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Closed-Form Solutions for Cauchy-Euler Differential Equations through the New Iterative Method (NIM)

Belal Batiha[1,*], *Ahmed Salem Heilat* [1] *and Firas Ghanim*[2]

[1]Faculty of Science and Information Technology, Jadara University, Irbid, Jordan
[2]College of Sciences, University of Sharjah, Sharjah, United Arab Emirates

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Abstract: In this paper, a new iterative method (NIM) is proposed for solving Cauchy-Euler differential equations. NIM simplifies the computational process and directly addresses both linear and nonlinear, homogeneous and non-homogeneous equations, while being user-friendly and accessible. The method's efficiency, reliability, and accuracy are demonstrated through four examples. NIM offers a versatile and practical solution, making it a valuable tool for mathematical analysis.

Keywords: New iterative method (NIM), Differential Equations, Cauchy-Euler equations

1 Introduction

The significance of differential equations lies in their ability to model and analyze a wide range of real-world phenomena. They provide a mathematical structure for comprehending and forecasting the behavior of systems, including those found in physical, biological, and social domains. However, obtaining numerical or theoretical solutions to these equations can be a daunting task, particularly in the case of nonlinear differential equations. These equations are notably intricate and their solutions are often challenging to obtain analytically.

To address this issue, numerous numerical methods have been developed in recent years to solve linear and nonlinear differential equations. These methods utilize advanced mathematical and computational techniques to approximate solutions to these equations. Numerical methods have become a crucial tool for researchers and practitioners in various fields, as they provide a way to solve complex problems that are otherwise intractable analytically.

In conclusion, differential equations continue to play a vital role in representing real-life phenomena, and the development of numerical methods has made it possible to solve linear and nonlinear differential equations that would otherwise be difficult to find analytically. These methods provide a valuable tool for researchers and practitioners, helping them to better understand and predict the behavior of complex systems [1,2,3,4,5,6,7].

In 2006, Daftardar-Gejji and Jafari introduced a new mathematical technique called the new iterative method (NIM), which is capable of solving both linear and nonlinear functional equations. This method has demonstrated high effectiveness in solving a wide range of nonlinear equations, including algebraic equations, integral equations, and ordinary or partial differential equations of both integer and fractional orders. What makes the NIM stand out is its simplicity and ease of implementation, making it accessible to those without advanced mathematical knowledge. Furthermore, it can be executed using computer software. In a comparative study conducted by Bhalekar in 2008, the NIM was found to produce better results than established techniques such as the Adomian Decomposition Method (ADM) [9], the Homotopy Perturbation Method (HPM) [10], the differential transformation method (DTM) [13], and the Variational Iteration Method (VIM) [11].

The aim of this paper is to use the new iterative method (NIM) to solve the Cauchy-Euler differential equation:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = g(x), \quad (1)$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, and Compare the numerical results to the exact solutions.

* Corresponding author e-mail: b.bateha@jadara.edu.jo

The defining characteristic of the Cauchy-Euler equation is that the degree of the monomial coefficients x^k ($k = n, n-1, \dots 1, 0$) matches the order of differentiation $d^k y/dx^k$. This type of equation is a linear equation with variable coefficients and its general solution can always be represented as a combination of powers of x, sines, cosines, and logarithmic functions.

2 The new iterative method (NIM)

In this section, the NIM numerical method will be outlined as follows [14, 15, 16, 17]:

$$\mathbf{y} = f + \mathbb{L}(\mathbf{y}) + \mathbb{N}(\mathbf{y}), \tag{2}$$

In the equation above, f is a given function, and $\mathbb{L}$ and $\mathbb{N}$ are linear and nonlinear operators, respectively. The solution to equation (2) is as follows:

$$\mathbf{y} = \sum_{i=0}^{\infty} \mathbf{y}_i. \tag{3}$$

Suppose we have,

$$H_0 = \mathbb{N}(\mathbf{y}_0), \tag{4}$$

$$H_m = \mathbb{N}\left(\sum_{i=0}^{m} \mathbf{y}_i\right) - \mathbb{N}\left(\sum_{i=0}^{m-1} \mathbf{y}_i\right). \tag{5}$$

then we get,

$$\begin{aligned} H_0 &= \mathbb{N}(\mathbf{y}_0),\\ H_1 &= \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1) - \mathbb{N}(\mathbf{y}_0),\\ H_2 &= \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2) - \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1),\\ H_3 &= \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3) - \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2) + \cdots. \end{aligned}$$

Therefore, $\mathbb{N}(\mathbf{y})$ can be decomposed as follows:

$$\begin{aligned} \mathbb{N}\left(\sum_{i=0}^{\infty} \mathbf{y}_i\right) &= \mathbb{N}(\mathbf{y}_0) + \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1) - \mathbb{N}(\mathbf{y}_0)\\ &+ \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2)\\ &- \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1) + \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3)\\ &- \mathbb{N}(\mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2) + \cdots. \end{aligned} \tag{6}$$

The recurrence relation is written in the following form:

$$\begin{aligned} \mathbf{y}_0 &= f\\ \mathbf{y}_1 &= L(\mathbf{y}_0) + H_0\\ \mathbf{y}_{m+1} &= L(\mathbf{y}_m) + H_m, \quad m = 1, 2, \cdots. \end{aligned} \tag{7}$$

By knowing that $\mathbb{L}$ is linear we will have:

$$\sum_{i=0}^{m} \mathbb{L}(\mathbf{y}_i) = \mathbb{L}\left(\sum_{i=0}^{m} \mathbf{y}_i\right). \tag{8}$$

So,

$$\begin{aligned} \sum_{i=0}^{m+1} \mathbf{y}_i &= \sum_{i=0}^{m} \mathbb{L}(\mathbf{y}_i) + \mathbb{N}\left(\sum_{i=0}^{m} \mathbf{y}_i\right).\\ &= \mathbb{L}\left(\sum_{i=0}^{m} \mathbf{y}_i\right) + \mathbb{N}\left(\sum_{i=0}^{m} \mathbf{y}_i\right), \quad m = 1, 2, \cdots \end{aligned} \tag{9}$$

Thus,

$$\sum_{i=0}^{\infty} \mathbf{y}_i = f + \mathbb{L}\left(\sum_{i=0}^{\infty} \mathbf{y}_i\right) + \mathbb{N}\left(\sum_{i=0}^{\infty} \mathbf{y}_i\right). \tag{10}$$

So, the solution will be given as follows:

$$\mathbf{y} = \sum_{i=0}^{k-1} \mathbf{y}_i. \tag{11}$$

3 The convergence of the NIM

Theorem 1: For any n and for some real $\mathbb{L} > 0$ and $||\mathfrak{u}_i|| \le M < \frac{1}{e}, i = 1, 2, \dots,$ if $\mathbb{N}$ is $C^{(\infty)}$ in the neighborhood of $\mathfrak{u}_0$ and $||\mathbb{N}^{(n)}(\mathfrak{u}_0)|| \le \mathbb{L}$, then $\sum_{n=0}^{\infty} H_n$ is convergent absolutely and $||H_n|| \le \mathbb{L} M^n e^{n-1}(e-1), \ n = 1, 2, \dots.$

Proof: The full details of the proof can be found in [18].

Theorem 2: The series $\sum_{n=0}^{\infty} H_n$ is convergent absolutely if $\mathbb{N}$ is $C^{(\infty)}$ and $||\mathbb{N}^{(n)}(\mathfrak{u}_0)|| \le M \le e^{-1}, \forall n.$

Proof: The full details of the proof can be found in [18].

4 Numerical results and discussion

To fully illustrate the effectiveness of the new iterative method (NIM), we shall consider four diverse test examples for the Cauchy-Euler equation. The Cauchy-Euler equation is a linear ordinary differential equation that occurs in a variety of mathematical modeling scenarios. Solving this equation numerically is crucial in obtaining solutions for practical problems. The NIM has been developed as a robust and efficient technique to tackle the Cauchy-Euler equation and many other differential equations. By examining these four test cases, we will be able to gauge the reliability and accuracy of the NIM in solving the Cauchy-Euler equation.

Example 1:

Initially, we shall introduce the analytical solution for the Cauchy-Euler differential equation.

$$x^2u'' + 3xu' = 0, \quad u(1) = 0, \quad u'(1) = 4. \tag{12}$$

The analytical solution to the aforementioned problem is expressed as follows, [13]:

$$u(x) = 2 - 2x^{-2}. \tag{13}$$

Initially, we must address the challenge posed by the singularity at $x = 0$. To accomplish this objective, we employ the transformation:

$$t = \ln x, \quad x = e^t, \tag{14}$$

so that

$$\frac{du}{dx} = \frac{1}{x}\frac{du}{dt}$$
$$\frac{d^2u}{dx^2} = \frac{1}{x^2}\frac{d^2u}{dt^2} - \frac{1}{x^2}\frac{du}{dt}. \tag{15}$$

Using (14) and (15) into (12) gives

$$\frac{d^2u}{dt^2} + 2\frac{du}{dt} = 0, \tag{16}$$

with the conditions

$$u(t=0) = 0, \quad u'(t=0) = 4. \tag{17}$$

To solve the Cauchy-Euler differential equation (16) with initial condition (17) we integrate eq. (16) and use eq. (17) to get:

$$u = -\int_0^t \int_0^t \left(2\frac{du}{dt}\right) dt dt \tag{18}$$

By using algorithm (7) we obtain:

$$u_0 = 4t,$$
$$u_1 = -4t^2,$$
$$u_3 = \frac{8t^3}{3},$$
$$u_4 = -\frac{4}{3}t^4,$$
$$u_5 = \frac{8t^5}{15}.$$
$$\vdots$$

The repetition formula's remaining components can be obtained easily by using computer algebra software like Mathematica.

Consequently, the closed form of the solution can be succinctly stated as follows:

$$\sum_{i=0}^{\infty} u_i = 4t - 4t^2 + \frac{8}{3}t^3 - \frac{4}{3}t^4 + \frac{8}{15}t^5 - \ldots \tag{19}$$
$$= 2 - 2e^{-2t}. \tag{20}$$

Recall that $x = e^t$ then

$$u(x) = 2 - 2x^{-2}. \tag{21}$$

Which is the exact solution.

The performance of the new iterative method (NIM) in solving the Cauchy-Euler differential equation Eq. (12) is demonstrated by its accuracy and efficiency, which are compared to the exact solution. The results of this comparison are presented in Table 1 and Figure 1, where the solutions obtained through the NIM and the exact method are compared.

Example 2:

At present, we shall examine the classical Cauchy-Euler equation below:

$$x^2u'' + xu' + u = 0, \quad u(1) = 1, \quad u'(1) = 2. \tag{22}$$

The analytical solution to the aforementioned problem is expressed as follows, [13]:

$$u(x) = \cos(\ln x) + 2\sin(\ln x). \tag{23}$$

we must address the challenge posed by the singularity at $x = 0$. To accomplish this objective, we employ the transformation:

$$t = \ln x, \quad x = e^t, \tag{24}$$

so that

$$\frac{du}{dx} = \frac{1}{x}\frac{du}{dt}$$
$$\frac{d^2u}{dx^2} = \frac{1}{x^2}\frac{d^2u}{dt^2} - \frac{1}{x^2}\frac{du}{dt}. \tag{25}$$

Using (24) and (25) into (22) gives

$$\frac{d^2u}{dt^2} + u = 0, \tag{26}$$

with the conditions

$$u(t=0) = 1, \quad u'(t=0) = 2. \tag{27}$$

To solve the Cauchy-Euler differential equation (26) with initial condition (27) we integrate eq. (26) and use eq. (27) to get:

$$u = -\int_0^t \int_0^t u\, dt dt \tag{28}$$

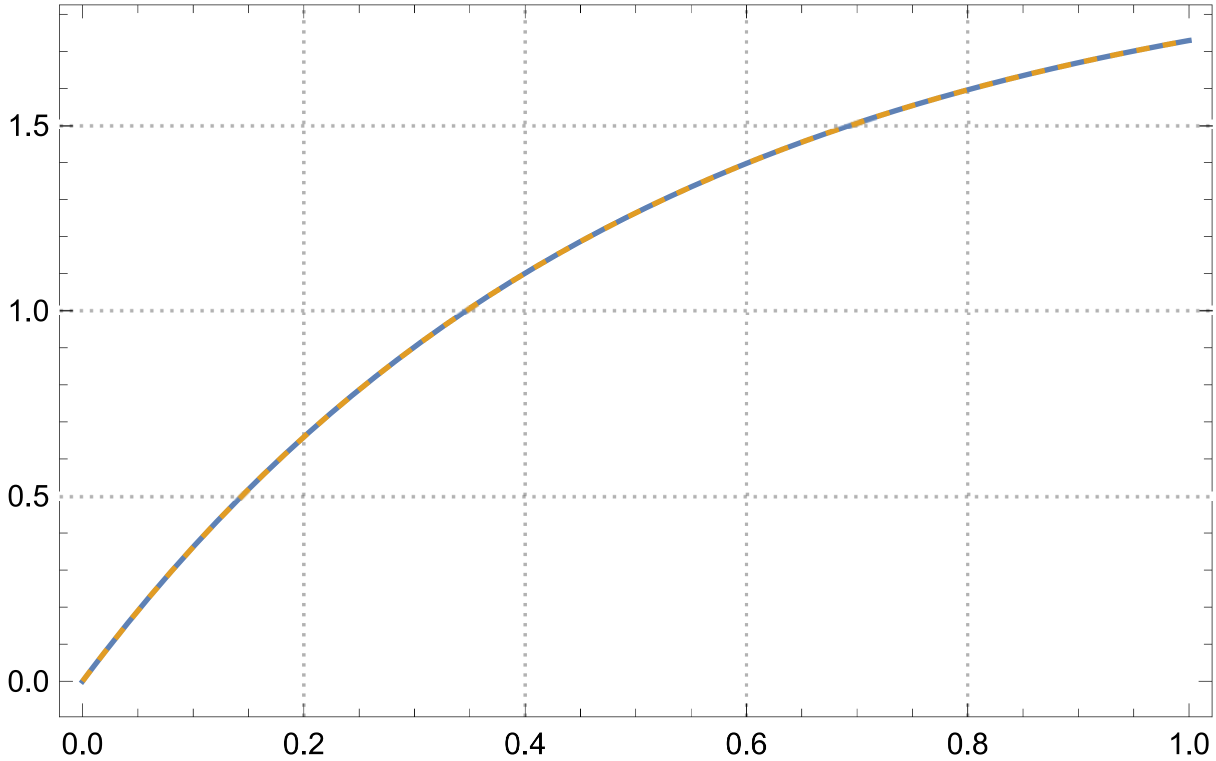

Fig. 1: A comparison between the exact solution and the numerical results for $u(x,y,t)$ obtained through the 10 iteration NIM solution.

Table 1: Comparison study between NIM with 10 iterations with exact solution.

t	Exact	NIM_{10}	Absolute error
0.1	0.362538494	0.362538494	$1.110223025\times10^{-16}$
0.2	0.659359908	0.659359908	$6.794564911\times10^{-14}$
0.3	0.902376728	0.902376728	$8.686718012\times10^{-12}$
0.4	1.101342072	1.101342072	$2.702287283\times10^{-10}$
0.5	1.264241118	1.264241122	3.875672894×10^{-9}
0.6	1.397611576	1.397611610	3.406407201×10^{-8}
0.7	1.506806072	1.506806286	2.135549633×10^{-7}
0.8	1.596206964	1.596208012	1.045538796×10^{-6}
0.9	1.669402224	1.669406462	4.238102103×10^{-6}
1.0	1.669402224	1.729344236	1.480248413×10^{-5}

By using algorithm (7) we obtain:

$$u_0 = 2t+1,$$

$$u_1 = -\frac{t^3}{3} - \frac{t^2}{2},$$

$$u_3 = \frac{1}{24}t^4 + \frac{1}{60}t^5,$$

$$u_4 = -\frac{t^6(2t+7)}{5040},$$

$$u_5 = \frac{t^8(2t+9)}{362880}$$

$$\vdots$$

Consequently, the closed form of the solution can be succinctly stated as follows:

$$\sum_{i=0}^{\infty} u_i = 1+2t-\frac{1}{2}t^2-\frac{1}{3}t^3+\frac{1}{24}t^4+\frac{1}{60}t^5-\dots \quad (29)$$

$$= \cos(t)+2\sin(t). \quad (30)$$

Recall that $t = \ln x$ then

$$u(x) = \cos(\ln x)+2\sin(\ln x). \quad (31)$$

Which is the exact solution.

The performance of the new iterative method (NIM) in solving the Cauchy-Euler differential equation Eq. (22)

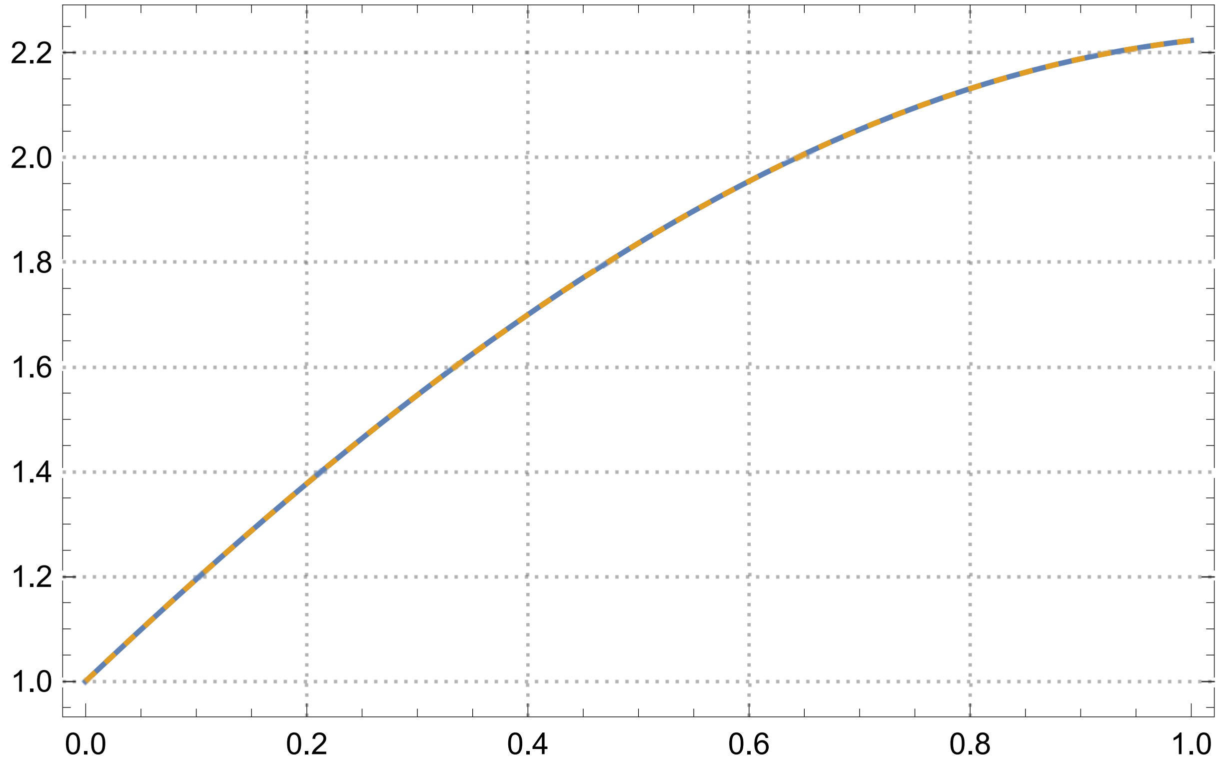

Fig. 2: A comparison between the exact solution and the numerical results for $u(x,y,t)$ obtained through the 10 iteration NIM solution.

Table 2: Comparison study between NIM with 10 iterations with exact solution.

t	Exact	NIM_{10}	Absolute error
0.1	1.194670999	1.194670999	$2.220446049\times10^{-16}$
0.2	1.377405239	1.377405239	0.000000000
0.3	1.546376902	1.546376902	0.000000000
0.4	1.699897679	1.699897679	$2.220446049\times10^{-16}$
0.5	1.836433639	1.836433639	0.000000000
0.6	1.954620562	1.954620562	0.000000000
0.7	2.053277562	2.053277562	$4.440892099\times10^{-16}$
0.8	2.131418891	2.131418891	0.000000000
0.9	2.188263788	2.188263788	$4.440892099\times10^{-16}$
1.0	2.223244275	2.223244275	$4.440892099\times10^{-16}$

is demonstrated by its accuracy and efficiency, which are compared to the exact solution. The results of this comparison are presented in Table 2 and Figure 2, where the solutions obtained through the NIM and the exact method are compared.

Example 3:

In this example, we will solve the second order Cauchy-Euler equation:

$$x^2u'' - 2xu' + 2u = 0, \tag{32}$$

subject to the initial conditions

$$u(1) = 2, \qquad u'(1) = 3. \tag{33}$$

With the exact solution, [13]:

$$u(x) = x + x^2. \tag{34}$$

To overcome the difficulties encountered by the singularity at $x = 0$, we use the transformation

$$t = \ln x, \quad x = e^t, \tag{35}$$

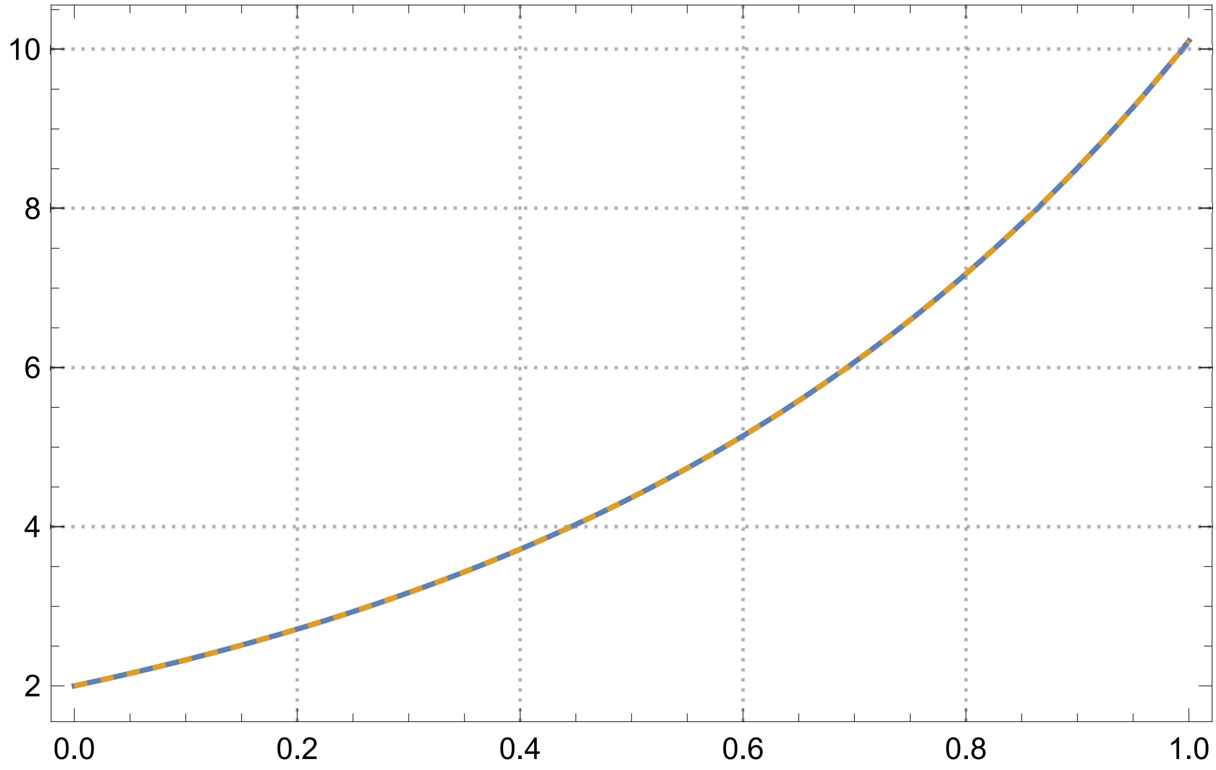

Fig. 3: A comparison between the exact solution and the numerical results for $u(x,y,t)$ obtained through the 10 iteration NIM solution.

Table 3: Comparison study between NIM with 10 iterations with exact solution.

t	Exact	NIM_{10}	Absolute error
0.1	2.326573676	2.326573676	$8.881784197\times10^{-16}$
0.2	2.713227456	2.713227456	$2.341682404\times10^{-12}$
0.3	3.171977608	3.171977608	$2.925757414\times10^{-10}$
0.4	3.717365626	3.717365617	8.892498293×10^{-9}
0.5	4.367003099	4.367002975	1.245670429×10^{-7}
0.6	5.142235723	5.142234654	1.068998975×10^{-6}
0.7	6.068952674	6.068946133	6.541528714×10^{-6}
0.8	7.178573353	7.178542102	3.125114243×10^{-5}
0.9	8.509250576	8.509127002	1.235730922×10^{-4}
1.0	10.10733793	10.10691702	4.209086768×10^{-4}

so that

$$\frac{du}{dx} = \frac{1}{x}\frac{du}{dt}$$
$$\frac{d^2u}{dx^2} = \frac{1}{x^2}\frac{d^2u}{dt^2} - \frac{1}{x^2}\frac{du}{dt}. \tag{36}$$

Using (35) and (36) into (32) gives

$$\frac{d^2u}{dt^2} - 3\frac{du}{dt} + 2u = 0, \tag{37}$$

with the conditions

$$u(t=0) = 2, \quad u'(t=0) = 3. \tag{38}$$

To solve the Cauchy-Euler differential equation (37) with initial condition (38) we integrate eq. (37) and use eq. (38) to get:

$$u = \int_0^t \int_0^t \left(3\frac{du}{dt} - 2u\right) dt dt \tag{39}$$

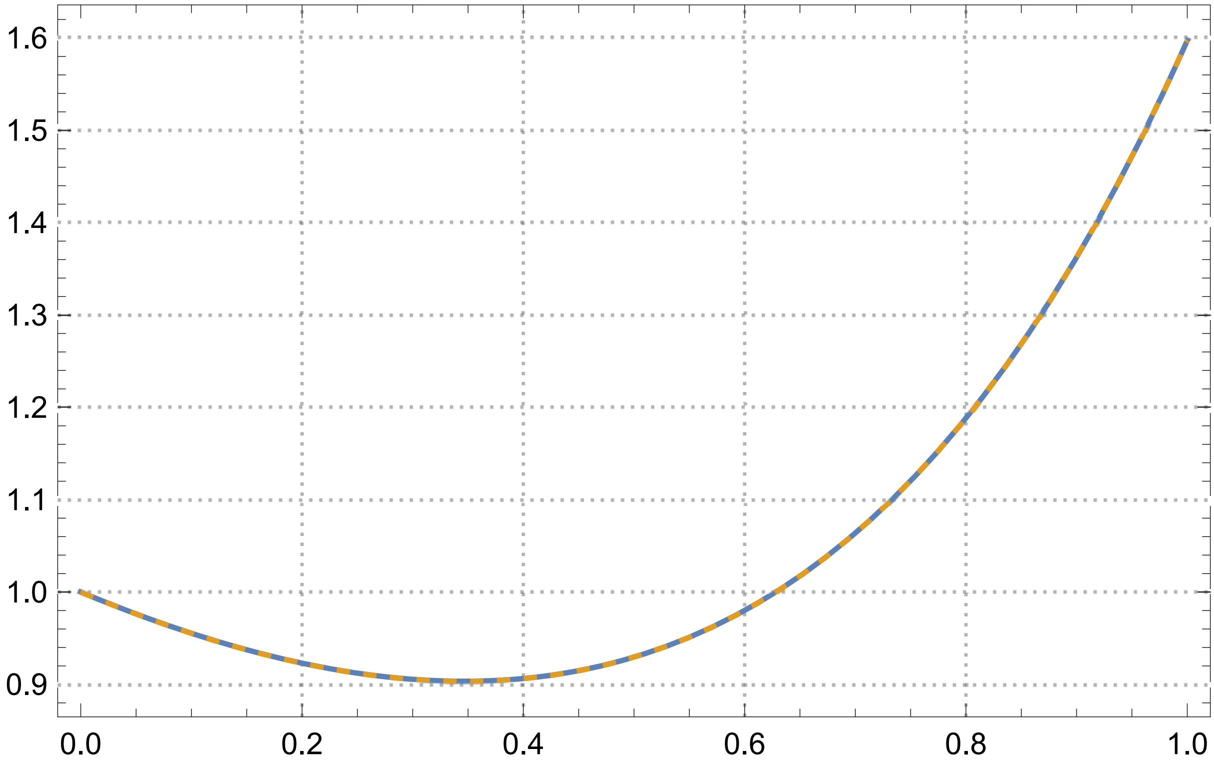

Fig. 4: A comparison between the exact solution and the numerical results for $u(x,y,t)$ obtained through the 10 iteration NIM solution.

Table 4: Comparison study between NIM with 10 iterations with exact solution.

t	Exact	NIM_{10}	Absolute error
0.1	0.95535068	0.95535068	1.110223×10^{-16}
0.2	0.92295617	0.92295617	1.110223×10^{-16}
0.3	0.90552970	0.90552970	1.110223×10^{-16}
0.4	0.90638523	0.90638523	1.110223×10^{-16}
0.5	0.92957045	0.92957045	0.0000000
0.6	0.98002923	0.98002923	1.110223×10^{-16}
0.7	1.063799992	1.0637999	0.0000000
0.8	1.188258106	1.1882581	2.220446×10^{-16}
0.9	1.362411866	1.36241186	2.220446×10^{-16}
1.0	1.597264025	1.59726402	0.0000000

By using algorithm (7) we obtain:

$$u_0 = 3t+2,$$
$$u_1 = \frac{5t^2}{2} - t^3,$$
$$u_3 = \frac{1}{30}t^3\left(3t^2-35t+75\right),$$
$$u_4 = -\frac{t^4\left(12t^3-322t^2+2394t-4725\right)}{2520},$$
$$u_5 = \frac{t^5\left(t^4-48t^3+756t^2-4536t+8505\right)}{7560}$$
$$\vdots$$

The performance of the new iterative method (NIM) in solving the Cauchy-Euler differential equation Eq. (32) is demonstrated by its accuracy and efficiency, which are compared to the exact solution. The results of this comparison are presented in Table 3 and Figure 3, where the solutions obtained through the NIM and the exact method are compared.

Example 4:

Consider the following Cauchy-Euler differential equation:

$$xu'' + u' = x, \tag{40}$$

subject to the initial conditions

$$u(1) = 1, \qquad u'(1) = -\frac{1}{2}. \tag{41}$$

With the exact solution, [13]:

$$u(x) = \frac{3}{4} + \frac{1}{4}x^2 - \ln x. \tag{42}$$

In order to change Eq. (40) to standard Cauchy-Euler form, we multiply (40) by x to get

$$x^2u'' + xu' = x^2, \tag{43}$$

Use the transformation

$$t = \ln x, \qquad x = e^t, \tag{44}$$

so that

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{x}\frac{du}{dt} \\ \frac{d^2u}{dx^2} &= \frac{1}{x^2}\frac{d^2u}{dt^2} - \frac{1}{x^2}\frac{du}{dt}. \end{aligned} \tag{45}$$

This gives

$$\frac{d^2u}{dt^2} - e^{2t} = 0, \tag{46}$$

with the conditions

$$u(t=0) = 1, \qquad u'(t=0) = -\frac{1}{2}. \tag{47}$$

To solve the Cauchy-Euler differential equation (46) with initial condition (47) we integrate eq. (46) and use eq. (47) to get:

$$u = \int_0^t \int_0^t e^{2t}\,dt\,dt \tag{48}$$

By using algorithm (7) we obtain:

$$\begin{aligned} u_0 &= 1 - \frac{1}{2}t, \\ u_1 &= \frac{1}{4}\left(-2t + e^{2t} - 1\right), \\ u_3 &= 0, \\ &\vdots \end{aligned}$$

The performance of the new iterative method (NIM) in solving the Cauchy-Euler differential equation Eq. (40) is demonstrated by its accuracy and efficiency, which are compared to the exact solution. The results of this comparison are presented in Table 4 and Figure 4, where the solutions obtained through the NIM and the exact method are compared.

5 Conclusions

The new iterative method (NIM) presented in this paper provides a versatile solution for solving Cauchy-Euler differential equations, both linear and nonlinear, homogeneous and non-homogeneous. The NIM is designed to simplify the size of calculations, without requiring linearization, perturbation, or assumptions. The method was outlined using linear and nonlinear operators and supported by a couple of theorems that demonstrate its convergence. To validate the proposed method, four examples were provided, solved, and analyzed with calculated absolute errors. The results show the efficiency, reliability, and accuracy of the NIM, making it an effective tool for mathematical analysis.

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Belal Batiha obtained his doctoral degree from the National University of Malaysia in 2008. His primary research interests lie in the development of novel numerical techniques for the resolution of differential equations.

Ahmed Salem Heilat is an Assistant professor at the Department of Mathematics, Faculty of Science and Information Technology, Jadara University. Heilat received his Ph.D. Mathematics from the Universiti Sains of Malaysia (USM, 2018. B.Sc. Degree in Mathematics, Faculty of Science, Irbid National University , Jordan, 2000. M.Sc. in Mathematics, Faculty of Science Mutah University, Jordan, 2007. Heilat is specific research area in Differential Equations and Numerical Analysis Applied Mathematics.

Firas Ghanim is a researcher in the Department of Mathematics at the University of Sharjah, with a focus on Complex Analysis, Hypergeometric Functions, Analytic Functions, Univalent Functions, Statistics, Fractional Calculus, and Applied Mathematics. He has been conducting research in the field of complex analysis since his Ph.D. studies in 2007, with a significant contribution towards resolving various issues. His work has been widely recognized, with an H- index of 16 and an i-index of 34 based on citations of his papers by other researchers in Google Scholar. Firas Ghanim has also reviewed over 200 papers for prominent journals and served as a member of the Scientific Committee for numerous conferences. He