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# ARA-Homotopy Perturbation Technique with Applications

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**Abstract:** In this study, we propose a novel combination method between the ARA integral transform and the homotopy perturbation approach to solve systems of nonlinear partial differential equations. The difficulty arising in solving nonlinear partial differential equations could simply be overcome by using He's polynomials during the application of the new method. The proposed technique can provide the solutions of the target problems without pre-assumptions or restrictive constrains in addition to avoiding the round-off errors. The efficiency of the new method is illustrated by applying it to solve different examples of systems of nonlinear partial differential equations. We discuss three interesting applications and solve them by the new approach, called ARA-homotopy perturbation method and get exact solutions, also the results are illustrated in figures.

**Keywords:** ARA transform; Homotopy analysis technique; Nonlinear partial differential equations; System of partial differential equations; Perturbation Method

## 1 Introduction

Nonlinear problems are equations in which the changes in the outputs are not proportional to the changes in the inputs. Nonlinear problems have reached much attention and interest by mathematicians, physicists, engineers, biologists and many other scientists, because most systems are originally nonlinear in their nature. In the last years, there have been increasing interests of researchers and scientists in dealing with nonlinear issues, such as solid-state physics, plasma physics, fluid mechanics and other dynamical systems. In many different areas of science and engineering, the purpose of analyzing nonlinear systems is to get precise or numerical solution of nonlinear partial differential equations (PDEs). Most of the solutions of the new nonlinear PDEs cannot be obtained in a closed form. Therefore, numerical techniques have been used extensively to deal with these equations. Several approximate methods have been proposed such as Adomian's decomposition method [1, 2], differential transformation method [3,4,5,6,7,8] and homotopy perturbation method [9,10,11,12,13,14].

The homotopy analysis technique was proposed in 1992 by Liao [15]. It is a semi-analytical process that is used to solve nonlinear differential equations. The homotopy analysis technique utilizes the topological concept of the homotopy to generate a convergent series solution for nonlinear systems. This is enabled by using a homotopy-Maclaurin series to handle the nonlinearities in the system. In the recent past, the homotopy analysis method has been used to solve a large wide of nonlinear differential equations in physics, engineering and finance. For example, nonlinear heat transfer [16], the solution of limit cycle of nonlinear dynamical equations [17], the Poisson–Boltzmann equation for semiconductor devices [18] and the option pricing under stochastic volatility [19].

The homotopy perturbation method was developed in 1998 by He [20,21,22,23]. The method was suggested for solving both linear and nonlinear differential equations, with initial and boundary value problems. It is based on merging two techniques, the concept of homotopy and the perturbation technique. The homotopy perturbation method was formulated by using the

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capability of the homotopy to overcome the difficulties arising in calculations in addition to the simplicity and easy execution of the perturbation techniques.

Inspiration and motivation from the ongoing research are developing a new method for solving system of nonlinear PDEs.

The new technique is based on combining the newly proposed ARA transform [24] with the homotopy perturbation method. The proposed algorithm can express the solution in a form of a fast convergent series which may lead to solve PDE in a closed form to the exact solution. The advantages of the new method are obvious in the ability of merging two powerful methods for obtaining exact solutions of nonlinear PDEs with simple steps and less computations in comparisons to other numerical methods. The simplicity of applying ARA transform and its merits in handling the singularities and easy computations makes the new method more powerful and applicable.

The rest of the paper is organized as follows. The ARA transform and some main properties are presented in Section 2. A brief description of the homotopy perturbation method is given in Section 3. In Section 4, we apply the proposed technique on several nonlinear PDEs.

## 2 ARA transform

In 2020, Saadeh and others presented a new integral transform called ARA transform [24], this transform has many properties that makes it a very powerful transform, that could be applied to solve various kinds of problems, by introducing double transform including it, or by merging it to other numerical methods such as residual power series method [25,26,27,28,29,30]. In this section, we present the definition and the basic properties of ARA transform that will be implemented in this study.

**Definition 1.** *The ARA integral transform of order $n$ of a continuous function $f(t)$ on the interval $(0,\infty)$ is defined as*

$$\mathcal{G}_n[f(t)](s) = F(n,s) = s\int_0^\infty t^{n-1}e^{-st}f(t)dt, \quad s>0.$$

*In this research, we consider $\mathcal{G}_1[f(t)]$, ARA transform of order one defined as*

$$\mathcal{G}_1[f(t)](s) = F(s) = s\int_0^\infty e^{-st}f(t)dt, \quad s>0.$$

*For simplicity, let us denote $\mathcal{G}_1[f(t)]$ by $\mathcal{G}[f(t)]$. The inverse ARA transform is given by*

$$\mathcal{G}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\,ds = f(t).$$

**Theorem 1.** **(Existence conditions)**. *Let $f(t)$ be a piecewise continuous in a finite interval $0\le t\le\alpha$ and satisfies*

$$\left|t^{n-1}f(t)\right| \le Me^{\alpha t},$$

*where $M$ is positive constant, then the ARA transform of order $n$ of the function $f(t)$ exists for all $s>\alpha$.*

**Proof of Theorem 1.** The definition of ARA transform, yields

$$|F(n,s)| = \left|s\int_0^\infty t^{n-1}e^{-st}f(t)dt\right|.$$

Using the property of improper integral, we get

$$\begin{aligned}
&|F(n,s)|\\
&= \left|s\int_0^\infty t^{n-1}e^{-st}f(t)dt\right| \le s\left|\int_0^\infty t^{n-1}e^{-st}f(t)dt\right|\\
&\le s\int_0^\infty e^{-st}\left|t^{n-1}f(t)\right|dt \le s\int_0^\infty e^{-st}Me^{\alpha t}dt\\
&= sM\int_0^\infty e^{-(s-\alpha)t}dt = \frac{sM}{s-\alpha}.
\end{aligned}$$

The integral is convergent for all $s>\alpha$. Thus, $\mathcal{G}_{n+1}[f(t)]$ is also exists.

Now, some main properties and characteristics of ARA transform of order one are stated. Assume that $F(s)=\mathcal{G}[f(t)]$ and $G(s)=\mathcal{G}[g(t)]$ and $a,b\in\mathbb{R}$. Then, we have

- $\mathcal{G}[af(t)+bg(t)] = a\mathcal{G}[f(t)] + b\mathcal{G}[g(t)]$.
- $\mathcal{G}^{-1}[aF(s)+bG(s)] = a\mathcal{G}^{-1}[F(s)] + b\mathcal{G}^{-1}[G(s)]$.
- $\mathcal{G}[t^\alpha] = \frac{\Gamma(\alpha+1)}{s^\alpha}$, $\alpha>0$.
- $\mathcal{G}[e^{at}] = \frac{s}{s-a}$, $a\in\mathbb{R}$.
- $\mathcal{G}[\sin at] = \frac{as}{s^2+a^2}$, $a\in\mathbb{R}$.
- $\mathcal{G}[\cos at] = \frac{s^2}{s^2+a^2}$, $a\in\mathbb{R}$.
- $\mathcal{G}[\sinh at] = \frac{as}{s^2-a^2}$, $a\in\mathbb{R}$.
- $\mathcal{G}[\cosh at] = \frac{s^2}{p^2-a^2}$, $a\in\mathbb{R}$.
- $\mathcal{G}[f\prime(t)] = s\,F(s) - sf(0)$.
- $\mathcal{G}\left[f^{(n)}(t)\right] = s^nF(s) - \sum_{k=0}^{n-1} p^{n-k}f^{(k)}(0)$.

For more details about the previous results, the reader can see [24].

*Remark.* Let $u(x,t)$ be a piece wise continuous function of the variables $x$ and $t$, in which ARA transform exists. Then

i. The ARA transform with respect to $t$ for the functions of several variables, can be expressed as follows:
  - $\mathcal{G}_t[u(x,t)] = U(x,s) = s\int_0^\infty e^{-ts}u(x,t)dt$.
  - $\mathcal{G}_t[u(x,y,t)] = U(x,y,s) = s\int_0^\infty e^{-ts}u(x,y,t)dt$.

ii. The ARA transform $\mathcal{G}_t$, of the partial derivatives can be expressed as:
  - $\mathcal{G}_t[u_t(x,t)] = sU(x,s) - su(x,0)$.
  - $\mathcal{G}_t[u_{tt}(x,t)] = s2U(x,s) - s2u(x,0) - sut(x,0)$.
  - $\mathcal{G}_t[u_x(x,t)] = \frac{\partial}{\partial x}U(x,s)$.
  - $\mathcal{G}_t[u_{xx}(x,t)] = \frac{\partial^2}{\partial x^2}U(x,s)$.

- $\mathscr{G}_t\left[u_t(x,y,t)\right] = sU(x,y,s) - su(x,y,0).$
- $\mathscr{G}_t\left[u_{tt}(x,y,t)\right] = s^2U(x,y,s) - s^2u(x,y,0) - su_t(x,y,0).$

## 3 Homotopy Perturbation Method

Let $f$ and $g: X \to Y$ be two continuous mappings of the topological space $X$ into the topological space $Y$, we call $f$ to be homotopic to $g$, if we can define a continuous mapping $\phi: X \times [0,1] \to Y$ such that $\phi(x,0) = f(x)$ and $\phi(x,1) = g(x)$, for each $x \in X$. Hence, the mapping is called a homotopy between $f$ and $g$.

To illustrate the main idea of the homotopy perturbation method, let us consider the following differential type equations

$$L(u) = 0, \tag{1}$$

where $L$ is a differential operator and $u$ is the unknown function. Now, we define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)\phi(u) + pL(u), \tag{2}$$

where $\phi(u)$ is a functional operator with initial guess $v_0$, and $p$ is a parameter between 0 and 1.

Hence, we have

$$H(u,p) = 0. \tag{3}$$

Thus, the HPM uses the embedding parameter $p$ as a small parameter and write the solution as a power series as follows

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots. \tag{4}$$

If $p \to 1$, then we can express the solution as

$$u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

We suppose Eq.(5) has a unique solution and by comparisons of the similar powers of $p$, we get the desired solutions of various orders [31,32,33,34].

The method assumes that the nonlinear term $N(u)$ can be expressed as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p\,H_1 + p^2H_2 + p^3H_3 + \dots.$$

Where $H_n$ are He's polynomials [31,32], which can be calculated by using the formula

$$H_n(u_0 + u_1 + u_2 + \ldots + u_n) = \frac{1}{n!}\frac{\partial^n}{\partial x^n}\left(N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right)_{p=0}, \quad n = 0,1,2,\cdots.$$

## 4 Algorithm of ARA-Homotopy Perturbation Approach

To illustrate the ARA- Homotopy Perturbation Method, let us consider the nonlinear PDE of the form

$$\mathscr{L}(u(x,t)) + \mathscr{N}(u(x,t)) = g(x,t), \tag{6}$$

with suitable initial conditions, where $\mathscr{L}$ is a differential linear operator and $\mathscr{N}$ is a nonlinear operator. The first step is to apply ARA transform on Eq. (6), to get

$$\mathscr{G}_t\left[\mathscr{L}(u(x,t))\right] + \mathscr{G}_t\left[\mathscr{N}(u(x,t))\right] = \mathscr{G}_t\left[g(x,t)\right],$$

which implies

$$\mathscr{G}_t\left[\mathscr{L}(u(x,t))\right] + \mathscr{G}_t\left[\mathscr{N}(u(x,t)) - g(x,t)\right] = 0, \tag{7}$$

with simple calculations and using the given conditions we can obtain an equation of the form

$$U(x,s) = h(x,s) + k(s)\mathscr{G}_t\left[\mathscr{N}(u(x,t)) - g(x,t)\right], \tag{8}$$

where $h(x,s)$ and $k(s)$ are functions to be determined depending on the linear operator $\mathscr{L}$.
Now, operating the inverse ARA-transform $\mathscr{G}_t$, Eq. (8) becomes

$$\begin{aligned} u(x,t) &= \mathscr{G}_t^{-1}\left[h(x,s)\right] \\ &+ \mathscr{G}_t^{-1}\left[k(s)\mathscr{G}_t\left[\mathscr{N}(u(x,t)) - g(x,t)\right]\right]. \end{aligned} \tag{9}$$

Then, apply the homotopy perturbation method and define the He's polynomials as presented in the previous section to get our result.

## 5 Numerical Applications

PDEs play an important role in modeling physical phenomena and describing some engineering's issues. So that many mathematicians have investigated the solutions of such problems. In this section, we introduce three interesting examples of systems of nonlinear PDEs and solve them by the proposed method.

**Application 1** *Let us consider the following system of nonlinear PDEs*

$$\begin{cases} u_t(x,t) + v(x,t)u_x(x,t) + u(x,t) = 1, \\ v_t(x,t) + u(x,t)v_x(x,t) - v(x,t) = -1. \end{cases} \tag{10}$$

*With the initial conditions (ICs)*

$$u(x,0) = e^x, \qquad v(x,0) = e^{-x}.$$

Applying ARA transform on both sides of Eq. (10) subject to the ICs, we have

$$\begin{cases} \mathscr{G}_t\left[u_t(x,t)\right] = -\mathscr{G}_t\left[v(x,t)u_x(x,t) + u(x,t) - 1\right], \\ \mathscr{G}_t\left[v_t(x,t)\right] = -\mathscr{G}_t\left[u(x,t)v_x(x,t) - v(x,t) + 1\right]. \end{cases}$$

Using the differential property of the ARA transform, we have:

$$\begin{cases} sU(x,s) - su(x,0) = -\mathcal{G}_t\left[v(x,t)\,u_x(x,t) + u(x,t) - 1\right], \\ sV(x,s) - sv(x,0) = -\mathcal{G}_t\left[u(x,t)\,v_x(x,t) - v(x,t) + 1\right]. \end{cases}$$

Rearranging the terms, we have

$$\begin{cases} U(x,s) = e^x - \frac{1}{s}\mathcal{G}_t\left[v(x,t)\,u_x(x,t) + u(x,t) - 1\right], \\ V(x,s) = e^x - \frac{1}{s}\mathcal{G}_t\left[u(x,t)\,v_x(x,t) - v(x,t) + 1\right]. \end{cases} \quad (11)$$

Applying the inverse ARA transform to Eq. (11), we have

$$\begin{cases} u(x,t) = e^x - \mathcal{G}_t^{-1}\left[\frac{1}{s}\mathcal{G}_t\left[v(x,t)\,u_x(x,t) + u(x,t) - 1\right]\right], \\ v(x,t) = e^{-x} - \mathcal{G}_t^{-1}\left[\frac{1}{s}\mathcal{G}_t\left[u(x,t)\,v_x(x,t) - v(x,t) + 1\right]\right]. \end{cases} \quad (12)$$

Now, we apply the homotopy perturbation technique, to get

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = e^x - p\left(\mathcal{G}_t^{-1}\left[\frac{1}{s}\mathcal{G}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = e^{-x} - p\left(\mathcal{G}_t^{-1}\left[\frac{1}{s}\mathcal{G}\left[\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right). \end{cases} \quad (13)$$

Where $H_n(u)$ and $H(v)$ are He's polynomials that express the nonlinear terms as follows. Thus,

$$H_n(u) : p\left[v(x,t)\,u_x(x,t) + u(x,t) - 1\right] = 0,$$
$$H_n(v) : p\left[u(x,t)\,v_x(x,t) - v(x,t) + 1\right] = 0,$$

where

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots,$$
$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots.$$

Now, we can express some components of He's polynomials, as follows

$$H_0(u) = v_0u_{0x} + u_0 - 1,$$
$$H_0(v) = u_0v_{0x} - v_0 + 1,$$
$$H_1(u) = v_0u_{1x} + v_1u_{0x} + u_1,$$
$$H_1(v) = u_0v_{1x} + u_1v_{0x} - v_1.$$
$$\vdots$$

By making comparisons between the coefficients of $p$ with same power to obtain

$$p^0: \quad u_0(x,t) = e^x, \quad v_0(x,t) = e^{-x},$$

thus

$$H_0(u) = v_0u_{0x} + u_0 - 1 = e^{-x}(e^x) + e^x - 1 = e^x,$$
$$H_0(v) = u_0v_{0x} - v_0 + 1 = e^x\left(-e^{-x}\right) - e^{-x} + 1 = e^{-x}.$$

$$\begin{aligned} p^1: \quad u_1(x,t) &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[H_0(u)\right]\right] \\ &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[e^x\right]\right] = -te^x, \\ v_1(x,t) &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[H_0(v)\right]\right] \\ &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[-e^{-x}\right]\right] = te^{-x}, \end{aligned}$$

thus,

$$\begin{aligned} H_1(u) &= v_0u_{1x} + u_{0x}v_1 + u_1 = e^{-x}\left(-te^x\right) + e^x\left(te^{-x}\right) \\ &\quad - te^x = -te^x, \\ H_1(v) &= u_0v_{1x} + v_{0x}u_1 - v_1 = e^x\left(-te^{-x}\right) - e^x\left(-te^{-x}\right) \\ &\quad - te^{-x} = -te^x. \end{aligned}$$

$$\begin{aligned} p^2: \quad u_2(x,t) &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[H_1(u)\right]\right] \\ &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[-te^x\right]\right] = \frac{t^2}{2}e^x, \\ v_2(x,t) &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[H_1(v)\right]\right] \\ &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[-te^x\right]\right] = \frac{t^2}{2}e^{-x}. \end{aligned}$$

Similarly, we can find

$$\begin{aligned} p^3: \quad u_3(x,t) &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[H_2(u)\right]\right] = -\frac{t^3}{3!}e^x, \\ v_3(x,t) &= -\mathcal{G}^{-1}\left[\frac{1}{s}\mathcal{G}\left[H_2(v)\right]\right] = \frac{t^3}{3!}e^{-x}. \\ &\vdots \end{aligned}$$

Therefore, the solutions $u(x,t)$, $v(x,t)$ of Application 1, are given by

$$u(x,t) = e^x\left(1 - t + \frac{t^2}{2}e^x - \frac{t^3}{3!} + \dots\right) = e^{x-t},$$
$$v(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2}e^x + \frac{t^3}{3!} + \right) = e^{t-x}.$$

The graphics and the contour solutions $u(x,t)$ and $v(x,t)$ at $t$ and $x \in [1,2]$ of Application 1 are presented in Figure 1 and Figure 2 below.

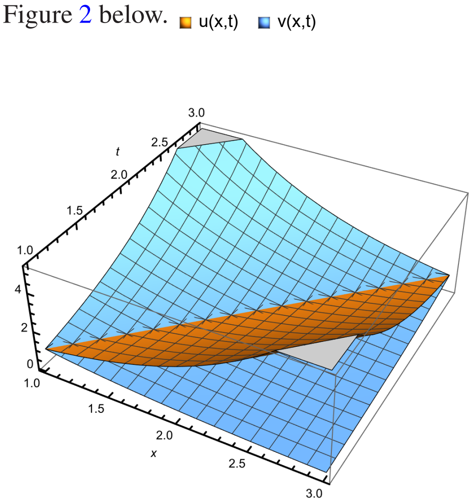


**Fig. 1:** The 3D graph of the solutions $u(x,t)$ and $v(x,t)$ of Application 1.

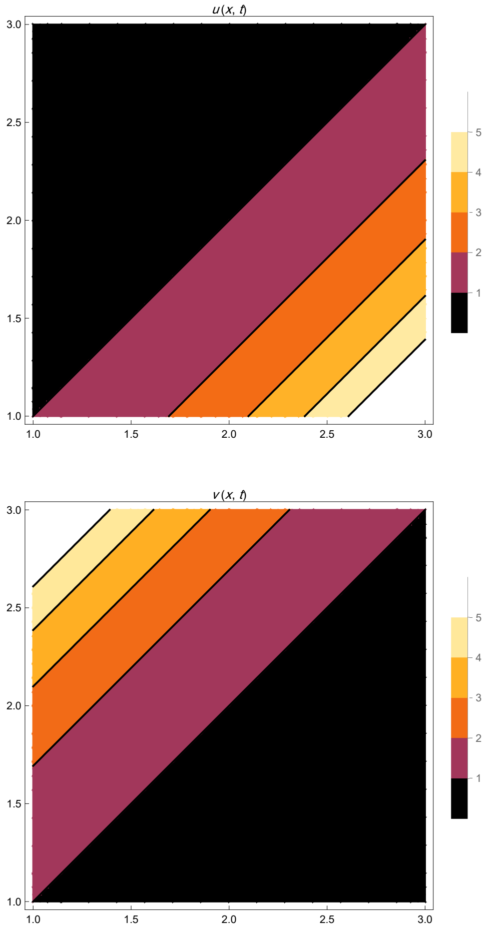


**Fig. 2:** The contour graph of the solutions $u(x,t)$ and $v(x,t)$ of Application 1.

**Application 2** *Consider the following Coupled Burger's system*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) - 2u(x,t)\,u_x(x,t) \\ \quad +(u(x,t)v(x,t))_x = 0, \\ v_t(x,t) - v_{xx}(x,t) - 2v(x,t)\,v_x(x,t) \\ \quad +(u(x,t)v(x,t))_x = 0. \end{cases} \tag{14}$$

*With the ICs*

$$u(x,0) = \sin x, \qquad v(x,0) = \sin x.$$

Applying ARA transform on both sides of Eq. (14) subject to the ICs, we have

$$\begin{cases} U(x,t) = \sin x + \mathscr{G}^{-1}\Big[\frac{1}{s}\mathscr{G}\big[u_{xx}(x,t) \\ \quad + 2u(x,t)\,u_x(x,t) - (u(x,t)\,v(x,t))_x\big]\Big], \\ V(x,t) = \sin x + \mathscr{G}^{-1}\Big[\frac{1}{s}\mathscr{G}\big[v_{xx}(x,t) \\ \quad + 2v(x,t)\,v_x(x,t) - (u(x,t)v(x,t))_x\big]\Big]. \end{cases} \tag{15}$$

Now, we apply the homotopy perturbation method, to obtain

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = \sin x \\ \quad + p\left(\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = \sin x \\ \quad + p\left(\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right). \end{cases} \tag{16}$$

Where $H_n(u)$ and $H_n(v)$ are He's polynomials that express the nonlinear terms as follows Thus,

$$H_n(u) : p\left[u_{xx} + 2u\, u_x - uv_x - u_x v\right] = 0,$$
$$H_n(v) : p\left[v_{xx} + 2v\, v_x - uv_x - u_x v\right] = 0,$$

where

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots,$$
$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots.$$

Now, we can express some components of He's polynomials, as follows

$$H_0(u) = u_{0xx} + 2u_0 u_{0x} - u_0 v_{0x} - u_{0x} v_0,$$
$$H_0(v) = v_{0xx} + 2v_0 v_{0x} - u_0 v_{0x} - u_{0x} v_0,$$
$$H_1(u) = u_{1xx} + 2u_0 u_{1x} + 2u_1 u_{0x} - u_0 v_{1x} - u_1 v_{0x} - u_{0x} v_1 - u_{1x} v_0,$$
$$H_1(v) = v_{1xx} + 2v_0 v_{1x} + 2v_1 v_{0x} - u_0 v_{1x} - u_1 v_{0x} - u_{0x} v_1 - u_{1x} v_0.$$
$$\vdots$$

By making comparisons between the coefficients of $p$ with same power to obtain

$$p^0 : \quad u_0(x,t) = \sin x, \quad v_0(x,t) = \sin x,$$

thus

$$H_0(u) = u_{0xx} + 2u_0 u_{0x} - u_0 v_{0x} - u_{0x} v_0 = -\sin x,$$
$$H_0(v) = v_{0xx} + 2v_0 v_{0x} - u_0 v_{0x} - u_{0x} v_0 = -\sin x.$$

$$p^1 : \quad u_1(x,t) = -\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[H_0(u)\right]\right] = -t\sin x,$$
$$v_1(x,t) = -\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[H_0(v)\right]\right] = -t\sin x,$$

thus,

$$H_1(u) = u_{1xx} + 2u_0 u_{1x} + 2u_1 u_{0x} - u_0 v_{1x} - u_1 v_{0x} - u_{0x} v_1 - u_{1x} v_0 = t\sin x,$$
$$H_1(v) = v_{1xx} + 2v_0 v_{1x} + 2v_1 v_{0x} - u_0 v_{1x} - u_1 v_{0x} - u_{0x} v_1 - u_{1x} v_0 = t\sin x.$$

$$p^2: \quad u_2(x,t) = -\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}[H_1(u)]\right] = \frac{t^2}{2}\sin x,$$
$$v_2(x,t) = -\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}[H_1(v)]\right] = \frac{t^2}{2}\sin x.$$

Similarly, we can find

$$p^3: \quad u_3(x,t) = -\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}[H_2(u)]\right] = -\frac{t^3}{3!}\sin x,$$
$$v_3(x,t) = -\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}[H_2(v)]\right] = -\frac{t^3}{3!}\sin x.$$
$$\vdots$$

Therefore, the solutions $u(x,t)$, $v(x,t)$ of Application 2, are given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots = e^{-t}\sin x,$$
$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \ldots = e^{-t}\sin x.$$

The graphics and the contour solutions $u(x,t)$ and $v(x,t)$ at $t$ and $x \in [1,2]$ of Application 2 are presented in Figure 3 and Figure 4 below.

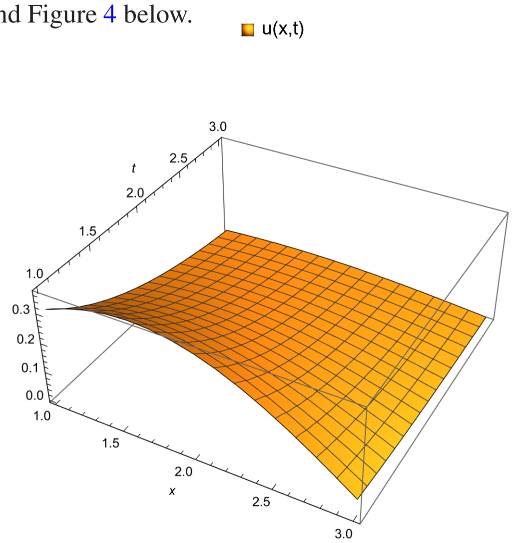


**Fig. 3:** The 3D graph of the solution $u(x,t)$ Application 2.

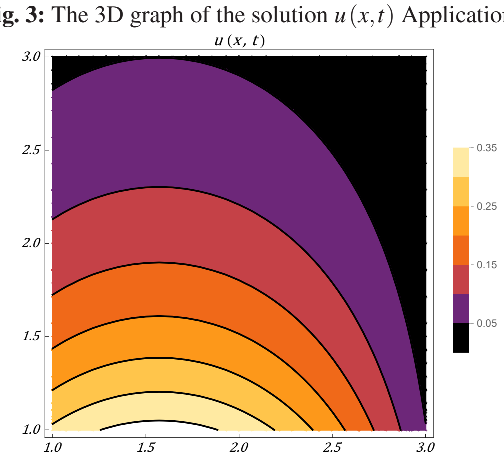


**Fig. 4:** The contour graph of the solution $u(x,t)$ of Application 2.

**Application 3** *Consider the following system of nonlinear PDEs*

$$\begin{cases} u_t(x,y,t) + v_x(x,y,t)\,w_y(x,y,t) \\ \quad -v_y(x,y,t)\,w_x(x,y,t) = -u(x,y,t), \\ v_t(x,y,t) + w_x(x,y,t)\,u_y(x,y,t) \\ \quad +w_y(x,y,t)\,u_x(x,y,t) = v(x,y,t), \\ w_t(x,y,t) + u_x(x,y,t)\,v_y(x,y,t) \\ \quad +u_y(x,y,t)\,v_x(x,y,t) = w(x,y,t). \end{cases} \tag{17}$$

*With the ICs*

$$u(x,y,0) = e^{x+y}, \quad v(x,y,0) = e^{x-y}, \quad w(x,y,0) = e^{-x+y}.$$

Applying ARA transform on both sides of Eq. (17) subject to the ICs, we have

$$\begin{cases} \mathscr{G}[u(x,y,t)] = e^{x+y} + \frac{1}{s}\mathscr{G}\big[v_y(x,y,t)\,w_x(x,y,t) \\ \quad -v_x(x,y,t)w_y(x,y,t) - u(x,y,t)\big], \\ \mathscr{G}[v(x,y,t)] = e^{x-y} + \frac{1}{s}\mathscr{G}\big[v(x,y,t) \\ \quad -w_x(x,y,t)\,u_y(x,y,t) - w_y(x,y,t)u_x(x,y,t)\big], \\ \mathscr{G}[w(x,y,t)] = e^{-x+y} + \frac{1}{s}\mathscr{G}\big[w(x,y,t) \\ \quad -u_x(x,y,t)\,v_y(x,y,t) - u_y(x,y,t)v_x(x,y,t)\big]. \end{cases} \tag{18}$$

Applying inverse ARA transform implies that

$$\begin{cases} u(x,y,t) = e^{x+y} + \mathscr{G}^{-1}\Big[\frac{1}{s}\mathscr{G}\big[v_y(x,y,t)\,w_x(x,y,t) \\ \quad -v_x(x,y,t)w_y(x,y,t) - u(x,y,t)\big]\Big], \\ v(x,y,t) = e^{x-y} + \mathscr{G}^{-1}\Big[\frac{1}{s}\mathscr{G}\big[v(x,y,t) \\ \quad -w_x(x,y,t)\,u_y(x,y,t) - w_y(x,y,t)u_x(x,y,t)\big]\Big], \\ w(x,y,t) = e^{-x+y} + \mathscr{G}^{-1}\Big[\frac{1}{s}\mathscr{G}\big[w(x,y,t) \\ \quad -u_x(x,y,t)\,v_y(x,y,t) - u_y(x,y,t)v_x(x,y,t)\big]\Big]. \end{cases} \tag{19}$$

Now, we apply the homotopy perturbation method, to obtain

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,y,t) = e^{x+y} \\ \qquad + p\left(\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \\ \sum_{n=0}^{\infty} p^n v_n(x,y,t) = e^{x-y} \\ \qquad p\left(\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right), \\ \sum_{n=0}^{\infty} p^n w_n(x,y,t) = e^{-x+y} \\ \qquad p\left(\mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[\sum_{n=0}^{\infty} p^n H_n(w)\right]\right]\right). \end{cases} \tag{20}$$

Where $H_n(u)$, $H_n(v)$ and $H_n(w)$ present the He's polynomials of the nonlinear terms as follows Thus,

$$H_n(u) : p\left[v_y w_x - v_x w_y - u\right] = 0,$$
$$H_n(v) : p\left[v - w_x u_y - w_y u_x\right] = 0,$$
$$H_n(w) : p\left[w - u_x v_y - u_y v_x\right] = 0,$$

where

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots,$$
$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots,$$
$$w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots.$$

By making comparisons between the coefficients of $p$ with same power to obtain

$$p^0 : \quad u_0(x,y,t) = e^{x+y},$$
$$v_0(x,y,t) = e^{x-y},$$
$$w_0(x,y,t) = e^{-x+y},$$

thus

$$H_0(u) = -e^{x+y}, \qquad H_0(v) = e^{x-y}, \qquad H_0(w) = e^{-x+y}.$$

$$p^1 : \quad u_1(x,u,t) = \mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[-e^{x+y}\right]\right] = -te^{x+y},$$
$$v_1(x,y,t) = \mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[e^{x-y}\right]\right] = te^{x-y},$$
$$w_1(x,y,t) = \mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[e^{-x+y}\right]\right] = te^{-x+y},$$

thus,

$$H_1(u) = te^{x+y},$$
$$H_1(v) = te^{x-y},$$
$$H_1(w) = te^{-x+y}.$$

$$p^2 : \quad u_2(x,y,t) = \mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[te^{x+y}\right]\right] = \frac{t^2}{2}e^{x+y},$$
$$v_2(x,y,t) = \mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[te^{x-y}\right]\right] = \frac{t^2}{2}e^{x-y},$$
$$w_2(x,y,t) = \mathscr{G}^{-1}\left[\frac{1}{s}\mathscr{G}\left[te^{-x+y}\right]\right] = \frac{t^2}{2}e^{-x+y}.$$
$$\vdots$$

Therefore, the solutions $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ of Application 3, are given by

$$\begin{aligned} u(x,y,t) &= u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \ldots \\ &= e^{x+y-t}, \\ v(x,y,t) &= v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \ldots \\ &= e^{x-y+t}, \\ w(x,y,t) &= w_0(x,y,t) + w_1(x,y,t) + W_2(x,y,t) + \ldots \\ &= e^{-x+y+t}, \end{aligned}$$

The graphics and the contour solutions $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ at $t = 3$ and $x, y \in [1,2]$ of Application 3 are presented in Figure 5 and Figure 6 below.

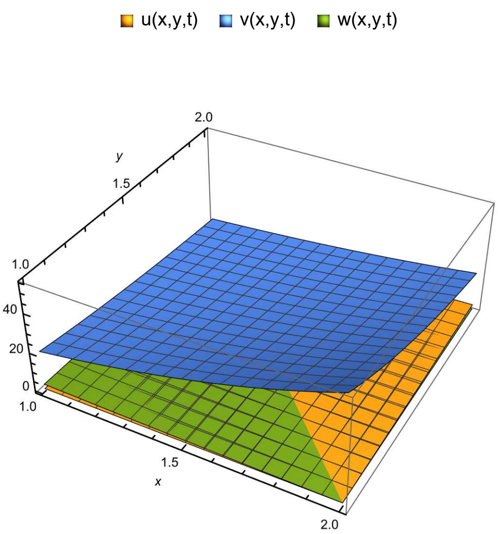


**Fig. 5:** The 3D graph of the solutions $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ of Application 3.

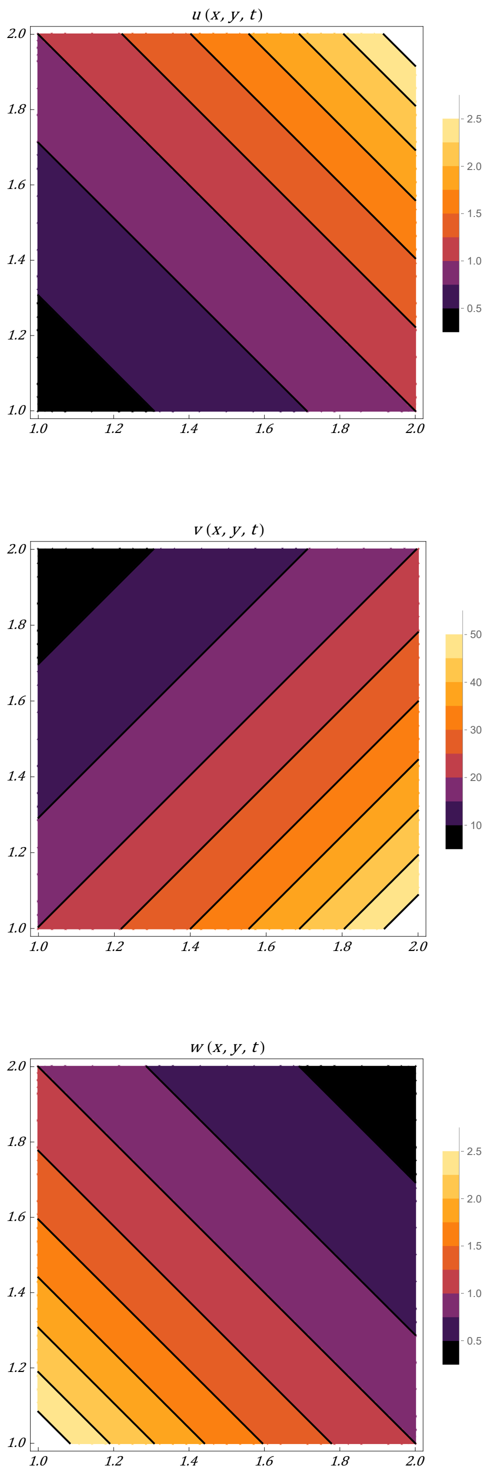


**Fig. 6:** The contour graph of the solutions the solutions $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ of Application 3

## 6 Conclusion

In this study, we introduced a new approach to solve nonlinear systems of PDEs, by combining ARA integral transform with the Adomian's decomposition method. We presented some fundamental properties of ARA transform, and the basic idea of the homotopy perturbation method. The proposed method is applied to solve some examples of systems of PDEs and we obtained the exact solutions and the graph of the solutions are presented. Our goal in this study is achieved, and we proved the efficiency of the method. In the future we intend to solve new fractional differential equations by the new method. In the future, we intend make comparisons with other numerical methods [35,36,37,38] and solve some new fractional models, such as in [39,40,41].

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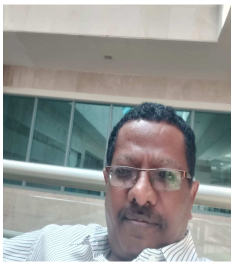

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