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On Weighted Discretized Fréchet-Weibull Distribution with Application to real life data

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Abstract: Weighted distributions have great practical importance in mathematics, probability and statistics. In this study a new discrete distribution which is a weighted version of Discretized Fréchet-Weibull distribution (DFWD), known as Weighted Discretized Fréchet-Weibull distribution (WDFWD), is proposed. It has been shown that the distribution is unimodal, positively skewed and suitable for modelling with overdispersed count data sets. This distribution has bathtub shape and decreasing hazard rate function. Various statistical properties and simulation of the proposed distribution are obtained. The estimation of parameters have been handled by the method of maximum likelihood estimation. Finally, the proposed model has been fitted to three real life data sets to test its goodness of fit. And to show its efficacy it is being compared with Discretized Fréchet-Weibull distribution, discrete generalized Weibull, discrete generalized inverse Weibull, discrete Burr and discrete Rayleigh distribution. It is established that the weighted version of DFWD gives better fit than the parent model DFWD, which indicates the importance of weighted distribution.

Keywords: Weighted distribution, Discretized Fréchet-Weibull distribution, Properties, Simulation, Estimation, Application

1 Introduction

In various studies it is observed that the sampling frames for plant, human, insect, wildlife and fish populations are not well defined, which leads the researchers to be unable to select the sampling units with equal probability. As a result, unless every observation is given an equal probability of being recorded, the observations on individuals in these populations are biased. As such biased data arise in all disciplines of science, the statisticians and researchers have found out solutions for correcting the biases in the recorded observations. In this regard, a standard approach for modeling biased data is given by the weighted distribution theory. The theory of weighted distributions is applied in various research areas related to biomedicine, reliability, ecology and branching processes.

The work of Fisher [1] is accountable for introducing the concept of weighted distributions. In his study a brief discussion on how methods of ascertainment can affect the form of distribution of recorded observations are discussed. Later, it was introduced and formulated in a more general way by Rao [2] with respect to the usual practice of using standard distributions for the purpose of modelling statistical data was found to be inappropriate. He explained when the recorded observations cannot be considered as a random sample from the original distribution then the weighted distributions can be applied. Such situations may occur due to non-observability of some events or a damage caused to original observations resulting in a reduced value or adoption of a sampling procedure which gives unequal chances to the units in original. It is obvious that when there are unequally likely observations in a population, then different events will have unequal probability of getting recorded. To deal with such a population a function is searched for expressing the proportionality in which the events are observed and here comes the role of weight function for this purpose.

Several authors have studied the various weighted probability models considering different forms of weight functions and illustrated their applicability in different fields. Patil and Rao [3] provided a comprehensive survey of examples of weighted distribution and explained how they come into existence in the domain of science. Patil and Rao [4] examined some general models leading to weighted distributions with weight functions not necessarily bounded by unity and studied length biased

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(size biased) sampling with applications to wildlife populations and human families. A detailed account of weight functions is discussed by Patil et al. [5], providing a list of weight functions that can be used to construct various forms of weighted distributions. Khatree [6] discussed the characteristics of many length biased distributions along with the results on preservation of stability and comparisons for weighted and length biased distributions. Gove [7] reviewed some of the results on size-biased distributions in connection to parameter estimation in forestry, with special emphasis on two- and three-parameter Weibull distribution. Saghir et al. [8] discussed a brief review work on weighted distributions available in the literature and presented a characterization criterion using a simple relationship between two truncated moments.

According to Patil and Rao [3], "although the situations that involve weighted distributions seem to occur frequently in various fields, but the underlying concept of weighted distributions as a major stochastic concept does not seem to have been widely recognized". Though many researchers have developed several weighted distributions to model such biased data in both continuous and discrete forms, but unfortunately a handful of work is seen on this topic.

Recently, the development of new weighted distributions has received much attention from the researchers. The weighted version of Generalized Inverse Weibull Distribution was proposed by Mudasir and Ahmad [9]. Bakouch [10] introduced a new discrete distribution called weighted negative binomial Lindley distribution, which is actually the weighted version of two parameter Lindley distribution. Para and Jan [11] proposed the Weighted Pareto type II distribution suitable for handling data from medical science background. Bhati and Joshi [12] introduced weighted geometric distribution for which the Negative Binomial is a limiting distribution. Also this distribution can be viewed as a discrete analog of weighted exponential distribution as discussed by Gupta and Kundu [13]. The Length-Biased weighted Exponentiated Lomax distribution was proposed by Moniem and Diab [14], of which the Length-Biased weighted Lomax distribution as derived by Ahmad et al. [15] is a sub-model. Rather and Subramanian [16] discussed the weighted Sushila distribution with various statistical properties and its applications.

Most recently, Dar et al. [17] proposed the Weighted Gamma-Pareto distribution, by considering the cumulative distribution function of Pareto distribution as a weight function. This distribution is a generalization of some of the well-known probability distribution viz., gamma-Pareto distribution, weighted exponential-Pareto distribution, Pareto distribution, weighted gamma distribution, gamma distribution, weighted exponential distribution, generalized exponential distribution and exponential distribution. Rather and Ozel [18] proposed the Weighted Power Lindley distribution with various statistical properties and its applications. A size-biased discrete Akash distribution with its properties and application to four real life data sets was discussed by Shanker and Sium [19]. Ganaie et al. [20] derived the weighted two parameter quasi Shanker distribution and established the efficiency of the proposed model than the parent model in application to two real life data sets. Ganaie and Rajagopalan [21] developed the Length biased Weighted New Quasi Lindley Distribution and using two applications the supremacy of the length-biased version than the parent distribution and three other competitive distributions are established.

In real application, weighted distributions can arise either because of observations from a sample are recorded with unequal probability by the sampling design or because of unequal probability of detection. When the observations fall either in the non-experimental or non-replicated or non-random categories, then the weighted distributions provide a unified approach for handling such problems. The weighted distributions occur frequently in every field of study including environmental science, reliability, econometrics, social science, biomedical science, human demography, family data, ecology, geology, forestry etc. In this article an attempt has been made to put a light in the theory of weighted distributions that enables us to obtain a solution for the problems of model specification whenever we deal with such kind of biased data.

As already discussed, though the concept of "weighted distribution" has a vast scope and the development of many more such distributions are awaited in the future. But focusing on the variable support on $\mathbf{Z}_+$ (set of positive integers), a handful of works are available in the literature based on weighted distributions. The most popular discrete weighted distribution is the weighted Poisson distribution. Several weighted Poisson distributions have already been studied in the literature (see [22],[3],[4],[5]). Chakraborty and Das [23] discussed some properties of a class of weighted quasi-binomial distribution. Shanker and Mishra [24] discussed size-biased quasi Poisson-Lindley distribution and established its efficiency in comparison to size-biased Poisson-Lindley distribution. Recently, Balakrishnan et al. [25] introduced another weighted Poisson distribution with its application to cure rate models. Weighted Negative Binomial Lindley [10], weighted Geometric [12] and size-biased discrete Akash [19] distributions are some of the researches on the weighted versions of their corresponding discrete distributions.

As it is known that manipulating the data sets leads to loss of information and there are abundance of such discrete data sets which can be fitted better with the newly developed weighted models. In practical situations, we may come across such biased data that would give better results when we treat them with some extreme value distribution rather than the weighted forms of Binomial, Poisson, Negative binomial, Geometric distribution. Moreover, it is also known that the generalised versions of a distribution are always superior than the parent model. Additionally, the lack of weighted distribution for count data also reassures the need of new developments of weighted model corresponding to a discretized distributions for biased extreme value datasets. The Weibull and Fréchet distributions are both applicable to extreme value theory and the generalized form of these distributions, named as Fréchet-Weibull distribution is more efficient than the original models. Since our main intention is to develop a new weighted distribution for count data so here, we shall concentrate on the discrete counterpart of the Fréchet-Weibull distribution. The selection of this distribution is entrenched on its flexible properties.

In reference to the above discussion the main motivation of this article is to make one realize how the weighted distributions enables us to express some random processes even though when we have already a number of existing distributions. The objective of this study is to introduce a new discrete weighted distribution based on the Discretized Fréchet-Weibull distribution. The proposed weighted distribution, named as Weighted Discretized Fréchet-Weibull distribution (WDFWD), is constructed by considering the cumulative distribution function of discrete Weibull distribution as the weight function. One interesting fact of the proposed distribution is that the weight function is so chosen that the total number of parameters is same as the parent distribution. The capability of exhibiting the increasing, decreasing and bathtub shaped hazard rate functions, which are sparingly noticed in count distributions, along with its variable support in the set of positive integers, serves as bounty for this proposed distribution. Besides this distribution is suitable to model with positively skewed data sets and can be used as an alternative for fitting overdispersed count data sets.

The rest of the paper is organized as follows: In Section 2 the development of Discretized Fréchet-Weibull distribution (DFWD) is discussed. The derivation of Weighted Discretized Fréchet-Weibull distribution (WDFWD) is illustrated in Section 3 and the various statistical properties of WDFWD are also derived. Maximum likelihood method for parameters estimation is discussed in Section 4. Simulation of WDFWD is carried out in Section 5. Applicability of this distribution to three real life data sets are illustrated in Section 6. Goodness of fit is carried out as a measure to establish the supremacy of the proposed model in comparison to the parent distribution and four other competitive distributions. Along with the Chi-square statistic, here we have also used the discrepancy coefficient as a measure of goodness of fit. Finally, it is seen that, in all the applications, WDFWD is more efficient than the parent distribution DFWD and the other four competitive distributions. Finally, a conclusive summary of the study has been presented in Section 7.

2 Discretized Fréchet-Weibull distribution (DFWD)

In real-life practice, we may encounter such situations where the continuous life time models are often recorded as discrete random variables rather than measuring on a continuous scale. These situations may happen either because of its inbuilt character or because of the restriction of measuring tools. This leads to the necessity of development of discrete analogue of the extant continuous distributions. Discretization of statistical models provides the researcher a basic field of study to operate with count data from diverse disciplines such as biological, medical and physical sciences, engineering, agriculture and many others.

A number of methods are present in the literature to construct the discretized version of continuous distributions. An encyclopedic survey of the different methods of discretization used to derive the discrete analogues of the continuous one has been presented by Chakraborty [26]. In his study a detailed account of discretized distributions are discussed under the various techniques of discretization available in the relevant works. The discretization technique that uses the survival function of the continuous model is the most used method for constructing the discrete analogues. This method is called the survival function approach of discretization and is defined as follows:

Definition 1. If $S_X(x)$ be the survival function of the considered continuous random variable X, then the random variable $Y = \lfloor X \rfloor =$ largest integer less than or equal to X will have the pmf

$$\begin{aligned} P(Y=k) &= P(k \leq X < k+1) \\ &= P(X \geq k) - P(X \geq k+1) \\ &= S_X(k) - S_X(k+1) \end{aligned} \tag{1}$$

Nakagawa and Osaki [27] was the first to introduce this method to develop the discrete Weibull (DW) distribution. The main advantage of this method is that the form of the survival function is conserved on its integer part, that is $S_Y(k) = S_X(k)$, where k is an integer. So using the form in Eq.(1) it is possible to develop the discrete version of a distribution corresponding to any given continuous distribution.

Teamah et al. [28] introduced a relatively new distribution called Fréchet-Weibull (FW) distribution and studied its statistical properties along with its application to earthquake data sets. Deka et al. [29] also studied this distribution with its application to two data sets related to mechanical engineering. The applications of this distribution showed its efficiency in comparison to the other extensions of Weibull distribution.

The probability density function (pdf) of the Fréchet-Weibull distribution (FWD) is given by

$$f_X(x) = \alpha k \beta^{\alpha} m^{\alpha k} x^{-1-\alpha k} exp\left\{ -\beta^{\alpha} \left(\frac{m}{x}\right)^{\alpha k} \right\} \tag{2}$$

and the corresponding survival function (sf) is given by

$$S_X(x) = 1 - exp\left\{ -\beta^{\alpha}\left(\frac{m}{x}\right)^{\alpha k}\right\} \quad (3)$$

where $x > 0$ and the parameters $(\alpha, \beta, m, k) > 0$. Also, α and k are shape parameters and β and m are scale parameters.

Using the survival function approach of discretization, a new discrete distribution called Discretized Fréchet-Weibull distribution (DFWD) is developed by Das and Das [30]. This distribution is suitable to be modelled with both positively and negatively skewed data. Also its hazard rate function can be increasing, decreasing and up-side down bathtub shaped, which are not much observed for count distributions. The pmf of the random variable Y having Discretized Fréchet-Weibull distribution, with the shape parameters α and k and the scale parameters β and m, is given by

$$P[Y = y] = exp\left\{ -\beta^{\alpha}\left(\frac{m}{y+1}\right)^{\alpha k}\right\} - exp\left\{ -\beta^{\alpha}\left(\frac{m}{y}\right)^{\alpha k}\right\} \quad (4)$$

where $y \in \mathbf{Z}_+$ and the parameters $(\alpha, \beta, m, k) > 0$. Let ξ be the parameter vector of DFWD, defined as $\xi = (\alpha, \beta, m, k)$, such that $\xi \in (R_+ \times R_+ \times R_+ \times R_+)$.

3 Weighted Discretized Fréchet-Weibull distribution (WDFWD)

Let Y be a count random variable with pmf $P[Y = y]$, where $y \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. Let $w(y)$ be a non-negative weight function on $\mathbf{Z}_+$ having a finite expectation

$$E[w(y)] = \sum_y w(y) \times P[Y = y] < \infty$$

Then to derive a new weighted distribution, the weight function $w(y)$ can be used to adjust the probability when $Y = y$ occur. Thus, the pmf of the weighted version of the random variable Y, which is the realization of count random variable Y_w is given by

$$P^w[Y_w = y] = \frac{w(y) \times P[Y = y]}{\sum_y w(y) \times P[Y = y]} = \frac{w(y) \times P[Y = y]}{E[w(y)]} \quad (5)$$

It is to be noted that similar definition can be stated for the continuous random variables. It is important to note that for a particular distribution, corresponding to different alternatives of the weight function $w(y)$ gives different forms of the weighted distribution.

As it is mentioned in Section 1, we shall consider the cumulative distribution function of discrete Weibull (DW) distribution, as developed by Nakagawa and Osaki [27] as our weight function. Thus the weight function is given by

$$w(y) = 1 - exp\left\{-\left(\frac{y+1}{m}\right)^k\right\} \; ; \; y \in \mathbf{Z}_+ \quad (6)$$

which is the cdf of discrete Weibull (DW) distribution with the parameters $(m, k) > 0$. The reason for considering this cdf as the weight function is that it is non-negative and as $k \to \infty, w(y) \to 1$. One more interesting fact behind the consideration of this weight function is that it doesn't increase the total number of parameters of the proposed weighted distribution. Thus the parameter vector of Weighted Discretized Fréchet-Weibull distribution (WDFWD) is $\xi = (\alpha, \beta, m, k)$, which is same as that of the parent distribution.

Considering the weight function $w(y)$ as in Eq.(6), the denominator in Eq.(5) can be written as

$$\begin{aligned} E[w(y)] &= \sum_{s=0}^{\infty} w(s) \times P[Y = s] \\ &= \sum_{s=0}^{\infty}\left[1 - exp\left\{-\left(\frac{s+1}{m}\right)^k\right\}\right] \times P[Y = s] \\ &= \sum_{s=0}^{\infty} q(s; \alpha, \beta, m, k) = Q(s; \xi) \;\; (say) \end{aligned} \quad (7)$$

where $s \in \mathbf{Z}_+$ and the parameter vector $\xi = (\alpha, \beta, m, k)$. Also,

$$q(s; \xi) = \left[1 - exp\left\{-\left(\frac{s+1}{m}\right)^k\right\}\right] \times \left[exp\left\{ -\beta^{\alpha}\left(\frac{m}{s+1}\right)^{\alpha k}\right\} - exp\left\{ -\beta^{\alpha}\left(\frac{m}{s}\right)^{\alpha k}\right\}\right] \quad (8)$$

and the summation of the term $q(s; \xi)$ over the entire range of $\mathbf{Z}_+$ is infinite and cannot be written in closed form and is termed as $Q(s; \xi)$.

Using the cdf of the Discrete Weibull distribution with parameters $(m, k) > 0$ given in Eq.(6) as the weight function to construct the Weighted Discretized Fréchet-Weibull distribution (WDFWD), then its pmf can be written as

$$P^w[Y_w = s] = \frac{q(s; \xi)}{\sum_{s=0}^{\infty} q(s; \xi)} = \frac{q(s; \xi)}{Q(s; \xi)} = P(s; \xi) \;\; (say) \quad (9)$$

where $s \in \mathbf{Z}_+$ and the parameter vector $\xi = (\alpha, \beta, m, k)$. And the terms $q(s; \xi)$ and $Q(s; \xi)$ is as defined in Eq.(8) and Eq.(7) respectively. For WDFWD also, α and k are the shape parameters and β and m are the scale parameters.

3.1 Cumulative distribution function

The cdf of the Weighted Discretized Fréchet-Weibull distribution (WDFWD) is given by

$$
\begin{aligned}
F_Y^w(s) = P^w[Y_w \leq s] &= \sum_{s=0}^{y} P^w[Y_w = s] \\
&= \sum_{s=0}^{y} \frac{q(s;\xi)}{Q(s;\xi)} \\
&= \frac{\sum_{s=0}^{y} q(s;\xi)}{Q(s;\xi)} = \frac{V(s;\xi)}{Q(s;\xi)} = C(s;\xi) \qquad (10)
\end{aligned}
$$

where $s \in \mathbf{Z}_+$ and the parameter vector $\xi = (\alpha, \beta, m, k)$. Also the term $Q(s;\xi)$ is as defined in Eq.(7).

3.2 Survival function

The survival function (sf) of Weighted Discretized Fréchet-Weibull distribution (WDFWD) is given by

$$
\begin{aligned}
S_Y^w(s) = P^w[Y_w \geq s] &= 1 - F^w(s) + P^w[Y_w = s] \\
&= 1 - C(s;\xi) + P(s;\xi) \\
&= 1 - \frac{V(s;\xi)}{Q(s;\xi)} + \frac{q(s;\xi)}{Q(s;\xi)} = S(s;\xi) \ \ (say) \qquad (11)
\end{aligned}
$$

where $s \in \mathbf{Z}_+$ and the parameter vector $\xi = (\alpha, \beta, m, k)$. Also the terms $Q(s;\xi)$, $q(s;\xi)$ and $V(s;\xi)$ are as defined in Eq.(7) , Eq.(8) and Eq.(10) respectively .

3.3 Hazard rate function

The hazard rate function (hrf) of Weighted Discretized Fréchet-Weibull distribution (WDFWD) is given by

$$
\begin{aligned}
H_Y^w(s) &= \frac{P^w[Y_w = s]}{S_Y^w(s)} \\
&= \frac{P(s;\xi)}{S(s;\xi)} = H(s;\xi) \ \ (say) \qquad (12)
\end{aligned}
$$

where $s \in \mathbf{Z}_+$ and the parameter vector $\xi = (\alpha, \beta, m, k)$. Also the terms $P(s;\xi)$ and $S(s;\xi)$ are as defined in Eq.(9) and Eq.(11) respectively .

3.4 Graphical representation of WDFWD

The possible shapes of the pmf and hrf of Weighted Discretized Fréchet-Weibull distribution (WDFWD) for different values of the parameter (α, β, m, k) are presented in Figure 1 and Figure 3 respectively. Also in Figure 2 we have graphically presented the impact in the pmf plot of WDFWD by increasing the shape parameters (α, k) and the scale parameters (β, k).

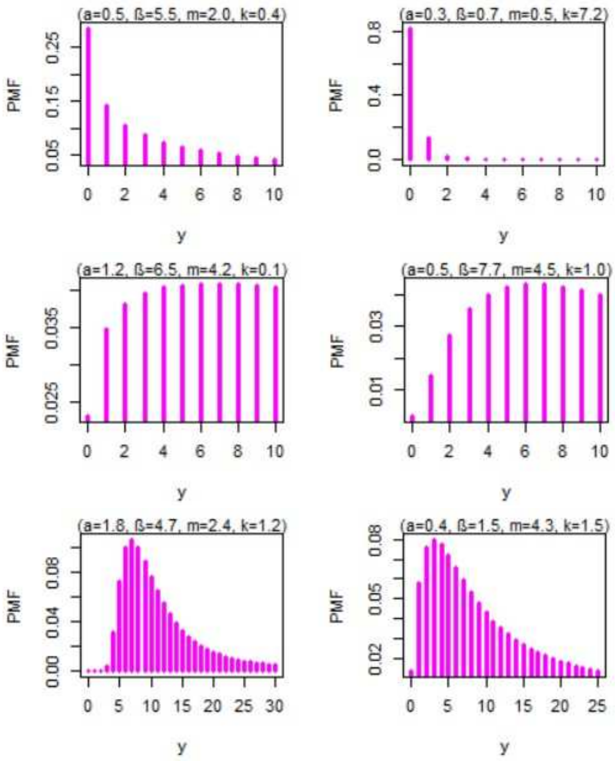


Fig. 1: PMF plot of WDFWD for different values the parameter (α, β, m, k).

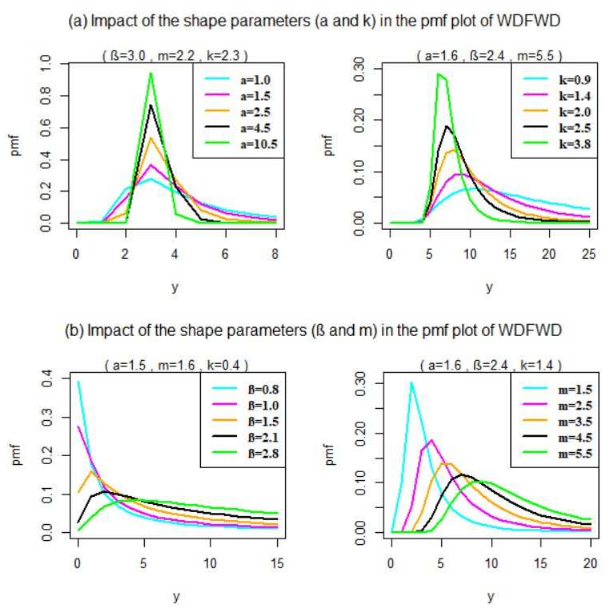


Fig. 2: Impact of shape and scale parameters in the pmf plot of WDFWD.

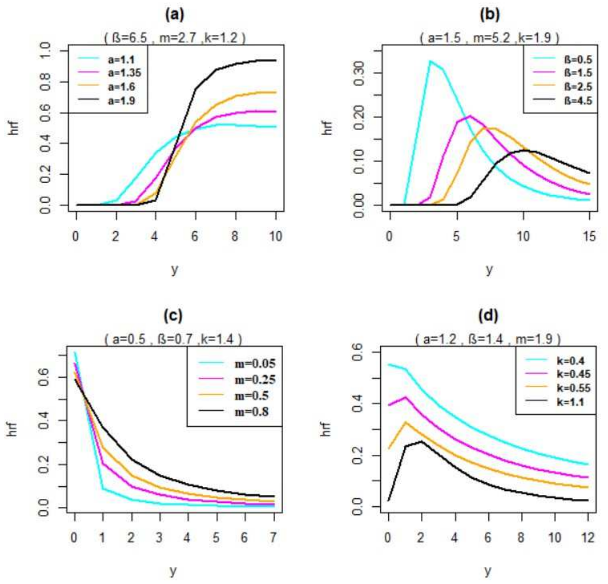


Fig. 3: HRF plot of WDFWD for different values the parameter (α,β,m,k).

We observe from Figure 1 that the pmf of the WDFWD for different values of the parameter (α,β,m,k) possess right long-tail. So, the proposed distribution is positive-skewed distribution. Also, it can be observed that the shape of the pmf of this distribution can be increasing, decreasing and unimodal.

In Figure 2 we have presented the effect changing of the shape and scale parameters over the pmf of the proposed distribution. In the Figure 2(a) the value of the shape parameters α and k increases and the rest parameters are kept fixed. Again in Figure 2(b) the the value of the scale parameters β and m are varying while the remaining parameters are kept fixed. From Figure 2(a) it can be clearly seen that the peak of the pmf of WDFWD sharpens gradually as the shape parameters α and k increases. Similarly, from Figure 2(b) it is clearly seen that as the scale parameters β and m increases the pmf of WDFWD flattens and stretches out more.
Figure 3 depicts the possible shapes of hazard rate function of WDFWD for different values of the parameter (α,β,m,k) are presented. In Figure 3 (a), (b), (c) and (d) the parameter values of α, β, m and k are varying respectively, while the remaining parameters are kept fixed. It can be seen that the hrf function of WDFWD can be increasing, decreasing or bathtub shaped depending upon the choice of parameter values. Also, it is to be noted that, in all cases it is less than 1. One obvious motivational fact of the proposed distribution is that it possess bathtub shaped hazard function along with the quality of exhibiting increasing and decreasing hazard rate shapes which are not often observed in count distributions.

3.5 Monotonic property

To understand the log-convexity or log-concavity behavior of WDFWD for the parameter set $\xi=(\alpha,\beta,m,k)$, it is sufficient if we can show that the ratio

$$\frac{P^w(y+1;\xi)}{P^w(y;\xi)}=\frac{q(y+1;\xi)}{q(y;\xi)}$$

for $y\in\mathbf{Z}_+$ and $\xi=(\alpha,\beta,m,k)$, is a non-decreasing function of y, then it implies that

$$\{P^w(y;\xi)\}^2\le P^w(y+1;\xi)\,.\,P^w(y-1;\xi)$$

for $y\in\mathbf{Z}_+$ and $\xi=(\alpha,\beta,m,k)$, then the distribution is log-convex, otherwise it is log-concave. But for the proposed distribution, depending upon the choice of the parameter values the ratio can be both increasing or decreasing function of y and consequently the WDFWD can act as both log-convex and log-concave.

Also it is known that, if a discrete distribution is log-convex (log-concave), then the hazard rate function is decreasing (increasing). Hence, WDFWD has both the decreasing failure rate (DFR) and increasing failure rate (IFR) distribution.

3.6 Infinite divisibility

The infinite divisibility is one of the valuable structural property of a distribution. According to Steutel and Van [31], the determination of infinite divisibility property of a distribution can be accomplished based on the Lemma 1, stated as

Lemma 1. A necessary condition for infinite divisibility of a discrete distribution p_y is that $p_0>0$.

From the pmf plot of WDFWD as shown in Figure 1, it is clearly seen that, there can be many such combinations of the parameter values, that satisfy the condition in Lemma 1. Hence WDFWD is infinite divisible.

Also as reported by Nekoukhou et al. [32], the property of self-decomposable and stable are direct consequences of infinitely divisibility of a distribution. Hence we conclude that WDFWD is self-decomposable and stable. Further, it is also known that if a count distribution is infinite divisible then it is over-dispersed. Thus, WDFWD is an over-dispersed distribution.

3.7 Moment generating function

To make the simplification easier let us rewrite the term $q(s;\xi)$ as mentioned in Eq.(8) by considering the following substitutions

$$D(y)=1-exp\left\{-\left(\frac{y+1}{m}\right)^k\right\}$$

and

$$E(y) = 1 - exp\left\{ -\beta^{\alpha} \left(\frac{m}{y} \right)^{\alpha k} \right\}$$

Then the Eq.(5) can be rewritten as

$$q(s;\xi) = D(y).\big[E(y) - E(y+1)\big] \quad (13)$$

The moment generating function (mgf) of WDFWD is obtained using the Eq.(13) in the pmf as defined in Eq.(9) and is given by

$$\begin{aligned} M_Y^w(t) = E[e^{ty}] &= \sum_{y=0}^{\infty} e^{ty}\, P^w[Y_w = y] \\ &= \frac{1}{Q(s;\xi)} \sum_{y=0}^{\infty} e^{ty}\, q(y;\xi) \\ &= \frac{1}{Q(s;\xi)} \sum_{y=0}^{\infty} e^{ty}\, D(y).\big[E(y) - E(y+1)\big] \\ &= \frac{1}{Q(s;\xi)} \Bigg[D(0) + \sum_{y=1}^{\infty} E(y) \times \\ &\quad \left\{ e^{ty}\, D(y) \ - \ e^{t(y-1)}\, D(y-1) \right\} \Bigg] \quad (14) \end{aligned}$$

where $y \in \mathbf{Z}_+$, $D(0) = 1 - exp\{-m^{-k}\}$ and the term $Q(s;\xi)$ is as defined in Eq.(7).

Now, differentiating the mgf of WDFWD as in Eq.(14), for 'r' times w.r.t 't', we get

$$\begin{aligned} M_Y^{w(r)}(t) &= \frac{d^r}{dt^r} M_Y^w(t) \\ &= \frac{1}{Q(s;\xi)} \sum_{y=1}^{\infty} E(y) \times \\ &\quad \left[y^r e^{yt}\, D(y) - (y-1)^r e^{(y-1)t}\, D(y-1) \right] \quad (15) \end{aligned}$$

Thus the r^{th} moment of WDFWD is given by

$$\begin{aligned} E(Y_w^r) &= M_Y^{w(r)}(t)\big|_{t=0} \\ &= \frac{1}{Q(s;\xi)} \sum_{y=1}^{\infty} E(y) \times \\ &\quad \left[y^r\, D(y) - (y-1)^r\, D(y-1) \right] \quad (16) \end{aligned}$$

where $y \in \mathbf{Z}_+$ and the term $Q(s;\xi)$ is as defined in Eq.(7).

3.8 Characteristic function

The characteristic function (cf) of WDFWD can be easily obtained by replacing 't' by 'it' in the (14), as follows

$$\begin{aligned} \phi_Y^w(t) = E[e^{ity}] &= \sum_{y=0}^{\infty} e^{ity}\, P^w[Y_w = y] \\ &= \frac{1}{Q(s;\xi)} \sum_{y=0}^{\infty} e^{ity}\, q(y;\xi) \\ &= \frac{1}{Q(s;\xi)} \Bigg[D(0) + \sum_{y=1}^{\infty} E(y) \times \\ &\quad \left\{ e^{ity}\, D(y) \ - e^{it(y-1)}\, D(y-1) \right\} \Bigg] \quad (17) \end{aligned}$$

where $y \in \mathbf{Z}_+$, $D(0) = 1 - exp\{-m^{-k}\}$ and the term $Q(s;\xi)$ is as defined in Eq.(7).

3.9 Probability generating function

The probability generating function (pgf) of WDFWD is derived by using the Eq.(13) in the pmf as defined in Eq.(9) and is given by

$$\begin{aligned} G_Y^w(t) = E[t^y] &= \sum_{y=0}^{\infty} t^y\, P^w[Y_w = y] \\ &= \frac{1}{Q(s;\xi)} \sum_{y=0}^{\infty} t^y\, q(y;\xi) \\ &= \frac{1}{Q(s;\xi)} \sum_{y=0}^{\infty} t^y\, D(y).\big[E(y) - E(y+1)\big] \\ &= \frac{1}{Q(s;\xi)} \Bigg[D(0) + \sum_{y=1}^{\infty} E(y) \times \\ &\quad \left\{ t^y\, D(y) \ - \ t^{(y-1)}\, D(y-1) \right\} \Bigg] \\ &= \frac{1}{Q(s;\xi)} \Bigg[D(0) + \sum_{y=1}^{\infty} t^{(y-1)} E(y) \times \\ &\quad \left\{ t\, D(y) \ - \ D(y-1) \right\} \Bigg] \quad (18) \end{aligned}$$

where $y \in \mathbf{Z}_+$, $D(0) = 1 - exp\{-m^{-k}\}$ and the term $Q(s;\xi)$ is as defined in Eq.(7).

Now, differentiating the pgf of WDFWD as in Eq.(18), for 'r' times w.r.t 't', we get

$$\begin{aligned} G_Y^{w(r)}(t) &= \frac{d^r}{dt^r} G_Y^w(t) \\ &= \frac{1}{Q(s;\xi)} \sum_{y=1}^{\infty} (y-1)_{(r-2)}\, t^{(y-r-1)}\, E(y) \times \\ &\quad \left[yt\, D(y) - (y-r)\, D(y-1) \right] \quad (19) \end{aligned}$$

where $(y-1)_{(r-2)} = (y-1)(y-2)...(y-r)(y-r-1)$ such that $(y-1)_0 = (y-1)$.
Thus the r^{th} factorial moment of WDFWD is given by

$$E(Y_w^{(r)}) = G_Y^{w(r)}(t)\big|_{t=1} = \frac{1}{Q(s;\xi)} \sum_{y=1}^{\infty} (y-1)_{(r-2)}\, E(y) \times \Big[y\, D(y) - (y-r)\, D(y-1)\Big] \tag{20}$$

where $y \in \mathbf{Z}_+$ and the term $Q(s;\xi)$ is as defined in Eq.(7). Also $(y-1)_{(r-2)} = (y-1)(y-2)...(y-r)(y-r-1)$.

3.10 Numerical computation

In this section we have carried out some numerical results of mean (μ), variance (σ^2), skewness (S_k), kurtosis (K_r) and index of dispersion (ID) of the WDFWD using R software, as presented in tables 1, 2, 3 and 4.

Table 1: Some descriptive statistics using WDFWD model as "α" increases

α	β	m	k	μ	σ^2	S_k	K_r	ID
2.1				4.68	46.4	4.44	28.5	9.9
2.8				3.01	6.74	2.01	7.59	2.2
3.9	1.5	2.1	0.9	2.73	3.55	1.59	5.71	1.3
4.5				2.40	2.47	1.48	5.75	1.0
5.5				2.20	2.11	1.39	6.04	0.9

Table 2: Some descriptive statistics using WDFWD model as "β" increases

α	β	m	k	μ	σ^2	S_k	K_r	ID
	3.0			4.88	6.76	2.38	9.92	1.4
	4.0			5.70	9.91	1.44	4.96	1.7
2.5	5.0	2.5	1.5	7.46	10.6	1.88	7.17	1.4
	7.0			9.14	12.7	0.73	2.66	1.4
	9.8			11.7	13.5	1.55	5.01	1.2

Table 3: Some descriptive statistics using WDFWD model as "m" increases

α	β	m	k	μ	σ^2	S_k	K_r	ID
		1.2		0.58	0.83	2.05	7.48	1.4
		1.8		1.67	1.48	1.89	8.01	0.9
3.0	1.8	2.2	1.5	2.54	6.09	4.03	20.7	2.4
		3.5		4.49	6.23	2.18	9.00	1.4
		4.5		6.54	7.06	1.92	0.94	1.1

Table 4: Some descriptive statistics using WDFWD model as "k" increases

α	β	m	k	μ	σ^2	S_k	K_r	ID
			1.2	6.54	32.7	3.83	23.3	5.0
			1.5	5.67	30.8	3.11	14.7	5.4
1.8	2.2	2.8	1.9	3.76	5.54	1.60	4.93	1.5
			2.2	3.66	5.15	2.81	13.4	1.4
			2.8	3.15	4.37	3.75	20.6	1.4

From the above Tables 1, 2, 3, and 4, the following observations can be noted:

- The proposed model is suitable of modelling positively skewed data sets.
- The proposed model is suitable for modelling with both the platykurtic (kurtosis<3) and leptokurtic (kurtosis>3) data sets.
- The mean and variance decrease with the increase in the shape parameters α and k.
- The mean and variance increase with the increase in the scale parameters β and m.
- The proposed model is suitable for modelling with overdispersed $(ID \geq 1)$ data sets.

4 Maximum likelihood estimation

In this section we use the maximum likelihood estimation method to obtain the estimates of the parameters (α, β, m, k) of WDFWD. Let $Y = (Y_1, Y_2, ..., Y_n)$ be a random sample of size 'n' from WDFWD with the pmf as defined in Eq.(9) and the corresponding observed values of Y as $y_1, y_2, ..., y_n$ respectively.
Then the log-likelihood function is given by

$$\begin{aligned} log\ L(y) &= log \prod_{i=1}^{n} P^w[Y_w = y_i] \\ &= \sum_{i=1}^{n} log\ q(y_i;\xi) - n\ log\ Q(y_i;\xi) \\ &= \sum_{i=1}^{n} log\ w(y_i) + \sum_{i=1}^{n} log\ P[Y = y_i] \\ &\quad - n\ log\ Q(y_i;\xi) \end{aligned} \tag{21}$$

where $P[Y = y_i]$, $w(y_i)$, and $Q(y_i;\xi)$ are as defined in Eq.(2), Eq.(6) and Eq.(7) respectively.

Now differentiating Eq.(19) wrt α, β, m and k, we get

$$\frac{\partial \, log \, L(y)}{\partial \alpha} = \sum_{i=1}^{n} \frac{E(y_i+1)\, F(y_i+1) - E(y_i)\, F(y_i)}{J(y_i)} \quad (22)$$

$$\frac{\partial \, log \, L(y)}{\partial \beta} = \sum_{i=1}^{n} \frac{E(y_i+1)\, G(y_i+1) - E(y_i)\, G(y_i)}{J(y_i)} \quad (23)$$

$$\frac{\partial \, log \, L(y)}{\partial m} = m^{-1} \left\{ \sum_{i=1}^{n} \frac{A(y_i)\, B(y_i)}{D(y_i)} \right\} + \sum_{i=1}^{n} \frac{E(y_i+1)\, H(y_i+1) - E(y_i)\, H(y_i)}{J(y_i)} \quad (24)$$

$$\frac{\partial \, log \, L(y)}{\partial k} = \left\{ \sum_{i=1}^{n} \frac{A(y_i)\, B(y_i)\, C(y_i)}{D(y_i)} \right\} + \sum_{i=1}^{n} \frac{E(y_i+1)\, I(y_i+1) - E(y_i)\, I(y_i)}{J(y_i)} \quad (25)$$

where

$$A(y_i) = exp\left\{ -\left(\frac{y_i+1}{m}\right)^k \right\},$$
$$B(y_i) = \left(\frac{y_i+1}{m}\right)^k,$$
$$C(y_i) = log_e\left(\frac{y_i+1}{m}\right),$$
$$D(y_i) = 1 - exp\left\{ -\left(\frac{y_i+1}{m}\right)^k \right\} = w(y_i),$$
$$E(y_i) = exp\left\{ -\beta^\alpha \left(\frac{m}{y_i}\right)^{\alpha k} \right\}$$
$$F(y_i) = \left\{ -\alpha\beta \left(\frac{m}{y_i}\right)^k \right\}^{\alpha-1},$$
$$G(y_i) = -\alpha\beta^{\alpha-1} \left(\frac{m}{y_i}\right)^{\alpha k},$$
$$H(y_i) = -\alpha k \beta^\alpha m^{\alpha k-1} y^{-\alpha k},$$
$$I(y_i) = -\alpha^2 \beta^\alpha k \left(\frac{m}{y_i}\right)^{\alpha k-1} \quad \text{and}$$
$$J(y_i) = P[Y = y_i]$$

By setting the non-linear Eq.(22), (23), (24) and (25) equal to zero and solving them iteratively, we obtain the MLEs $\hat{\xi} = (\hat{\alpha}, \hat{\beta}, \hat{m}, \hat{k})$ for the parameter vector $\xi = (\alpha, \beta, m, k)$. These equations do not have explicit solutions and they have to be obtained numerically by using statistical software like optim package in R programming.

The Fisher's information matrix $\mathscr{I}(\xi)$ is very useful and we require this for interval estimation on the parameters and is given by

$$\mathscr{I}(\xi) = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial m}\right) & E\left(\frac{\partial^2 L}{\partial \alpha \partial k}\right) \\ E\left(\frac{\partial^2 L}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial \beta^2}\right) & E\left(\frac{\partial^2 L}{\partial \beta \partial m}\right) & E\left(\frac{\partial^2 L}{\partial \beta \partial k}\right) \\ E\left(\frac{\partial^2 L}{\partial m \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial m \partial \beta}\right) & E\left(\frac{\partial^2 L}{\partial m^2}\right) & E\left(\frac{\partial^2 L}{\partial m \partial k}\right) \\ E\left(\frac{\partial^2 L}{\partial k \partial \alpha}\right) & E\left(\frac{\partial^2 L}{\partial k \partial \beta}\right) & E\left(\frac{\partial^2 L}{\partial k \partial m}\right) & E\left(\frac{\partial^2 L}{\partial k^2}\right) \end{bmatrix}$$

Since the exact evaluation of $\mathscr{I}(\xi)$ may be cumbersome and can be computed using the approximation

$$\mathscr{I}(\hat{\xi}) = \begin{bmatrix} -\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial \alpha \partial \beta} & -\frac{\partial^2 L}{\partial \alpha \partial m} & -\frac{\partial^2 L}{\partial \alpha \partial k} \\ -\frac{\partial^2 L}{\partial \beta \partial \alpha} & -\frac{\partial^2 L}{\partial \beta^2} & -\frac{\partial^2 L}{\partial \beta \partial m} & -\frac{\partial^2 L}{\partial \beta \partial k} \\ -\frac{\partial^2 L}{\partial m \partial \alpha} & -\frac{\partial^2 L}{\partial m \partial \beta} & -\frac{\partial^2 L}{\partial m^2} & -\frac{\partial^2 L}{\partial m \partial k} \\ -\frac{\partial^2 L}{\partial k \partial \alpha} & -\frac{\partial^2 L}{\partial k \partial \beta} & -\frac{\partial^2 L}{\partial k \partial m} & -\frac{\partial^2 L}{\partial k^2} \end{bmatrix}\Bigg|_{(\hat{\alpha}, \hat{\beta}, \hat{m}, \hat{k})}$$

where $\hat{\alpha}, \hat{\beta}, \hat{m}$ and $\hat{k}$ are the MLEs of the parameters α, β, m and k respectively. Computation of $\mathscr{I}(\hat{\xi})$ enables us to obtain the approximate confidence intervals of the parameters. For example, the $100(1-\eta)\%$ asymptotic confidence interval for the j^{th} parameter ξ_j is given by

$$(\hat{\xi}_j - z_{\frac{\eta}{2}} \sqrt{\mathscr{I}_{j,j}} \quad , \quad \hat{\xi}_j + z_{\frac{\eta}{2}} \sqrt{\mathscr{I}_{j,j}})$$

where $\mathscr{I}_{j,j}$ is the j^{th} diagonal element of $\mathscr{I}^{-1}(\hat{\xi})$, for $j = 1,2,3,4$ and $z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}$ point of standard normal distribution.

5 Simulation

In order to assess the performance of the MLEs, a simulation study is performed utilizing the statistical software R. Then 1000 replications of samples of size $n = 50, 100, 150$ and 200 from WDFWD with known values of the parameters α, β, m, k have been generated for the following two cases:

$$\text{(i)}\alpha = 1.8, \beta = 5.5, m = 2.0, k = 0.4$$
$$\text{(ii)}\alpha = 1.5, \beta = 2.5, m = 1.5, k = 0.5.$$

For each sample size the evaluation of the estimates was performed based on the empirical biases and the mean squared errors (MSEs), which are calculated utilizing the R package and are defined as

$$Bias(\theta) = \frac{1}{1000} \sum_{s=1}^{1000} (\hat{\theta}_s - \theta)$$

and

$$MSE(\theta) = \frac{1}{1000}\sum_{s=1}^{1000}(\hat{\theta}_s - \theta)^2$$

Table 5: The Avg Est, biases and MSEs for case I

Case I:	$\alpha = 1.8\,,\ \beta = 5.5\,,\ m = 2.0\,,\ k = 0.4$				
Avg Est	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{m}$	$\hat{k}$
	50	2.144	5.0285	2.515	0.219
	100	1.999	5.419	2.353	0.294
	150	1.942	5.385	2.173	0.309
	200	1.810	5.494	2.002	0.429
Bias	50	0.344	-0.472	0.515	-0.180
	100	0.199	-0.081	0.353	-0.106
	150	0.142	-0.115	0.173	-0.091
	200	0.010	-0.006	0.002	0.029
MSE	50	0.523	1.632	0.962	0.145
	100	0.183	0.174	0.318	0.152
	150	0.236	0.019	0.186	0.153
	200	0.088	0.026	0.093	0.004

Table 6: The Avg Est, biases and MSEs for case II

Case II:	$\alpha = 1.5\,,\ \beta = 2.5\,,\ m = 1.5\,,\ k = 0.5$				
Avg Est	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{m}$	$\hat{k}$
	50	2.041	3.321	1.828	0.266
	100	1.651	2.454	1.850	0.261
	150	1.644	2.688	1.632	0.512
	200	1.529	2.453	1.572	0.505
Bias	50	0.541	0.821	0 .328	-0.234
	100	0.151	-0.046	0.350	-0.239
	150	0.144	0.188	0.132	0.012
	200	0.029	-0.047	0.072	0.004
MSE	50	0.496	1.259	0.792	0.120
	100	0.115	0.009	0.235	0.121
	150	0.036	0.089	0.019	0.003
	200	0.002	0.006	0.008	0.0002

The average estimates, biases and MSEs for the above two cases are presented in Table 5 and 6 respectively. From the table values it is observed that as the sample size n increases the biases decreases to zero and also the MSEs diminishes to zero with increase in the sample size n. This shows consistency and unbiasedness of the MLEs.

6 Application

Finally, the proposed model has been fitted to three real life datasets to test its goodness of fit. And to show its efficacy it is being compared with Discretized Fréchet-Weibull distribution (DFWD) [30], discrete Burr (DBurr) [33], discrete generalised Weibull (DGW) [34], discrete generalised Inverse Weibull (DGIW) [35] and discrete Rayleigh [36] distribution. It is shown that the weighted version of DFWD gives better fit than the parent model DFWD, which indicates the importance of weighted distribution.

Although the Pearson Chi-square (χ^2) test-statistic with its corresponding p-value is a measure to check the goodness of fit of the fitted models. But the value of the Pearson χ^2 statistics increases gradually with the increase in the sample size N (which may often be hundreds of thousands or even more). Under such circumstances making conclusion over Pearson χ^2 statistics is not much encouraged. Makcutek [37] used the discrepancy coefficient $C = \frac{\chi^2}{N}$ as the goodness of fit criterion in his study, as the values of Pearson χ^2 statistics and N were very large.

In general the greater the value of p-value, the better is the fit. But in case of discrepancy coefficient, the value of C more nearer to zero, provides the better fit of the data set (see Grzybek [38], Nekoukhou et al. [32]).

6.1 Application 1

For the first application, we have considered an experimental data as reported by Bliss and Fisher [39]. This dataset represents the total number of borers per hill in each plot for a control group. Four treatments (viz. 1,2,3,4) were arranged in 15 randomized blocks, to carry out the field experiment of insect pests on the corn borer. At the end of the season, from each plot eight hills of corn were selected at random and the count of borers were recorded from each hill. Here we have used the data corresponding to the Treatment 2 and this data is extracted from Beall [40] (table II).

Dataset I:

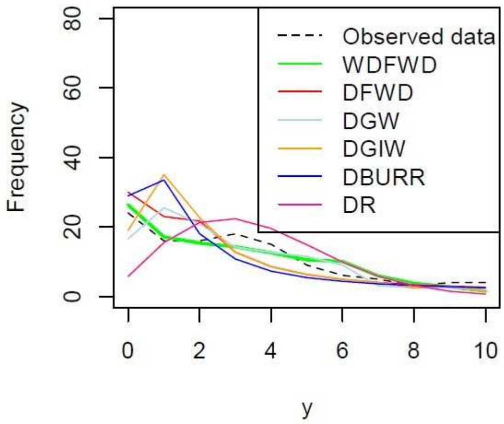
X	0	1	2	3	4	5
Frequency	24	16	16	18	15	9
X	6	7	8	9	≥ 10	Total
Frequency	6	5	3	4	4	120

For this dataset its mean $= 3.15000 <$ variance $= 7.50672$, therefore its is overdisperssed. Also this is right skewed (as skewness $= 0.81468$) and platykurtic (as kurtosis $= 2.93118$) in nature. Thus this dataset is suitable to model with the WDFWD distribution.
In Table 7, the maximum likelihood estimates with their corresponding standard Errors (SEs) and 95% confidence intervals of the parameters for all fitted models in application to dataset I are shown. And in Table 8, the summary of goodness of fit criteria: p-value of χ^2 and C, for all the fitted models are shown.

From the Table 8, it can be seen that for the dataset I the p-value is the largest and the value of C is least for WDFWD distribution. Hence, WDFWD provides a better

Table 7: MLE's and standard errors for dataset I

Distribution	Estimates	SE	95% CI
DR	$\hat{\theta} = 0.953$	$\hat{\theta} = 0.004$	(0.952, 0.954)
DBurr	$\hat{\alpha} = 2.384$	$\hat{\alpha} = 0.397$	(2.313, 2.454)
	$\hat{\theta} = 0.685$	$\hat{\theta} = 0.046$	(0.677, 0.694)
DGIW	$\hat{\alpha} = 1.178$	$\hat{\alpha} = 0.098$	(1.160, 1.195)
	$\hat{\beta} = 1.258$	$\hat{\beta} = 8.920$	(-0.329, 2.846)
	$\hat{\theta} = 0.234$	$\hat{\theta} = 12.138$	(-1.927, 2.394)
DGW	$\hat{\alpha} = 1.276$	$\hat{\alpha} = 0.104$	(1.258, 1.294)
	$\hat{\beta} = 1.017$	$\hat{\beta} = 6.656$	(-0.168, 2.201)
	$\hat{\theta} = 0.836$	$\hat{\theta} = 1.493$	(0.571, 1.102)
DFWD	$\hat{\alpha} = 1.101$	$\hat{\alpha} = 6.452$	(-0.047, 2.250)
	$\hat{\beta} = 1.077$	$\hat{\beta} = 5.676$	(0.066, 2.087)
	$\hat{m} = 1.614$	$\hat{m} = 8.322$	(0.133, 3.095)
	$\hat{k} = 1.069$	$\hat{k} = 6.266$	(-0.046, 2.185)
WDFWD	$\hat{\alpha} = 2.110$	$\hat{\alpha} = 1.097$	(1.915, 2.306)
	$\hat{\beta} = 4.132$	$\hat{\beta} = 3.568$	(3.497, 4.767)
	$\hat{m} = 3.177$	$\hat{m} = 26.217$	(-1.490, 7.843)
	$\hat{k} = 0.024$	$\hat{k} = 0.028$	(0.019, 0.029)



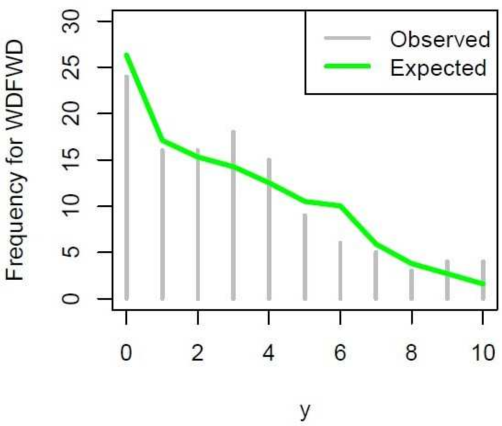


Fig. 4: Observed and Estimated frequency graph of different models for dataset I.

fit for the dataset I as compared to the remaining five distributions. In Figure 4, the observed and estimated frequency graphs of the WDFWD, DFWD, DGIW, DGW, DBurr and DR distribution are presented.

6.2 Application 2

The second dataset is extracted from Altun et al. [41] representing the numbers of fires in Greece for the time period of 1 July 1998 to 31 August 1998.

Dataset II:

X	0	1	2	3	4
Frequency	16	13	14	9	11
X	5	6	7	8	9
Frequency	13	8	4	9	6
X	10	11	12	15	16
Frequency	3	4	6	4	1
X	≥ 20	Total			
Frequency	2	123			

The skewness and kurtosis for the dataset II are 2.9916 and 19.4989 respectively. Clearly the given dataset is highly right skewed and leptokurtic in nature. Also, it is over-dispersed (as mean = 5.39980 < variance = 30.0449), hence suitable to model with the WDFWD distribution.

Table 9: MLE's and standard errors for dataset II

Distribution	Estimates	SE	95% CI
DR	$\hat{\theta} = 0.985$	$\hat{\theta} = 0.001$	(0.984, 0.985)
DGIW	$\hat{\alpha} = 1.035$	$\hat{\alpha} = 0.079$	(1.021, 1.049)
	$\hat{\beta} = 0.893$	$\hat{\beta} = 7.323$	(-0.394, 2.181)
	$\hat{\theta} = 0.058$	$\hat{\theta} = 24.17$	(-4.191, 4.307)
DBurr	$\hat{\alpha} = 2.503$	$\hat{\alpha} = 0.487$	(2.418, 2.589)
	$\hat{\theta} = 0.762$	$\hat{\theta} = 0.043$	(0.754, 0.769)
DFWD	$\hat{\alpha} = 0.811$	$\hat{\alpha} = 4.573$	(0.007, 1.615)
	$\hat{\beta} = 1.576$	$\hat{\beta} = 2.409$	(1.152, 1.999)
	$\hat{m} = 1.719$	$\hat{m} = 4.735$	(0.886, 2.551)
	$\hat{k} = 1.276$	$\hat{k} = 7.192$	(0.0114, 2.541)
DGW	$\hat{\alpha} = 1.131$	$\hat{\alpha} = 0.082$	(1.116, 1.145)
	$\hat{\beta} = 1.903$	$\hat{\beta} = 18.78$	(-1.399,5.205)
	$\hat{\theta} = 0.767$	$\hat{\theta} = 2.954$	(0.248, 1.287)
WDFWD	$\hat{\alpha} = 2.232$	$\hat{\alpha} = 0.970$	(2.061, 2.403)
	$\hat{\beta} = 4.248$	$\hat{\beta} = 3.165$	(3.692, 4.805)
	$\hat{m} = 3.031$	$\hat{m} = 24.77$	(-1.323, 7.386)
	$\hat{k} = 0.018$	$\hat{k} = 0.017$	(0.015, 0.021)

In Table 9, the maximum likelihood estimates with their corresponding standard Errors (SEs) and 95% confidence intervals of the parameters for all fitted models in application to dataset II are shown. And in Table 10,

Table 8: Summary of goodness of fit criteria for dataset I

X	OBSERVED FREQUENCY	EXPECTED FREQUENCY					
		DR	DBURR	DGIW	DGW	DFWD	WDFWD
0	24	5.69554	28.89545	18.99977	16.53446	30.00631	26.31246
1	16	15.46959	33.44313	34.99378	25.45639	22.99415	17.10072
2	16	21.27022	17.97208	22.64835	20.87194	21.64661	15.29123
3	18	22.32989	10.73755	12.70072	14.20254	12.69936	14.25879
4	15	19.56471	7.28814	8.60158	12.59455	8.60063	12.52945
5	9	14.80981	5.42271	6.29648	11.47304	6.29542	10.49733
6	6	9.85367	4.30761	4.89308	8.95499	4.89253	10.03157
7	5	5.82206	3.58931	3.98291	3.01615	3.98247	5.90906
8	3	3.07779	3.09967	2.36235	2.57396	3.36202	3.79468
9	4	1.46727	2.75086	2.92197	2.53019	2.92167	2.68762
≥ 10	4	0.63945	2.49349	1.59901	1.79179	2.59883	1.58709
N	120	120	120	120	120	120	120
χ^2		87.99271	28.2759	26.39326	15.95434	14.64121	8.22509
Df		7	5	5	6	5	7
p-value		≤ 0.00001	0.00003	0.00007	0.01400	0.01201	**0.31317**
C		0.73327	0.23563	0.21994	0.13295	0.12201	**0.06854**

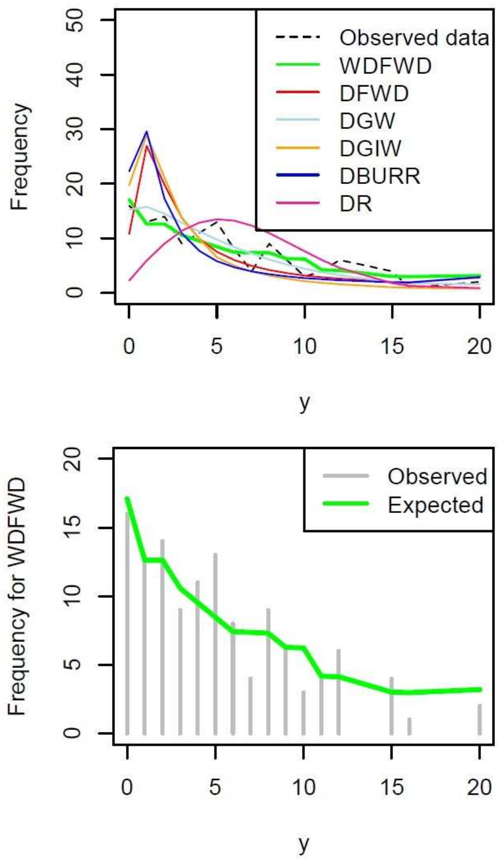


Fig. 5: Observed and Estimated frequency graph of different models for dataset II.

the summary of goodness of fit criteria: p-value of χ^2 and C, for all the fitted models are shown. Also,from the Table 10, it can be seen that the proposed model WDFWD provides a better fit for the dataset II (with largest p-value $= 0.46447$ and smallest value of $C = 0.07911$). In Figure 5, the observed and estimated frequency graphs of the WDFWD, DFWD, DGIW, DGW and DR distribution are presented.

6.3 Application 3

The third data set contains the observations on the establishment of Pyrausta nubilalis Hubn. (for the year 1983). This data is also extracted from Beall [40] (table VII). This dataset is right skewed (as skewness $= 0.45763$) and platykurtic in nature (as kurtosis $= 1.804009$). Also, the mean $= 3.11300$ and variance $= 7.56386$, hence it is over-dispersed and thus suitable to model with the WDFWD distribution.

Dataset III:

X	0	1	2	3	4
Frequency	12	8	7	6	3
X	5	6	7	≥ 8	Total
Frequency	3	4	6	4	53

In Table 11, the maximum likelihood estimates with their corresponding standard Errors (SEs) and 95% confidence intervals of the parameters for all fitted models in application to dataset III are shown. And in Table 12, the summary of goodness of fit criteria: p-value of χ^2 and C, for all the fitted models are shown.

From the Table 12, we see that the largest p-value and the least value of C belongs to the proposed model

Table 10: Summary of goodness of fit criteria for dataset II

X	OBSERVED FREQUENCY	EXPECTED FREQUENCY					
		DR	DGIW	DBURR	DFWD	DGW	WDFWD
0	16	2.27411	19.76927	22.31093	10.78624	15.28633	17.09294
1	13	5.88999	28.97079	29.63127	26.98343	15.79917	12.60858
2	14	9.01211	21.16475	17.30548	19.87027	14.49479	12.61829
3	9	11.39299	13.70417	10.90672	13.72199	12.88692	10.57896
4	11	12.89005	9.91402	7.66795	9.93264	11.26907	9.52725
5	13	13.4769	6.54017	5.83406	7.55914	9.75187	8.47276
6	8	13.23373	4.98178	4.69541	6.00089	8.37921	7.41875
7	4	12.32092	3.91181	3.93781	4.93095	7.16439	7.36639
8	9	10.94347	3.14845	3.40656	4.16761	6.10543	7.31614
9	6	9.31455	2.58612	3.01851	3.60525	5.19261	6.26806
10	3	7.62519	2.16055	2.72567	3.17967	4.41259	6.22213
11	4	6.02441	1.83108	2.49873	2.85017	3.75077	4.17823
12	6	4.61095	1.57099	2.31896	2.59005	3.19254	4.13624
15	4	1.80867	1.05177	1.95606	2.07077	2.00681	3.02043
16	1	1.30647	0.93467	1.87291	1.95364	1.73657	2.98482
≥ 20	2	0.87549	0.75961	2.91297	2.79729	1.57093	3.19003
N	123	123	123	123	123	123	123
χ^2		112.3571	66.35459	46.41465	32.68862	11.40105	9.73027
Df		11	5	5	6	9	10
p-value		≤ 0.00001	≤ 0.00001	≤ 0.00001	0.00001	0.24922	**0.46447**
C		0.91347	0.53947	0.37735	0.26576	0.09269	**0.07911**

Table 11: MLE's and standard errors for dataset III

Distribution	Estimates	SE	95% CI
DR	$\hat{\theta} = 0.952$	$\hat{\theta} = 0.006$	(0.951, 0.954)
DBurr	$\hat{\alpha} = 2.169$	$\hat{\alpha} = 0.534$	(2.027, 2.313)
	$\hat{\theta} = 0.657$	$\hat{\theta} = 0.071$	(0.638 ,0.676)
DGIW	$\hat{\alpha} = 1.123$	$\hat{\alpha} = 0.145$	(1.084, 1.162)
	$\hat{\beta} = 1.151$	$\hat{\beta} = 8.085$	(-1.015, 3.316)
	$\hat{\theta} = 0.234$	$\hat{\theta} = 11.469$	(-2.838, 3.306)
DFWD	$\hat{\alpha} = 1.148$	$\hat{\alpha} = 35.657$	(-8.403, 10.70)
	$\hat{\beta} = 1.001$	$\hat{\beta} = 13.500$	(-2.615, 4.616)
	$\hat{m} = 1.604$	$\hat{m} = 22.159$	(-4.331, 7.539)
	$\hat{k} = 0.979$	$\hat{k} = 30.41$	(-7.166, 9.123)
DGW	$\hat{\alpha} = 1.204$	$\hat{\alpha} = 0.153$	(1.163, 1.245)
	$\hat{\beta} = 0.938$	$\hat{\beta} = 14.226$	(-2.873, 4.748)
	$\hat{\theta} = 0.832$	$\hat{\theta} = 3.367$	(-0.071, 1.733)
WDFWD	$\hat{\alpha} = 2.239$	$\hat{\alpha} = 1.989$	(1.707, 2.772)
	$\hat{\beta} = 3.808$	$\hat{\beta} = 5.347$	(2.376, 5.240)
	$\hat{m} = 1.850$	$\hat{m} = 25.83$	(-5.068, 8.767)
	$\hat{k} = 0.023$	$\hat{k} = 0.041$	(0.012, 0.034)

WDFWD. Hence, it is clear that the WDFWD distribution provides a better fit as compared to the parent distribution DFWD and the remaining four competitive distributions. In Figure 6, the observed and estimated frequency graphs of the WDFWD, DFWD, DGIW, DGW and DR distribution are presented.

7 Conclusion

In this article, the weighted version of Discretized Fréchet-Weibull distribution is studied, named as Weighted Discretized Fréchet-Weibull distribution (WDFWD). For the construction of this new model, a new weight function is considered here which is the cdf of discrete Weibull distribution. We have derived some distributional properties of this model and it is seen from numerical computations that this model possesses both the platykurtic and leptokurtic nature. The shapes of the hazard rate function of this distribution can be increasing, decreasing and bathtub shaped depending upon the choice of parameters, which are rarely noticed in discrete probability distributions. Also the proposed model is suitable for modeling with overdispersed data sets. It is known that by increasing the parameters the distribution becomes more flexible and gives more better fit. But in this study we have considered the weight function such that the total number of parameters in the proposed weighted model is same as that of the parent distribution. And thus it has been shown that with the same number of parameters, the weighted version gives better fit than the parent distribution. It is also illustrated with the help of three count datasets, that the proposed weighted distribution is efficient to give better results of fitting. Thus it can be asserted that WDFWD is capable to exhibit its supremacy over the parent distribution DFWD and the other four competitive distributions viz. discrete generalised Weibull (DGW), discrete generalised Inverse Weibull (DGIW), discrete Burr (DBurr) and discrete

Table 12: Summary of goodness of fit criteria for dataset III

X	OBSERVED FREQUENCY	EXPECTED FREQUENCY					
		DR	DBurr	DGIW	DFWD	DGW	WDFWD
0	12	2.64705	14.16725	10.46178	10.45558	9.92442	10.77052
1	8	7.01406	14.49738	15.3920	15.3913	10.28837	9.14844
2	7	9.58223	7.92885	8.81706	8.81849	8.75871	6.09689
3	6	10.01493	4.87819	5.52389	5.52526	7.03619	6.01777
4	3	8.74039	3.40499	3.85417	3.85529	5.47722	5.93716
5	3	6.59604	2.60174	2.91693	2.91787	4.18306	4.86041
6	4	4.38661	2.11926	2.34482	2.34561	3.16014	3.78852
7	6	2.60807	1.80758	1.97223	1.97292	2.37784	3.72145
≥ 8	4	1.41062	1.59476	1.71703	1.71764	1.79405	2.65884
N	53	53	53	53	53	53	53
χ^2		50.42054	18.73986	16.81294	16.80514	11.35708	4.66689
Df		5	2	3	3	4	4
p-value		≤ 0.0001	0.00008	0.00077	0.00078	0.02283	**0.32321**
C		0.95133	0.35358	0.31723	0.31708	0.21428	**0.08805**

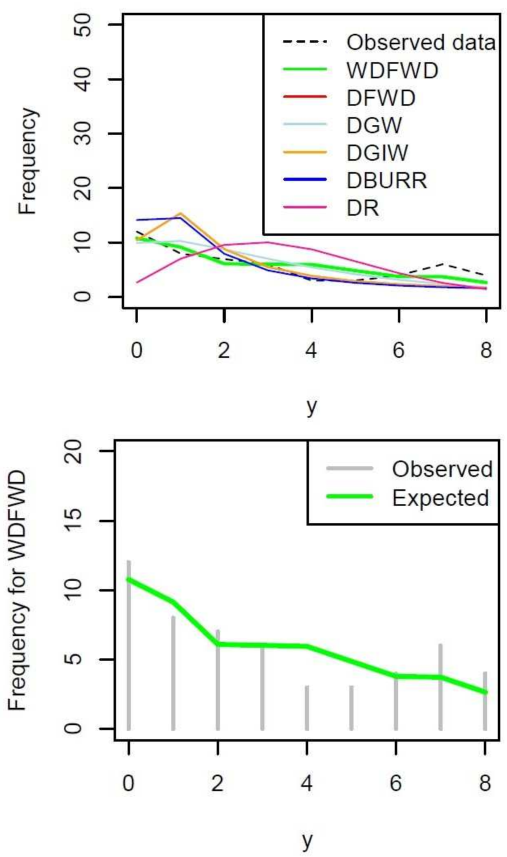


Fig. 6: Observed and Estimated frequency graph of different models for dataset III.

Rayleigh (DR) distribution in reference to the above discussed application cases.

Further as a continuation of this study one can develop the bivariate and multivariate extensions of the proposed distribution. Along with that, some work on statistical inference can also be applied. As we know variation in the choice of weight functions leads to different forms of weighted distributions, which will later give different characteristic for the newly developed one. Thus, for this distribution a study with various weight alternatives is also appreciated. Moreover, the weighted distributions have a vast field of practical situations where it is applicable. So this model can also be further applied to such real-life datasets which can preferably be claimed to give better fit than the existing distributions.

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Code Availability

R Programming software.

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