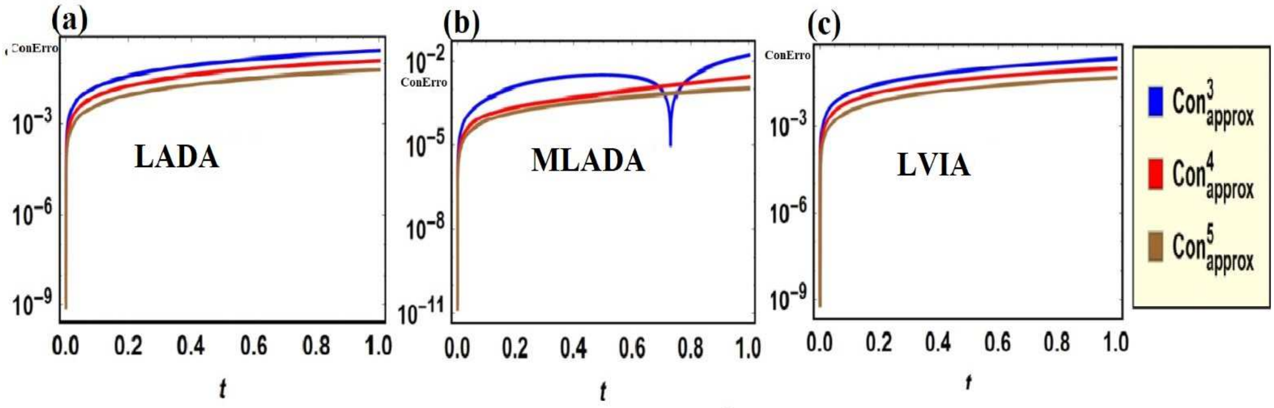

Figure 3. Graphics of the consecutive errors $Con^n(t)$for several values of $n(n = 3,\ 4,\ 5)$ for (a) LADA (b) MLADA (c) LVIA when $0 \le t \le 1$, for Example 4.1.

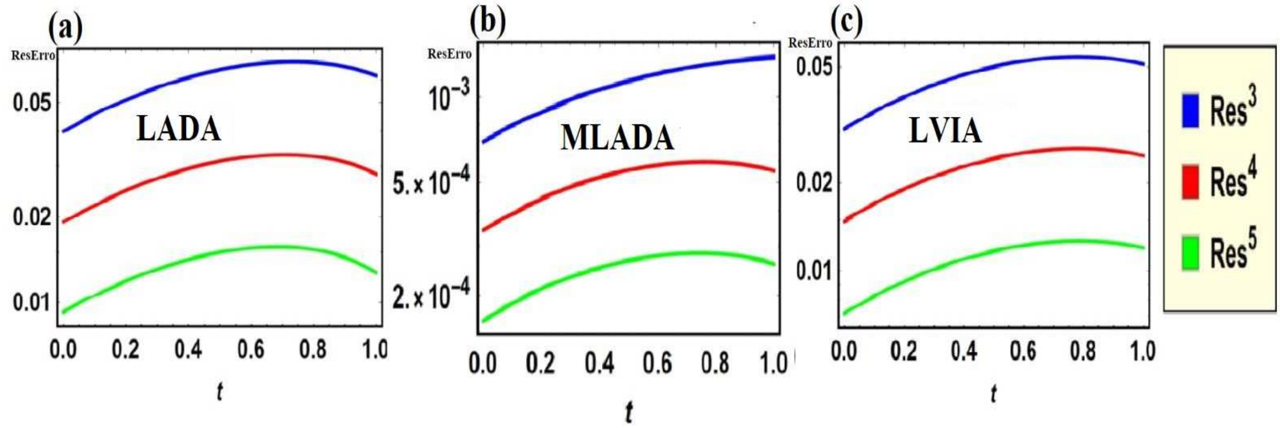

Figure 4. Graphics of the residual errors $Res^n(t) = \left(\left|v_n'(t) - f(t,\ v_n(t),\ v_n(h(t)),\ I[K,\ v_n])\right|\right)$ for (a) LADA (b) MLADA (c) LVIA at different values of $n(n = 3,\ 4,\ 5)$ when $0 \le t \le 1$, for Example 4.1.

As a result, the series solution is as follows:

$$v(\mathrm{t}) = \sum_{n=0}^{\infty} v_n(\mathrm{t}). \tag{44}$$

MLADA solution for Example 4.2

We divided source function $g(t) = -\frac{11}{4}\sin(t) + t\cos(t) + \sin\left(\frac{t}{2}\right)$ into two functions are $\mathrm{g}_1(\mathrm{t}) = -\frac{11}{4}\sin(t)$ and $\mathrm{g}_2(\mathrm{t}) = t\cos(t) + \sin\left(\frac{t}{2}\right)$
By applying MLADA, we get

$$\begin{aligned} v_0(t) = v_0(\mathrm{t}) &= \pounds_t^{-1}\{\frac{1}{s^2 - \frac{3}{4}}(1 + G_1(s))\} \\ &= \frac{1}{21}(-4\sqrt{3}\,e^{-\frac{\sqrt{3}t}{2}}(-1 + e^{\sqrt{3}t}) + 33\sin(t)), \end{aligned} \tag{45}$$

$$\begin{aligned} v_1(t) &= \pounds_t^{-1}\left\{\frac{1}{s^2 - \frac{3}{4}}(G_2(s)\right. \\ &\quad \left. + \pounds_t\left[v_0\left(\frac{t}{2}\right)\right] + \pounds_t\left[I\left[K,\ v_0(t)\right]\right]\right)\Bigg\} \\ &= \frac{1}{9261}2\,e^{-\frac{\sqrt{3}t}{2}}[-625\sqrt{3} + 1568\sqrt{3}\,e^{\frac{\sqrt{3}t}{4}} \\ &\quad - 1568\sqrt{3}\,e^{\frac{3\sqrt{3}t}{4}} + 625\sqrt{3}\,e^{\sqrt{3}t} + 1764t \\ &\quad + 1764\,e^{\sqrt{3}t}t + 294\sqrt{3}t^2 - 294\sqrt{3}\,e^{\sqrt{3}t}t^2 \\ &\quad + 1512e^{\frac{\sqrt{3}t}{2}}t\cos(t) + 2646e^{\frac{\sqrt{3}t}{2}}\sin(\frac{t}{2}) \\ &\quad - 5886e^{\frac{\sqrt{3}t}{2}}\sin(t)], \end{aligned} \tag{46}$$

$$\vdots$$

As a result, the series solution is as follows:

$$v(\mathrm{t}) = \sum_{n=0}^{\infty} v_n(\mathrm{t}).$$

It turns out that MLADA yields the analytical solution from first iteration.

LVIA solution for Example 4.2

The first iterations of LVIA solution can be determined by applying the initial conditions in Eqs. (32), which are:

$$v_0(t) = \pounds_t^{-1}\left\{\frac{1}{s^2-\frac{3}{4}}\left(-sv(0)-v'(0)\right)\right\} = \frac{e^{-\frac{\sqrt{3}t}{2}}(-1+e^{\sqrt{3}t})}{\sqrt{3}},$$

$$v_1(t) = v_0(t)+\pounds_t^{-1}\left\{\frac{1}{s^2-\frac{3}{4}}\left(G(s)+\pounds_t\left[v_0\left(\frac{t}{2}\right)\right]\right.\right.$$
$$\left.\left.+\pounds_t\left[I\left[K,\ v_0(t)\right]\right]\right)\right\} = \frac{1}{2646}\,e^{-\frac{\sqrt{3}t}{2}}$$
$$\times\,(-257\sqrt{3}-1568\sqrt{3}\,e^{\frac{\sqrt{3}t}{4}}+1568\sqrt{3}\,e^{\frac{3\sqrt{3}t}{4}}$$
$$+\,257\sqrt{3}\,e^{\sqrt{3}t}-1764t-1764\,e^{\sqrt{3}t}t-294\sqrt{3}t^2$$
$$+\,294\sqrt{3}\,e^{\sqrt{3}t}t^2-1512\,e^{\frac{\sqrt{3}t}{2}}t\cos(t)$$
$$-\,2646\,e^{\frac{\sqrt{3}t}{2}}\sin(\frac{t}{2})+5886\,e^{\frac{\sqrt{3}t}{2}}\sin(t)), \qquad (47)$$

$$\vdots$$

Hence, the analytical solution is

$$v(t) = \lim_{n\to\infty} v_n(t)\ .$$

Numerical results of Example 4.2

Graphics of the analytical solution and the semi-analytical solutions for $n = 1,\ 2,\ 3$, which obtained by three analytical methods (LADA, MLADA and LVIA) at $0 \le t \le 1$, are specified in Figure 5(a, b, c), respectively. This improvement in the accuracy of the semi-analytical solutions can also be understood visually from this figure. Values of the absolute errors, the consecutive errors and the residual error for $n = 4$ at several points are given in Table 1.

<u>Example 4.3:</u> Consider the first order neutral Volterra delay integrodifferential equation (NVDIDE) [16]

$$v'(t-1) = 1-\frac{t^3}{6}+\int_0^t (t-x)\,v(x)\,dx, \quad 0\le t\le 1 \qquad (48)$$

with initial conditions

$$v(0) = 0, \qquad (49)$$

and the analytical solution is $v(t) = t$.

LADA solution for Example 4.3

We use the algorithms solution of using LADA for solving Example 4.3

Firstly, we apply Laplace transform to both sides of Eq.(48), and use definition 2.3, we get

$$e^{-s}\left(sV(s)-v(0)+\overline{\Phi}(s)\right) = G(s)+\pounds_t\left[I\left[K,\ v(t)\right]\right]. \qquad (50)$$

Where $K(x,\ t) = t-x,\ V(s) = \pounds_t[v(t)],\ g(t) = 1-\frac{t^3}{6}$, $G(s) = \pounds_t[g(t)],\ I[K,\ v(t)] = \int_0^t (t-x)\,v(x)\,d$ and $\overline{\Phi}(s) = \int_{-1}^0 e^{-sy}\varphi'(y)\,dy$.
By using initial condition's Example 4.3

$$e^{-s}\left(sV(s)+\frac{e^s}{s}-\frac{1}{s}\right) = G(s)+\pounds_t\left[I\left[K,\ v(t)\right]\right], \qquad (51)$$

$$V(s) = \frac{1}{s^2}-\frac{e^s}{s^2}+\frac{e^s}{s}G(s)+\frac{e^s}{s}\pounds_t\left[I\left[K,\ v(t)\right]\right].$$

By using Padé approximation of exponential function as $e^s = \frac{1+\frac{s}{2}}{1-\frac{s}{2}}$

$$V(s) = \frac{1}{s^2}-\frac{2+s}{s^2(2-s)}+\frac{2+s}{s(2-s)}G(s)$$
$$+\frac{2+s}{s(2-s)}\pounds_t\left[I\left[K,\ v(t)\right]\right]. \qquad (52)$$

Second, Infinite series of components are used by LADA to define the solutions,

$$v(\mathrm{t}) = \sum_{n=0}^{\infty} v_n(\mathrm{t}).$$

The decomposition analysis approach is applied to (52) to create a set of recursive equations that are provided by

$$v_0(\mathrm{t}) = \pounds_t^{-1}\left\{\frac{1}{s^2}-\frac{2+s}{s^2(2-s)}+\frac{2+s}{s(2-s)}G(s)\right\},$$
$$v_{k+1}(\mathrm{t}) = \pounds_t^{-1}\left\{\frac{2+s}{s(2-s)}\pounds_t\left[I\left[K,\ v(t)\right]\right]\right\}.\ k\ge 0 \qquad (53)$$

Hence,

$$v_0(t) = t+\frac{1}{24}(-3+3e^{2t}-6t-6t^2-4t^3-t^4),$$

$$v_1(t) = -\frac{7}{32}+\frac{1}{32}e^{2t}(7-2t)+\frac{5t}{8}$$
$$-\frac{t^2(1575+t(840+t(315+t(84+t(14+t)))))}{5040},$$

$$\vdots$$

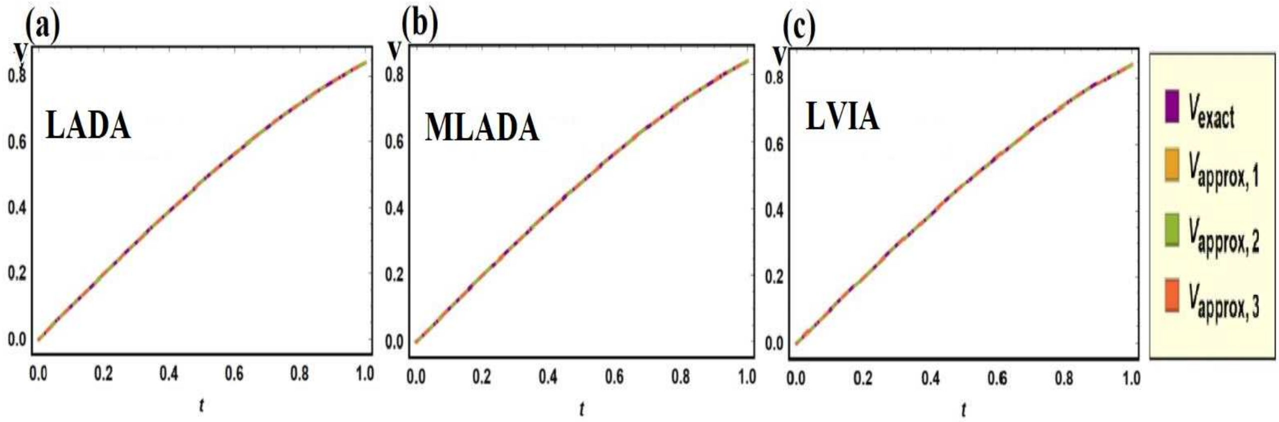

Figure 5. The behaviours of $v_{approx,\ n}(t)$ obtained by (a) LADA, (b) MLADA, (c) LVIA for various values of "n" of the components of the solution series when $0 \leq t \leq 1$, for Example 4.2.

Table 1: Error analysis for Example 4.2

t	Ext^k_{method}			Con^k_{method}			Res^k_{method}		
	Ext^4_{LADA}	Ext^4_{MLADA}	Ext^4_{LVIA}	Con^4_{LADA}	Con^4_{MLADA}	Con^4_{LVIA}	Res^4_{LADA}	Res^4_{MLADA}	Res^4_{LVIA}
0.0	**1.4 E-13**	**2.5 E-13**	9.2 **E-13**	**1.4 E-13**	**2.5 E-13**	**9.1 E-13**	**2.5 E-13**	**2.7 E-13**	**3.3 E-12**
0.1	**2.3 E-12**	**1.1 E-12**	2.6 **E-12**	**2.3 E-12**	**1.0 E-12**	**2.6 E-12**	**9.6 E-12**	**4.5 E-11**	**7.8 E-11**
0.2	**2.7 E-12**	**8.5 E-13**	1.3 **E-12**	**2.7 E-12**	**8.5 E-13**	**1.2 E-12**	**7.5 E-12**	**1.2 E-11**	**2.6 E-11**
0.3	**9.4 E-12**	**5.9 E-13**	9.8 **E-13**	**9.5 E-12**	**5.7 E-13**	**1.0 E-12**	**1.2 E-10**	**2.3 E-11**	**3.4 E-11**
0.4	**3.4 E-12**	**1.2 E-12**	8.2 **E-13**	**3.4 E-12**	**1.2 E-12**	**8.0 E-13**	**4.2 E-10**	**4.2 E-11**	**7.5 E-11**
0.5	**4.5 E-12**	**1.2 E-12**	3.2 **E-12**	**4.4 E-12**	**1.3 E-12**	**3.3 E-12**	**2.1 E-10**	**3.3 E-11**	**6.5 E-11**
0.6	**7.1 E-12**	**3.2 E-12**	3.2 **E-12**	**6.9 E-12**	**4.4 E-12**	**4.2 E-12**	**3.6 E-10**	**8.9 E-12**	**1.7 E-11**
0.7	**7.7 E-12**	**6.1 E-13**	1.6 **E-12**	**6.8 E-12**	**2.6 E-12**	**5.0 E-12**	**2.9 E-11**	**1.1 E-12**	**1.7 E-13**
0.8	**8.0 E-12**	**2.0 E-12**	2.4 **E-12**	**8.1 E-12**	**2.4 E-12**	**3.1 E-12**	**8.5 E-11**	**4.5 E-11**	**8.1 E-11**
0.9	**1.2 E-11**	**8.0 E-13**	1.7 **E-12**	**5.3 E-12**	**3.4 E-11**	**5.9 E-11**	**1.1 E-10**	**8.7 E-11**	**1.4 E-10**
1	**2.9 E-11**	**1.7 E-12**	2.6 **E-12**	**1.5E-10**	**2.3 E-10**	**4.0 E-10**	**3.9 E-10**	**9.3 E-11**	**1.6 E-10**

As a result, the series solution is as follows:

$$v\,(\mathrm{t}) = \sum_{n=0}^{\infty} v_n\,(\mathrm{t}). \tag{54}$$

MLADA solution for Example 4.3

We divided source function $g\,(t) = 1 - \frac{t^3}{6}$ into two functions are $\mathrm{g}_1\,(\mathrm{t}) = 1$ and $\mathrm{g}_2\,(\mathrm{t}) = -\frac{t^3}{6}$. By applying MLADA, we get

$$v_0\,(t) = \pounds_t^{-1}\left\{\frac{1}{s^2} - \frac{2+s}{s^2\,(2-s)} + \frac{2+s}{s\,(2-s)}G_1\,(s)\right\}\} = t,$$

$$v_1\,(t) = v_2\,(t) = v_3\,(t) = \cdots = 0. \tag{55}$$

Hence, the solution in series form is provided by

$$v\,(\mathrm{t}) = \sum_{n=0}^{\infty} v_n\,(\mathrm{t}) = t.$$

It turns out that MLADA yields the analytical solution from first iteration.

LVIA solution for Example 4.3

The first iterations of LVIA solution can be determined by applying the initial conditions in Eqs. (32), which are:

$$v_0\,(t) = \pounds_t^{-1}\left\{\frac{2+s}{s\,(2-s)}\,(v\,(0))\right\} = 0,$$

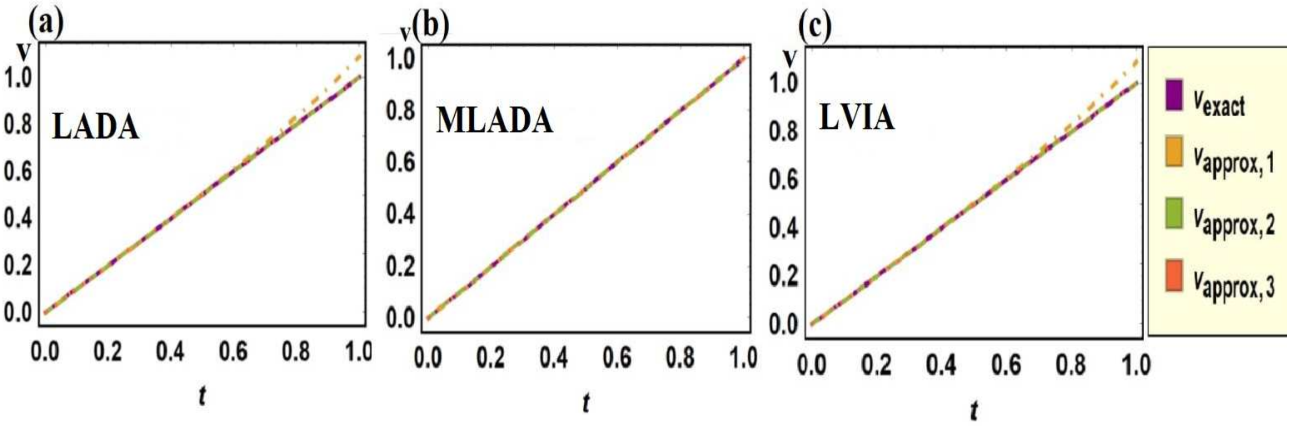

Figure 6. The behaviours **of** $v_{approx,\,n}(t)$ **by using LADA, MLADA and LVIA for various values of "n "of components (a)** n=1, **(b)** n=2 **and** (c) n=3 **when** , 0≤t≤1, **with the analytical solution for Example 4.3.**

Table 2: Comparison of the absolute errors of Example 4.3

t	Ext^{k}_{method}								
	Ext^{1}_{LADA}	Ext^{1}_{MLADA}	Ext^{1}_{LVIA}	Ext^{3}_{LADA}	Ext^{3}_{MLADA}	Ext^{3}_{LVIA}	Ext^{6}_{LADA}	Ext^{6}_{MLADA}	Ext^{6}_{LVIA}
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	3.1 E-9	0.0	7.8E-5	0.0	0.0	3.5 E-14	1.9 E-16	0.0	3.1 E-16
0.4	4.8 E-7	0.0	1.5 E-3	1.7 E-15	0.0	4.4 E-11	6.7 E-16	0.0	8.3 E-16
0.6	9.9 E-6	0.0	8.6 E-3	4.1 E-13	0.0	3.2 E-9	6.7 E-16	0.0	5.6 E-16
0.8	9.0 E-5	0.0	3.2 E-2	2.2 E-11	0.0	6.9 E-8	9.9 E-16	0.0	9.9 E-16
1	5.1 E-4	0.0	9.0 E-2	7.9 E-10	0.0	7.9 E-7	1.3 E-15	0.0	1.8 E-15
t	Con^{k}_{method}								
	Con^{1}_{LADA}	Con^{1}_{MLADA}	Con^{1}_{LVIA}	Con^{3}_{LADA}	Con^{3}_{MLADA}	Con^{3}_{LVIA}	Con^{5}_{LADA}	Con^{5}_{MLADA}	Con^{5}_{LVIA}
0.0	0.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.
0.2	3.1 E-9	0.0	7.8 E-5	6.1 E-17	0.0	3.5 E-14	9.9 E-17	0.0	1.7 E-16
0.4	4.8 E-7	0.0	1.5E-3	1.8 E-15	0.0	4.4E-11	5.3 E-16	0.0	5.8 E-16
0.6	9. E-6	0.0	8.7 E-3	4.1 E-13	0.0	3. 2 E-9	4.2 E-16	0.0	3.8 E-16
0.8	9.0 E-5	0.0	3.2 E-2	2.2 E-11	0.0	6.9 E-8	6.6 E-16	0.0	3.9E-15
1	5.2 E-4	0.0	9.1 E-2	4.9 E-10	0.0	7.9E-7	2.4 E-15	0.0	1.5 E-13
t	Res^{k}_{mothod}								
	Res^{1}_{LADA}	Res^{1}_{MLADA}	Res^{1}_{LVIA}	Res^{3}_{LADA}	Res^{3}_{MLADA}	Res^{3}_{LVIA}	Res^{6}_{LADA}	Res^{6}_{MLADA}	Res^{6}_{LVIA}
0.0	3.9 E-4	0.0	4.9 E-2	5.7 E-10	0.0	7.6 E-7	2.5 E-13	0.0	5.6 E-16
0.2	1.4 E-4	0.0	3.4 E-2	5.0 E-11	0.0	1.3 E-7	7.5 E-12	0.0	6.7 E-16
0.4	3.1 E-5	0.0	1.9 E-2	9.9 E-10	0.0	1.3 E-8	4.2 E-10	0.0	1.7 E-16
0.6	3.5 E-6	0.0	7.0 E-3	4.5 E-8	0.0	4.5 E-10	3.6 E-10	0.0	1.1 E-16
0.8	6.2 E-7	0.0	1.7 E-3	6.9 E-7	0.0	2.9 E-10	8.5 E-11	0.0	1.3 E-15
1	6.0 E-6	0.0	2.4 E-3	6.0 E-6	0.0	5.1 E-9	3.9 E-10	0.0	2.2 E-16

$$v_1(t) = v_0(t) + \pounds_t^{-1}\left\{ \frac{2+s}{s(2-s)} \left(G(s) + \frac{2+s}{s(2-s)} \pounds_t \left[I\left[K,\ v_0(t)\right]\right] \right) \right\}$$
$$= \frac{1}{24}\left(-3+3e^{2t}+18t-6t^2-4t^3-t^4\right) \quad (56)$$
$$\vdots$$

Hence, the analytical solution is

$$v(t) = \lim_{n\to\infty} v_n(t) \ .$$

Numerical results of Example 4.3

We construct our solutions with the analytical solution for several values of n in Figure 6(a, b, c). The exact error $(Ext^n(t))$, the Consecutive error $(Con^n(t))$ and the Residual error $(Res^n(t))$, for the three proposed methods at several values of t and n in Tables $2-4$. It is observed that MLADA solution coincides with the analytical solution while LADA Solution and LVIA solution convergence to the analytical solution. Table 2 lists the exact error $(Ext^n(t))$, for the three proposed methods at different values of t and. Table 3 lists the Consecutive error $(Con^n(t))$, for the three proposed methods. While Table 4 lists the Residual error $(Res^n(t))$, for the three proposed methods. The tabulated data demonstrate that the three suggested algorithms' solutions converge to the analytical solution obtained by applying a few terms of $v_{approx,\ n}(t)$. MLADA solution converges to the analytical solution faster than LADA and LVIA solutions, as seen in the comparison. For distinct values $n = 1,2,3$ of the series solution terms, Figure 6(a, b, c) shows the curves of the analytical solution v_{exact} and the semi-analytical solution $v_{approx,n}$ derived by LADA, MLADA, and LVIA. The recommended procedures are shown in this figure to be effective, simple and accurate.

4 Conclusion

Our main objective was to construct semi-analytical solutions to functional integrodifferential equations involving variable delays. The three algorithms we used to accomplish this aim were LADA, MLADA, and LVIMA. These strategies provide semi-analytical solutions for the problems that are close to adequate when compared with analytical solutions. The visionary findings' accuracy and applicability are shown in the tables and figures, which prove that the procedures were done correctly. Simple coding, apparent calculations and algorithm are among the key advantages of these algorithms. Furthermore, the error analysis comprises three categories of errors: exact consecutive, and residual corrections, which are explained and implemented in examples. These numerical studies and methodologies can be developed in the future to solve a system of functional Volterra integrodifferential equations with variable delays.

Acknowledgements

The authors would like to thank the reviewers for their suggestions and constructive comments to improve the paper.

References

[1] V.V. Vlasov and R. Perez Ortiz, Spectral Analysis of Integro-Differential Equations in Viscoelasticity and Thermal Physics, Math. Notes, **98(4)**, 689-693 (2015).
[2] M. Dehghan and F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic field using Hes homotopy perturbation method, Prog. Electromagnet. Res. PIER, **78**, 361-376 (2008).
[3] Y. Chu, F. You, J.M. Wassick and A. Integrated planning and scheduling under production uncertainties: Bi-level model formulation and hybrid solution method,Comput. Chem. Eng., **72**, 255-272 (2015).
[4] M.R. Ramadan and M.R. Ali, Hybrid Orthonormal Bernstein and Block-Pulse Functions for solving VolterraFredholm integral equations, Asian Res. J. Math., **4(4)**, 1-14 (2017).
[5] G.A. Bocharov, F.A. Rihan, Numerical modelling in biosciences with delay differential equations, J. Comput. Appl. Math., **125**, 183-199 (2000).
[6] W.M. Abd-Elhameed and Y.H. Youssri, Numerical solutions for Volterra-Fredholm-Hammerstein integral equations via second kind Chebyshev quadrature collocation Algorithm, Adv. math. sci. appl. **24(1)**, 129-141 (2014).
[7] A. Rohul, N. Shah and G.M. Ivan, A Collocation Method for Numerical Solution of Nonlinear Delay Integro-Differential Equations for Wireless Sensor Network and Internet of Things, Sensors, **20(2020)**, doi:10.3390/s20071962.
[8] M.T. Rashed, Numerical solution of functional differential integral and integro-differential equations. Appl. Math. Comput., **156**, 485-492 (2004).
[9] Z. Jackiewicz, The numerical solution of Volterra functional differential equations of neutral type. SIAM J. Numer. Anal., 1**8(4)**, 615–626 (1981).
[10] S. Reutskiy, The backward substitution method for multipoint problems with linear Volterra-Fredholm integro-differential equations of the neutral type, J. Comput. Appl. Math., **296**, 724-738 (2016).
[11] Y. Liu, Numerical solution of implicit neutral functional differential equations. SIAM J. Numer. Anal., **36(2)**, 516–528 (1999).
[12] W.H. Enright, M. Hu, Continuous Runge–Kutta methods for neutral Volterra integro-differential equations with delay. Appl. Numer. Math., **24(2-3)**, 175–190 (1997).
[13] T. Koto, Stability of Runge–Kutta methods for delay integrodifferential equations. J. Comput. Appl. Math. **145(2)**, 483-492 (2002).

[14] S. Sedaghat, Y. Ordokhani and M. Dehghan, On spectral method for Volterra functional integro-differential equations of neutral type. Numer. Funct. Anal. Optim., **35(2)**, 223–239 (2014).

[15] Y. Suayip, and K. Murat, A Numerical Approach for Solving High-Order Linear Delay Volterra Integro-Differential Equations, Int. J. Comput. Methods, **15(3)**, (2018).

[16] N.I. Ismail, Z. Abdul Majid, N. Senu and N. Wahi, Numerical method in solving neutral and retarded Volterra delay integro-differential equations, J.Phys: Conference Series, 1988 (2021) 012-033,doi:10.1088/1742-6596/1988/1/012033.

[17] S.A. Khuri, A Laplace decomposition algorithm applied to class of non-linear differential equations, J. Math. Appl., **4**, 141-155 (2001).

[18] E. Yusufoglu, Numerical solution of Duffing equation by the Laplace decomposition algorithm, Appl. Math. Comput., **177**, 572-580 (2006).

[19] V. Daftardar-Gejji, and H. Jafari, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl., **316(2)**, 753-763 (2006).

[20] A.M. Wazwaz, The combined Laplace transform Adomain Decomposition method for handling nonlinear Volterra integro-differential equations, Appl. Math. Comput., **216**, 1304-1309 (2010).

[21] H. Ahmed, M.S.M. Bahgat and M. Zaki, Analytical approaches to space-and time-fractional coupled Burgers'equations, J. Pramana, **92(3)**, 1-14 (2019).

[22] Y. Khan, A effective modification of the Laplace decomposition method for nonlinear equations, In. J. Nonline. Sci. Num. Simul., **10(11-12)**, 1373-1376 (2009).

[23] M. Hussain and M. Khan, Modified Laplace decomposition method, Appl. Math. Scie., **4**, 1769-1783 (2010).

[24] G.C. Wu and D. Beleanu, Varirational iteration method for fractional calculus – a universal approach by Laplace transform, Adv. Differ. Equ., 2013(1), (2013).

[25] M.S.M. Bahgat, Approximate analytical solution of the linear and nonlinear multi-pantograph delay differential equations, Phys. Scr., **95**, Article ID 055219 (2020).

[26] M.S.M. Bahgat and A.M. Sebaq, An Analytical Computational Algorithm for Solving a System of Multipantograph DDEs Using Laplace Variational Iteration Algorithm, ADV. ASTRON., **2021**, Article ID 7741166, 16 pages, https://doi.org/10.1155/2021/7741166.

[27] J.L. Schiff, The Laplace Transform Theory and Applications, New York, Springer-Verlag (1999).

[28] E. Cimen and S. Uncu, On the solution of the Delay Differential Equation via Laplace Transform, Commun. Math. Appl., **10(3)**, 379-387 (2020).

M. S. M. Bahgat is presently working as an Associate Professor in the Department of Mathematics, Faculty of Science, Minia University. He received his Ph.D. in the year 2000 through membership in the channel system between Koeln university (German) and El Minia University (Egypt). His research interests in numerical analysis. He has published more than 20 research papers in journals of international repute.

M. Zaki is Lecturer of Pure Mathematics at Mathematics Department, Faculty of Science, Minia University, Minia, Egypt. She received her Ph.D. in the year 2020. Her research interests in Fraction Analysis. She has published four research papers in journals of international.