Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/170318

Adapted Shifted ChebyshevU Operational Matrix of Derivatives: Two Algorithms for Solving Even-Order BVPs

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Received: 12 Feb. 2022, Revised: 22 Mar. 2023, Accepted: 22 Apr. 2023
Published online: 1 May 2023

Abstract: Herein, modified orthogonal polynomials are introduced. These polynomials are generated from the second kind of shifted Chebyshev polynomials on the interval $[\alpha, \beta]$. The operational matrix of its derivative is constructed. The Tau and Galerkin method with the proposed orthogonal polynomials is used to solve the boundary value problems (BVPs) with even order. The effectiveness of these methods is proved through their application to several BVPs.

Keywords: Second-kind Chebyshev polynomials; Galerkin method; Tau method; Boundry value problems; Even-order BVPs.

1 Introduction

BVPs have risen in importance since they are used in various domains, including chemistry [1], physics [2], biology [3], fluid dynamics [4], engineering [5], and diseases [6]. Most of these applications haven't exact solutions. That's why we turn to the approximate solutions—one of those approximate solutions is the numerical solution. There are several numerical methods available, each one having advantages and disadvantages. The finite difference approach, for example, implies dividing a continuous domain into a grid of discrete points and estimating derivatives using finite differences between neighboring grid points [7, 8]. The finite element method includes expressing the domain as a collection of smaller, simpler subdomains and then solving the problem by merging the solutions for each subdomain. This approach is very beneficial for issues involving irregularly shaped domains or complex boundary conditions [9]. On the other hand, spectral methods have been most prevalent because of their advantages. Like, higher accuracy, speed, convergence, and the ability to deal with complex geometries and boundary conditions. These techniques implied the approximate solution as a summation of unknown constants times suitable basis functions. There are three kinds of spectral methods. The first is the Galerkin method used in [10, 11]. A critical condition to using the Galerkin method is the chosen basis functions that should satisfy the initial and boundary conditions of the given BVP. In contrast, the Tau method hasn't any conditions but needs a suitable weight function [12–15]. The collocation (pseudospectral) method is the third kind [16–19]. Its technique depends on differentiation matrices.

The base function mentioned may be orthogonal or not. Some examples of these polynomials were presented. The first kind of Chebysheve polynomials was used in [20–26]. The second kind of Chebyshev polynomials was applied in [27, 28]. Similar to the authors in [29], they use the Chebyshev polynomials' first derivative, orthogonal polynomials. The authors in [30] used the Legendre polynomials. Additionally, the second derivative Legendre polynomials were utilized as a base function in [10, 31]. The authors [32, 33] choose the ultraspherical polynomials to be their base functions.

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In this paper, we modified the second kind of shifted Chebyshev polynomials in the interval $[\alpha,\beta]$ (SCH2-Ps) and named it Modified shifted second kind Chebyshev polynomial (MSCH2-Ps). These novel polynomials were used to solve the even BVPs using Galerkin and Tau method.

This paper consists of five sections. All necessary definitions and relations are presented in the second section. Section 3 consists of 2 parts; in the first part, we defined the new orthogonal polynomial, its recurrence relations, and orthogonality relations. The second part constructed the operation matrix of the differentiation of order k. The two proposed methods, Galerkin and Tau, are presented in detail to solve even-order BVPs in section 4. In section 5, the efficiency and accuracy of the two techniques were proved by solving even-order BVPs and comparing the obtained results with other methods. Some of the solutions to these problems achieved the exact solution. Finally, the paper will be ended by conclusion and the future work.

2 Some needed relations

In this section, important and essential relationsCH2-Ps are listed. The CH2-Ps, $U_j(y)$ of order $j\geq 0$, are defined on interval $[-1,1]$ as [35,36]:

$$U_j(y)=\frac{\sin(j+1)\theta}{\sin\theta}\qquad ;y=\cos\theta.$$

Its recurrence relation:

$$U_j(y)=2yU_{j-1}(y)-U_{j-2}(y)\quad j\geq 2, \tag{1}$$

with its initials $U_0(y)=1$ and $U_1(y)=2y$.

$\{U_j(y)\}_{j\geq 0}$ are orthogonal concerning their weight function $w(y)=\sqrt{1-y^2}$:

$$\int_{-1}^{1}U_i(y)U_j(y)\,w(y)\,dy=\begin{cases}\frac{\pi}{2}, & i=j,\\ 0, & i\neq j.\end{cases} \tag{2}$$

The boundaries of $U_j(y)$ satsisfy:

$$U_j(-1)=(-1)^j(j+1),\quad U_j(1)=1+j, \tag{3}$$

$$U_j'(-1)=\frac{(-1)^{j-1}j(j-1)}{2},\quad U_j'(1)=\frac{j(j-1)}{2}, \tag{4}$$

$$|U_j(y)|\leq 1,\quad |U_j'(y)|\leq\frac{j(j-1)}{2}. \tag{5}$$

The formula of $U_j(y)$ as series defined as:

$$U_j(y)=\sum_{i=0}^{\lfloor j/2\rfloor}(-1)^i\frac{2^{j-2i}(j-i)!}{(i)!(j-2i)!}y^{j-2i}. \tag{6}$$

Moreover,the SCH2-Ps, $U_j^*(y)$; $y\in[\alpha,\beta]$, of order j can be defined as:

$$U_j^*(y)=U_j\left(\frac{2y-\beta-\alpha}{\beta-\alpha}\right),\quad j=0,1,2,... \tag{7}$$

The orthogonality relation of $\{U_j^*(y)\}_{j=0}^N$ with respect to its weight function $w^*(y)=\sqrt{(y-\alpha)(\beta-y)}$ formed as:

$$\int_\alpha^\beta U_i^*(y)U_j^*(y)w^*(y)dy=\begin{cases}0, & i\neq j,\\ \frac{\pi}{8}(\beta-\alpha)^2, & i=j.\end{cases} \tag{8}$$

The linearization relation of two SCH2-Ps is represented as:

$$U_i^*(y)U_j^*(y)=\sum_{\substack{l=|i-j|\\ step\,2}}^{i+j}U_l^*(y). \tag{9}$$

In the next section, the presented novel polynomials will be introduced. Consequently, the operational matrix of derivatives will be constructed.

3 Modified shifted second kind Chebyshev polynomials

The following section will be divided into two parts. In the first part, we will establish novel orthogonal polynomials based on SCH2-Ps. The new polynomials will be called modified shifted second-kind Chebyshev polynomials (MSCH2-Ps). Furthermore, all relations of these polynomials are investigated in that part. While in the second part, the operational matrix of these polynomials' derivatives will be introduced.

3.1 Modified shifted second kind Chebyshev polynomial

In the beginning, we defined the investigated polynomials MSCH2-Ps on the interval $[\alpha,\beta]$.

Definition 1.*The set of MSCH2-Ps* $\{\psi_{r,j}(y)\}$*;* $r,j=0,1,2,\cdots$, $y\in[\alpha,\beta]$ *will be generated as:*

$$\psi_{r,j}(y)=(y-\alpha)^r(\beta-y)^rU_j^*(y), \tag{10}$$

where

$$\psi_{r,0}(y)=(y-\alpha)^r(\beta-y)^r, \tag{11}$$

$$\psi_{r,1}(y)=2(y-\alpha)^r(\beta-y)^r(\frac{2y-a-b}{b-a}), \tag{12}$$

$$\psi_{r,2}(y)=(y-\alpha)^r(\beta-y)^r\left(\frac{16y^2-16(\alpha+\beta)y+3(\alpha+\beta)^2+4\alpha\beta}{(\beta-\alpha)^2}\right). \tag{13}$$

From the recurrence relation (1), the recurrence relation of MSCH2-Ps defined as:

$$\psi_{r,j+2}(y)=2\left(\frac{2y-\alpha-\beta}{\beta-\alpha}\right)\psi_{r,j+1}(y)-\psi_{r,j}(y),\quad j=0,1,2,\cdots. \tag{14}$$

Here are some of the initials and boundaries of $\psi_{r,j}(y)$:

$$\psi_{r,j}(\alpha) = \psi_{r,j}(\beta) = 0, \quad r > 0, \tag{15}$$
$$\psi'_{r,j}(\alpha) = \psi'_{r,j}(\beta) = 0, \quad r > 1. \tag{16}$$

The set of polynomials $\{\psi_{r,j}(y)\}_{r,j\geq 0}$ are orthogonal with respect to its weight function $\hat{w}(y)$ as:

$$\int_{\alpha}^{\beta} \psi_{r,i}(y)\psi_{r,j}(y)\hat{w}(y)dy = \begin{cases} 0, & i \neq k, \\ \frac{\pi}{8}(\beta-\alpha)^2, & i = j, \end{cases} \tag{17}$$

such that $\hat{w}(y) = \sqrt{(y-\alpha)^{1-4r}(\beta-y)^{1-4r}}$.

In addition, the linearization of MSCH2-Ps is defined in the following remark.

*Remark.*The product of two MSCH2-Ps expressed as:

$$\psi_{r,i}(y)\psi_{r,j}(y) = (x-\alpha)^r(\beta-x)^r \sum_{\substack{l=|i-j| \\ step\,2}}^{i+j} \psi_{r,l}(y) \tag{18}$$

3.2 Operational matrix of MSCH2-Ps

In this part, the first derivative of $\psi_{r,j}(y)$ will be defined in terms of itself. Based on that, the first derivative operational matrix of MSCH2-Ps will be created. Then, the k^{th} derivative operational matrix will be inducted.

Theorem 1.*The first derivative of MSCH2-Ps can be interpreted as:*

$$\frac{d}{dy}\psi_{r,j}(y) = \sum_{n=0}^{j} \left(4\, e_n\, \eta_j\, R_{j,n} + \xi_{r,j}(y)\right) \psi_{r,j}(y), \tag{19}$$

where:

$$e_n = \frac{n+1}{\beta-\alpha}, \tag{20}$$

$$\xi_{n,j}(y) = r\left(\frac{1}{y-\alpha} - \frac{1}{\beta-y}\right), \tag{21}$$

$$\eta_j = \begin{cases} 0 & j = 1, \\ 1 & otherwise, \end{cases} \tag{22}$$

$$R_{j,n} = \begin{cases} 1 & n+j\ odd, \\ 0 & n+j\ even. \end{cases} \tag{23}$$

*Proof.*This theorem can be proved by using mathematical induction: at $j = 0$

$$\psi'_{r,0}(y) = -r\left[(\beta-y)(y-\alpha)\right]^{r-1}(2y-\alpha-\beta). \tag{24}$$

Consider that the summation of Eq.(19) is true at $j = v$. So, by differentiate (Eq.1) of index $j = v-1$

$$\begin{aligned} \frac{d}{dy}\psi_{r,v+1}(y) &= \frac{4}{\beta-\alpha}\psi_{r,v} \\ &+ 2\left(\frac{2y-\alpha-\beta}{\beta-\alpha}\right) \times \\ &\sum_{n=0}^{v} \left(4\, e_n\, \eta_v\, R_{v,n} + \xi_{r,v}(y)\right)\psi_{r,v}(y) \\ &- \sum_{n=0}^{v-1} \left(4\, e_n\, \eta_{v-1}\, R_{v-1,n} + \xi_{r,v-1}(y)\right)\psi_{r,v-1}(y) \end{aligned} \tag{25}$$

By simplifying the previous equation, the theorem was proved.

On the other hand, the derivative of $\psi_{r,j}(y)$ can be formed as a matrix.

Corollary 1.*The first derivative of $\psi_{r,j}(y)$ can be expressed as:*

$$\psi'(y) = M.\psi(y), \tag{26}$$

where $\psi'(y) = [\psi'_{r,0}(y), \psi'_{r,1}(y), \cdots, \psi'_{r,N}(y)]^T$, $\psi(y) = [\psi_{r,0}(y), \psi_{r,1}(y), \cdots, \psi_{r,N}(y)]^T$, and $M = (m_{jn})_{j,n=0}^{N}$ is the $(N+1 \times N+1)$ square matrix such that:

$$m_{jn} = \begin{cases} 0 & n > j \\ 4\, e_n\, \eta_j\, R_{j,n} + \xi_{r,j}(y) & otherwise \end{cases} \tag{27}$$

For example, if $N = 3$ and $r = 2$

$$M = \begin{bmatrix} \frac{2}{y-\alpha}+\frac{2}{y-\beta} & 0 & 0 & 0 \\ \frac{4}{\beta-\alpha} & \frac{2}{y-\alpha}+\frac{2}{y-\beta} & 0 & 0 \\ 0 & \frac{8}{\beta-\alpha} & \frac{2}{y-\alpha}+\frac{2}{y-\beta} & 0 \\ \frac{4}{\beta-\alpha} & 0 & \frac{12}{\beta-\alpha} & \frac{2}{y-\alpha}+\frac{2}{y-\beta} \end{bmatrix}. \tag{28}$$

Similar to for $N = 3$ and $r = 3$

$$M = \begin{bmatrix} \frac{3}{y-\alpha}+\frac{3}{y-\beta} & 0 & 0 & 0 \\ \frac{4}{\beta-\alpha} & \frac{3}{y-\alpha}+\frac{3}{y-\beta} & 0 & 0 \\ 0 & \frac{8}{\beta-\alpha} & \frac{3}{y-\alpha}+\frac{3}{y-\beta} & 0 \\ \frac{4}{\beta-\alpha} & 0 & \frac{12}{\beta-\alpha} & \frac{3}{y-\alpha}+\frac{3}{y-\beta} \end{bmatrix}. \tag{29}$$

Corollary 2.*The k^{th} derivative of $\psi(y)$ can be expressed as:*

$$\psi^{(k)}(y) = [M \, . \, \psi(y)]^{(k)}, \quad k = 1, \cdots, N. \tag{30}$$

After constructing the operation matrix of the derivative of $\psi(y)$, we can used it via two spectral methods; Galerkin and Tau methods. These will be presented in the next section.

4 The two Methods for solving BVPs

Assumethe BVP of the even-order p:

$$Z^{(p)}(y) = \mathbb{F}\left(y, Z(y), Z'(y), ..., Z^{(p-1)}\right), \tag{31}$$

where $y \in [\alpha, \beta]$. The homogeneous initial and boundary conditions are:

$$\begin{aligned} Z(\alpha) = Z'(\alpha) = Z''(\alpha) = \cdots = U^{(\frac{p}{2}-1)}(\alpha) = 0, \\ Z(\beta) = U'(\beta) = Z''(\beta) = \cdots = Z^{(\frac{p}{2}-1)}(\beta) = 0. \end{aligned} \tag{32}$$

Considering the approximate solution of Eq.(31) as:

$$Z(y) \simeq \sum_{j=0}^{N} c_j \psi_{r,j}(y). \tag{33}$$

Applying Theorem (1) and Corollary (2) to Eq.(31) and determining its residual:

$$\Re(y) = \sum_{j=0}^{N} c_j \psi_{r,j}^{(p)}(y) - \mathbb{F}\left(y, \sum_{j=0}^{N} c_j \psi_{r,j}(y), \sum_{j=0}^{N} c_j \psi'_{r,j}(y), ..., \sum_{j=0}^{N} c_j \psi_{r,j}^{(p-1)}(y)\right). \tag{34}$$

4.1 Galerkin spectral method via MSCH2-Ps (MSCH2-Gal)

As we know, the basis functions should be verified the initial and boundary conditions to meet the use of Galerkin method. From Definition (1), the basis function $\psi_{r,j}(y)$ and its derivative equal zero at α and β for suitable choice of r.

Collocating the residual (34) by $N+1$ points, y_s, and applying the Galerkin method to get:

$$\Re(y_s) = 0; \quad r = 0, 1, \cdots, N, \tag{35}$$

to get system of $(N+1)$ equations.: The collocation point $y_s \in [\alpha, \beta]$ were chosen as the equidistant points, zeros of SCH2-Ps, or any suitable points. Any numerical solver may be used to solve the previous system to find the constant c_j.

4.2 Tau spectral method via MSCH2-Ps (MSCH2-Tau)

In the Tau method, the algebraic system will be generated by solving the Tau integral:

$$\int_{\alpha}^{\beta} \Re(y)\phi_j(y)W(y)dy = 0, \quad j = 0, 1, \cdots, N - BC. \tag{36}$$

where $W(y)$ is any trial function, $W(y)$ be any suitable weight function, and BC is the number of the non-zero initial and boundary conditions. We choose the trial function similar to the base function and the weight function will specified as $W(x) = \sqrt{(y-\alpha)^{1-2r}(\beta-y)^{1-2r}}$ So, 36 can be written as:

$$\int_{\alpha}^{\beta} \Re(y)\psi_{r,j}(y)W(y)dy = 0, \quad j = 0, 1, \cdots, N. \tag{37}$$

The $N+1$ algebraic system (37) of $N+1$ unknowns c_i can be solved using any appreciate method

The homogeneity of initial and boundary conditions is the primary condition to use the investigated polynomials. However, there are some BVPs whose conditions are non-homogeneous. The following remark fix the non-homogeneous conditions.

*Remark.*The non-homogeneous initial and boundary conditions can be transformed into homogeneous as:

$$z(y) = Z(y) + \sum_{v=0}^{p-1} G_v y^v, \tag{38}$$

such that

$$\begin{aligned} z(\alpha) = z'(\alpha) = z''(\alpha) = \cdots = z^{(\frac{p}{2}-1)}(\alpha) = 0, \\ z(\beta) = z'(\beta) = z''(\beta) = \cdots = z^{(\frac{p}{2}-1)}(\beta) = 0, \end{aligned} \tag{39}$$

where, G_v are constants determined by solving Eqs.(38,39).

In the next sections, some BVPs will solved using MSCH2-Ps via the two introduced spectral methods.

5 Numerical Examples

*Example 1.*Consider fourth-order linear BVP:

$$\begin{aligned} &Z^{(4)}(y) = Z(y) + Z''(y) + e^y(y-3), \\ &Z(0) = 1, Z(1) = 0, Z''(0) = -1, Z''(1) = -2e, \end{aligned} \tag{40}$$

and its exact solution $Z(y) = (1-y)e^y$. The example's initial and boundary conditions are non-homogeneous, so we need to convert it to be homogeneous at $r = 2$. Table (1) shows the absolute error (AE) of the two proposed methods and other method at $N = 5$. While Fig (1) shows the stability of the solution using MSCH2-Gal

*Example 2.*Consider fourth-order linear BVP:

$$\begin{aligned} Z^{(4)}(y) =& Z(y) - 4(2y\cos(y) + 3\sin(y)), \quad y \in [0,1], \\ & Z(0) = 0, \; Z(1) = 0, \\ & Z''(0) = 0, \; Z''(1) = -2\sin(1) + 4\cos(1), \end{aligned} \tag{41}$$

and its exact solution $Z(y) = (y^2 - 1)\sin(y)$.

To satisfy the homogeneity, we use Remark (4.2) at $r = 3$. Table (2) compares the two proposed methods and another author's method at $N = 5$.

Table 1: The AE Example 1 at $N = 5$.

y	MSCH2-Gal	MSCH2-Tau	[38]
0.1	$2.76 * 10^{-10}$	$3.62 * 10^{-12}$	$7.69 * 10^{-10}$
0.2	$4.90 * 10^{-10}$	$9.95 * 10^{-11}$	$1.46 * 10^{-9}$
0.3	$6.48 * 10^{-10}$	$1.40 * 10^{-10}$	$2.02 * 10^{-9}$
0.4	$7.53 * 10^{-10}$	$1.13 * 10^{-11}$	$2.39 * 10^{-9}$
0.5	$7.93 * 10^{-10}$	$1.13 * 10^{-10}$	$2.53 * 10^{-9}$
0.6	$7.77 * 10^{-10}$	$1.83 * 10^{-11}$	$2.44 * 10^{-9}$
0.7	$6.89 * 10^{-10}$	$1.43 * 10^{-10}$	$2.10 * 10^{-9}$
0.8	$5.38 * 10-10$	$1.04 * 10^{-10}$	$1.55 * 10^{-9}$
0.9	$3.11 * 10^{-10}$	$3.20 * 10^{-12}$	$8.27 * 10^{-9}$

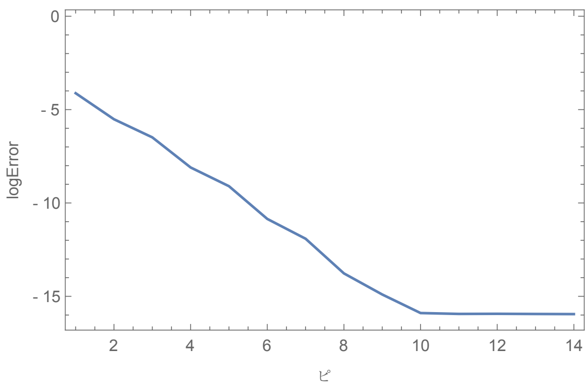

Fig. 1: The log error via MSCH2-Gal of Example 1.

Table 2: The AE for Example 2 at $N = 5$.

y	MSCH2-Gal	MSCH2-Tau	[38]
0.1	$1.27 * 10^{-13}$	$6.00 * 10^{-14}$	$1.54 * 10^{-9}$
0.2	$8.76 * 10^{-13}$	$1.05 * 10^{-13}$	$2.95 * 10^{-9}$
0.3	$1.32 * 10^{-12}$	$5.11 * 10^{-13}$	$4.10 * 10^{-9}$
0.4	$1.56 * 10^{-12}$	$3.32 * 10^{-14}$	$4.88 * 10^{-9}$
0.5	$1.73 * 10^{-12}$	$6.48 * 10^{-13}$	$5.21 * 10^{-9}$
0.6	$1.59 * 10^{-12}$	$6.05 * 10^{-14}$	$5.04 * 10^{-9}$
0.7	$1.38 * 10^{-12}$	$5.29 * 10^{-13}$	$4.36 * 10^{-9}$
0.8	$9.23 * 10-13$	$1.26 * 10^{-13}$	$3.22 * 10^{-9}$
0.9	$1.21 * 10^{-13}$	$6.29 * 10^{-14}$	$1.72 * 10^{-9}$

Example 3. Consider the Lane–Emden–Fowler equation of the first type:

$$Z^{(4)}(y)+\frac{4}{y}Z^{(3)}(y) = -Z^m(y), \quad y \in [0,1], \qquad Z(0) = Z''(0) = 0 \tag{42}$$

and its exact solution at $m = 0$ is $Z(y) = 1 - \frac{1}{120}\,y^4$. In this example, the initial conditions are homogeneous. However, the conditions at the $y = 1$ are non-homogeneous. The convert all conditions to homogeneous conditions, Remark (4.2) at $r = 2$. Let:

$$Z(y)_1 = c_0\psi_{2,0}(y) + c_1\psi_{2,1}(y). \tag{43}$$

Applying MSCH2-Gal to get the algebraic system:

$$\begin{aligned} 120\ c_0 + 720\ c_1 &= -1 \\ 120\ c_0 + 120\ c_1 &= -1 \end{aligned} \tag{44}$$

Solving the previous system to get $c_0 = \frac{-1}{120}$ and $c_1 = 0$. Substitute into Eq.(43) to get $Z_1(y) = -\frac{y^4}{120} + \frac{y^3}{60} - \frac{y^2}{120}$, which is the exact solution of the homogeneous BVP. The same exact solution can be achieved by using MSCH2-Tau.

Example 4. Consider the tenth-order BVP:

$$Z^{(10)}(y) = e^{-y}(Z(y))^2, \quad y \in [0,1] \qquad Z^{(2m)}(0) = 1, Z^{(2m)}(1) = e, \tag{45}$$

where $m = 0, 1, 2, 3, 4$. Its exact solution is $Z(y) = e^y$. Table (3) shows the accuracy of MSCH2-Gal.

Table 3: The AE for Example 4.

y	MSCH2-Gal	[39]
0	0	0
0.2	$6.52 * 10^{-18}$	$7.37 * 10^{-7}$
0.4	$3.07 * 10^{-16}$	$1.18 * 10^{-6}$
0.6	$4.26 * 10^{-15}$	$1.26 * 10^{-6}$
0.8	$2.63 * 10^{-14}$	$9.11 * 10^{-7}$
1	$1.04 * 10^{-13}$	0

6 Conclusions

In this study, we introduce a new set of orthogonal polynomials called MSCH2-Ps. These polynomials are generated by shifting the second kind of Chebyshev polynomials. The paper investigates and proves some meaningful relationships of MSCH2-Ps before constructing the operational matrix of the k^{th} derivative. The resulting matrix is then used with the Galerkin and Tau methods to solve even-order BVPS. The accuracy and efficiency of the presented methods are demonstrated through the successful solution of several even-order BVPs, with results compared to those obtained using other techniques.

Acknowledgment

The authors would like to thank the anonymous reviewers for carefully reading the article and for their constructive and valuable comments, which have improved the paper's present form. The authors also sincerely thank the Helwan School of Numerical Analysis in Egypt (HSNAE) members for their valuable effort and support.

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