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Abdelfattah Mustafa
Mathematics Department, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia\\ Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, amelsayed@mans.edu.eg

M. I. Khan
Mathematics Department, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia, amelsayed@mans.edu.eg

Maher. A. Alraddadi
Mathematics Department, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia, amelsayed@mans.edu.eg

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Improving the Performance of a Series-Parallel System Based on Lindley Distribution

Abdelfattah Mustafa[1,2,*], *M. I. Khan*[1] *and Maher. A. Alraddadi*[1]

[1]Mathematics Department, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia
[2]Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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Abstract: In this article, the performance of a series-parallel system is improved. The system components are assumed to follows independently and identically Lindley distributed with three parameters. The system reliability for the given system will be improved by using reduction method, hot, cold and imperfect duplication method. Some reliability measures are derived. Two types of reliability equivalence factors and gamma fractiles are calculated. A numerical example is introduced to explain the theoretical results.

Keywords: Lindley distribution, reliability equivalence, series-parallel, improving methods.

1 Introduction

Råde [1] obtained the reliability equivalence factors (REF) for some simple systems. Sarhan [2,3] is provided four methods:

(i) Reduction method (RM): the failure rates are reduced by a factor $\rho, 0 < \rho < 1$;
(ii) Hot duplication method (HDM): It assumes that some components of the system will be connected to components in a parallel system (one for each).
(iii) Cold duplication method (CDM): In this method cold coupling is used which assumes that some components will be connected to components via a perfect switch (one for each).
(iv) Imperfect duplication method (IDM): It will differ from the previous method, CDM, in that the switch used in the connection process is an imperfect switch. The switch has lifetime distribution.

Various systems are improved by applying the concept of REF, see [4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22]
A random variable T has a three-parameters Lindley distribution (TPLD), if it has the pdf given by

$$f(t;\alpha,\beta,\theta) = \frac{\theta^2(\alpha+\beta t)}{\alpha\theta+\beta} e^{-\theta t}, \quad t \geq 0, \qquad (1)$$

where $\theta > 0, \alpha > 0, \alpha\theta+\beta > 0$.

The TPLD can be easily expressed as

$$f(t;\alpha,\beta,\theta) = pg_1(t) + (1-p)g_2(t),$$

where $g_1(t) = \theta e^{-\theta t} \sim Exp(\theta)$, $g_2(t) = \theta^2 t e^{-\theta t} \sim Gamma(2,\theta)$ and $p = \frac{\alpha\theta}{\alpha\theta+\beta}$.
The TPLD has the following cumulative distribution function (CDF),

$$F(t;\alpha,\beta,\theta) = 1 - \left(1 + \frac{\theta\beta t}{\alpha\theta+\beta}\right) e^{-\theta t}, \quad t \geq 0. \qquad (2)$$

Many interesting properties of TPLD and its applications are discussed in [23]. The TPLD contains some models:

1. The TPLD, reduced to two-parameter quasi-Lindley distribution if $\beta = \theta$, [24],
2. If $\beta = 1$, we have two-parameter Lindley distribution, [25],
3. When $\alpha = 1$, two-parameter Lindley distribution, is obtained, [26],
4. A new two-parameter quasi-Lindley distribution is obtained if $\alpha = \theta, \beta = \alpha$, [27],
5. If $\alpha = \beta = 1$, we have Lindley distribution, [28],
6. The TPLD is reduced to Gamma $(2,\theta)$ distribution, when $\alpha = \theta$,
7. The exponential distribution is a special model of TPLD, if $\beta = 0$,

* Corresponding author e-mail: amelsayed@mans.edu.eg

The TPLD has the following failure rate

$$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{\theta^2(\alpha+\beta t)}{\beta+\theta(\alpha+\beta t)}.$$

The $\lambda(t)$ is a function of time. Since

$$\frac{d}{dt}\lambda(t) = \left(\frac{\beta\theta}{\alpha\theta+\beta+\theta\beta t}\right)^2 > 0, \quad \text{for all}\, t \geq 0.$$

Therefore, $\lambda(t)$ is increasing failure rate function.

2 Original system

The series-parallel system (SPS) consists n subsystems connected in series. Each subsystem has m_i components in parallel mode, such that $M = \sum_{i=1}^{n} m_i$, see Figure 1, [29,30,31].

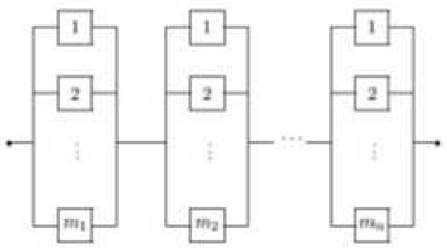

Fig. 1: SPS structure

The lifetime of the system components is independent and identically distributed with a TPLD. The survival function (SF) for a component j is

$$S_{ij}(t) = 1-F(t) = \left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t},\ t \geq 0, \quad (3)$$

where $\alpha,\beta,\theta > 0$, and $j = 1,\cdots,m_i, i = 1,2,\cdots,n$.
The SF of the subsystem i, $S_i(t)$, can be expressed as,

$$S_i(t) = 1-\prod_{i=1}^{m_i} F_{ij}(t) = 1-\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i}. \quad (4)$$

Let SF of the SPS, $S(t)$, is

$$S(t) = \prod_{i=1}^{n} S_i(t) = \prod_{i=1}^{n}\left\{1-\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i}\right\}. \quad (5)$$

The mean time to failure (MTTF) to the SPS is calculated by, [32]

$$m = \int_0^\infty S(t)dt. \quad (6)$$

Some numerical techniques can be used to calculate the MTTF.

3 The Improved Systems

The SPS are improved by using the following methods.

3.1 The RM

To improve the system reliability, the failure rates of r will be reduced, where $0 \leq r \leq M$. Here, the failure rate will be reduced by reducing the scale parameter only, by the factor $\rho, 0 < \rho < 1$. From each subsystem, r_i components, $0 \leq r_i \leq m_i$, will be improved, such that $r = \sum_{i=1}^{n} r_i$.
The SF of the component j in the subsystem i, $S_{ij,\rho}(t)$, after reducing its failure rate is given as

$$S_{ij,\rho} = \left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)e^{-\rho\theta t}. \quad (7)$$

The SF of the subsystem i after reducing the failure rates of r_i components, $S_{r_i,\rho}$, is obtained as follows.

$$\begin{aligned} S_{r_i,\rho}(t) &= 1-\left[1-S_{ij,\rho}(t)\right]^{r_i}\left[1-S_{ij}(t)\right]^{m_i-r_i} \\ &= 1-\left[1-\left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)e^{-\rho\theta t}\right]^{r_i} \times \\ &\quad \left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-r_i}. \end{aligned} \quad (8)$$

The SF of the improved system when the failure rate of r components are reduced, is

$$\begin{aligned} S_{r,\rho}(t) &= \prod_{i=1}^{n} S_{r_i,\rho}(t) \\ &= \prod_{i=1}^{n}\left\{1-\left[1-\left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)e^{-\rho\theta t}\right]^{r_i} \times \right. \\ &\quad \left.\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-r_i}\right\}. \end{aligned} \quad (9)$$

We can calculate the MTTF of the improved system, say $m_{r,\rho}$, by

$$m_{r,\rho} = \int_0^\infty S_{r,\rho}(t)dt. \quad (10)$$

We can calculate the above integral by using some numerical techniques to find $m_{r,\rho}$.

3.2 The HDM

The system will be improved by duplicating ℓ, $0 \leq \ell \leq M$, components, where each component is duplicated by a hot redundant identical standby component. From each subsystem ℓ_i components will be improved by HDM, such that $0 \leq \ell_i \leq m_i$, and $\ell = \sum_{i=1}^{n} \ell_i$.
Let $S_{\ell_i}^H(t)$, be the SF of the improved subsystem by HDM, then

$$S_{\ell_i}^H(t) = 1-\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i+\ell_i}. \quad (11)$$

The SF of the improved system by improving ℓ components according to the HDM, can be obtained as

$$S_{\ell}^{H}(t) = \prod_{i=1}^{n} S_{\ell_i}^{H}(t)$$
$$= \prod_{i=1}^{n} \left\{ 1 - \left[1 - \left(1 + \frac{\beta\theta t}{\alpha\theta+\beta} \right) e^{-\theta t} \right]^{m_i+\ell_i} \right\}. \quad (12)$$

From equation (12), we can calculate the MTTF as follows.

$$m_{\ell}^{H} = \int_{0}^{\infty} S_{\ell}^{H}(t)dt. \quad (13)$$

Some numerical techniques can be used to calculate the above integral to find m_{ℓ}^{H}.

3.3 The CDM

In this method, consider each component of the ℓ is connected with an identical component via a perfect switch. The SF, $S_{\ell_i}^{C}(t)$, of the improved subsystem i, according to CDM is

$$S_{\ell_i}^{C}(t) = 1 - \left[1 - S_{ij}^{C}(t)\right]^{\ell_i} \left[1 - S_{ij}(t)\right]^{m_i-\ell_i}, \quad (14)$$

where

$$S_{ij}^{C}(t) = S_1(t) + \int_0^t f_1(x)S_2(t-x)dx = \left[1 + \frac{\beta\theta t}{\beta+\alpha\theta} + \frac{\theta^2[6\alpha(\beta+\alpha\theta)+3\beta(\beta+2\alpha\theta)t+\theta\beta^2t^2]t}{6(\beta+\alpha\theta)^2}\right] e^{-\theta t}. \quad (15)$$

Then, the SF of the improved system by use CDM to improve ℓ components is given as

$$S_{\ell}^{C}(t) = \prod_{i=1}^{n} S_{\ell_i}^{C}(t)$$
$$= \prod_{i=1}^{n} \left\{ 1 - \left[1 - \left(1 + \frac{\beta\theta t}{\alpha\theta+\beta} \right) e^{-\theta t} \right]^{m_i-\ell_i} \left[1 - \left(\frac{\theta^2[6\alpha(\beta+\alpha\theta)+3\beta(\beta+2\alpha\theta)t+\theta\beta^2t^2]t}{6(\beta+\alpha\theta)^2} + \frac{\beta\theta t}{\beta+\alpha\theta} + 1 \right) e^{-\theta t} \right]^{\ell_i} \right\}. \quad (16)$$

The MTTF to the improved system can be calculated by

$$m_{\ell}^{C} = \int_{0}^{\infty} S_{\ell}^{C}(t)dt. \quad (17)$$

By using some Mathematical Programs, (17) can be calculated.

3.4 The IDM

Suppose each component of ℓ is connected with an identical component via an imperfect switch. The switch has TPLD with parameters α, β and ν. Let $S_{\ell_i}^{I}(t)$ be the SF of subsystem i, after improved by the IDM, we have

$$S_{\ell_i}^{I}(t) = 1 - \left[1 - S_{ij}^{I}(t)\right]^{\ell_i} \left[1 - S_{ij}(t)\right]^{m_i-\ell_i}, \quad (18)$$

where

$$S_{ij}^{I}(t) = S_1(t) + \int_0^t f_1(x)S_{sw}(x)S_2(t-x)dx$$
$$= \left(1 + \frac{\beta\theta t}{\beta+\alpha\theta}\right) e^{-\theta t} + \frac{\theta^2 e^{-(\theta+\nu)t}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3} \times \{(-1+e^{\nu t})\alpha^3\theta\nu^3 + \alpha\beta^2\nu[-\theta(-1+\nu t)(2+\nu t) - \nu(3+2\nu t-3e^{\nu t}) + \theta(-2+3\nu t)e^{\nu t}] + \alpha^2\beta\nu^2[-\nu-2\theta(1+\nu t)+(\nu+\theta(2+\nu t))e^{\nu t}] + \beta^3[-\nu(3-3e^{\nu t}+(3+\nu t)\nu t)+\theta(8+(5+\nu t)\nu t + (-8+3\nu t)e^{\nu t})]\}. \quad (19)$$

Substituting from (19) into (18), the SF, $S_{\ell}^{I}(t)$, of the improved system by IDM, has the following form

$$S_{\ell}^{I}(t) = \prod_{i=1}^{n} S_{\ell_i}^{I}(t)$$
$$= \prod_{i=1}^{n} \left\{ 1 - \left[1 - \left(1 + \frac{\beta\theta t}{\alpha\theta+\beta} \right) e^{-\theta t} \right]^{m_i-\ell_i} \left[1 - \left(1 + \frac{\beta\theta t}{\beta+\alpha\theta} \right) e^{-\theta t} - \frac{\theta^2 e^{-(\theta+\nu)t}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3} \{(-1+e^{\nu t})\alpha^3\theta\nu^3 + \alpha\beta^2\nu[-\theta(-1+\nu t)(2+\nu t) - \nu(3+2\nu t-3e^{\nu t}) + \theta(-2+3\nu t)e^{\nu t}] + \alpha^2\beta\nu^2[-\nu-2\theta(1+\nu t)+(\nu+\theta(2+\nu t))e^{\nu t}] + \beta^3[-\nu(3-3e^{\nu t}+(3+\nu t)\nu t)+\theta(8+(5+\nu t)\nu t + (-8+3\nu t)e^{\nu t})]\} \right]^{\ell_i} \right\}. \quad (20)$$

The MTTF to the improved system by IDM is derived by

$$m_{\ell}^{I} = \int_{0}^{\infty} S_{\ell}^{I}(t)dt. \quad (21)$$

The integration in (21), can be calculated numerically by using some numerical techniques.

4 The γ-Fractiles

The performance of the systems reliability can be compared by using the γ-fractiles measure. The γ-fractiles of the SPS, $\mathscr{F}(\gamma)$, can be found as a solution $\mathscr{F} = \mathscr{F}(\gamma)$ of the following equation:

$$S\left(\frac{\mathscr{F}(\gamma)}{\Theta}\right) = \gamma, \quad (22)$$

where $\Theta = M\theta$, $M = \sum_{i=1}^{n} m_i$.
Substituting from (5) into (22), $\mathscr{F} = \mathscr{F}(\gamma)$ satisfies the following non-linear equation

$$\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\theta\mathscr{F}}{(\alpha\theta+\beta)\Theta}\right)e^{-\frac{\theta}{\Theta}\mathscr{F}}\right]^{m_i}\right\} - \ln(\gamma) = 0. \quad (23)$$

For the duplication methods, the γ-fractiles, $\mathscr{F}_\ell^D(\gamma)$, are the solution of the equation,

$$S_\ell^D\left(\frac{\mathscr{F}(\gamma)}{\Theta}\right) = \gamma, \qquad D = H, I, \text{ and } C. \quad (24)$$

From equations (12) and (24), $\mathscr{F} = \mathscr{F}_\ell^H(\gamma)$ can be derived as a solution of

$$\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\theta\mathscr{F}}{(\alpha\theta+\beta)\Theta}\right)e^{-\frac{\theta}{\Theta}\mathscr{F}}\right]^{m_i+\ell_i}\right\} - \ln(\gamma) = 0. \quad (25)$$

For $D = C$, and from equations (16) and (24), $\mathscr{F} = \mathscr{F}_\ell^C(\gamma)$ is the solution of

$$\sum_{i=}^{n} \ln\Bigg\{1-\Bigg[1-\Bigg(1+\frac{\beta\theta\mathscr{F}}{(\beta+\alpha\theta)\Theta}+\frac{\theta^2\mathscr{F}}{6\Theta^3(\beta+\alpha\theta)^2}\times [6\alpha(\beta+\alpha\theta)\Theta^2+3\beta(\beta+2\alpha\theta)\Theta\mathscr{F}+\theta\beta^2\mathscr{F}^2]\Bigg)e^{-\frac{\theta}{\Theta}\mathscr{F}}\Bigg]^{\ell_i} \left[1-\left(1+\frac{\beta\theta\mathscr{F}}{(\alpha\theta+\beta)\Theta}\right)e^{-\frac{\theta}{\Theta}\mathscr{F}}\right]^{m_i-\ell_i}\Bigg\} - \ln(\gamma) = 0. \quad (26)$$

Substituting from (20) into (24), $\mathscr{F} = \mathscr{F}_\ell^I(\gamma)$ is obtained by solving the following equation.

$$\sum_{i=1}^{n} \ln\Bigg\{1-\left[1-\left(1+\frac{\beta\theta\mathscr{F}}{\Theta(\alpha\theta+\beta)}\right)e^{-\frac{\theta}{\Theta}\mathscr{F}}\right]^{m_i-\ell_i} \times \Bigg[1-\left(1+\frac{\beta\theta\mathscr{F}}{\Theta(\beta+\alpha\theta)}\right)e^{-\frac{\theta}{\Theta}\mathscr{F}} - \frac{\theta^2 e^{-\frac{(\theta+\nu)}{\Theta}\mathscr{F}}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3} \Big\{(-1+e^{\frac{\nu}{\Theta}\mathscr{F}})\alpha^3\theta\nu^3+\alpha\beta^2\nu\Big[-\theta(-1+\frac{\nu}{\Theta}\mathscr{F})(2+\frac{\nu}{\Theta}\mathscr{F}) -\nu(3+\frac{2\nu}{\Theta}\mathscr{F}-3e^{\frac{\nu}{\Theta}\mathscr{F}})+\theta(-2+\frac{3\nu}{\Theta}\mathscr{F})e^{\frac{\nu}{\Theta}\mathscr{F}}\Big]+ \alpha^2\beta\nu^2\left[-\nu-2\theta(1+\frac{\nu}{\Theta}\mathscr{F})+\left(\nu+\theta(2+\frac{\nu}{\Theta}\mathscr{F})\right)e^{\frac{\nu}{\Theta}\mathscr{F}}\right]+ \beta^3\Big[-\nu\left(3-3e^{\frac{\nu}{\Theta}\mathscr{F}}+(3+\frac{\nu}{\Theta}\mathscr{F})\frac{\nu}{\Theta}\mathscr{F}\right)+ \theta\left(8+(5+\frac{\nu}{\Theta}\mathscr{F})\frac{\nu}{\Theta}\mathscr{F}+(-8+\frac{3\nu}{\Theta}\mathscr{F})e^{\frac{\nu}{\Theta}\mathscr{F}}\right)\Big]\Big\}\Bigg]^{\ell_i}\Bigg\} - \ln(\gamma) = 0. \quad (27)$$

The equations (23), (25) – (27) can be solved numerically by some numerical technique.

5 The REFs

The REFs are derived in this section. The REFs of TPLD are a function of time t. The $\lambda(t)$ is reduced by the factor $r(t)$. For convenience of calculation, the scale parameter, θ is reduced to $\rho\theta$ only. That is

$$r(t)\lambda(t) = \frac{\rho^2\theta^2(\alpha+\beta t)}{\beta+\rho\theta(\alpha+\beta t)}. \quad (28)$$

In this section, we will deduce two types of REFs of the SPS: (i) the survival reliability equivalence factor (SREF), (ii) mean reliability equivalence factor (MREF) as follows.

5.1 The SREF

The SREF, $\rho_{r,\ell}^D(\gamma)$, is obtained by equating the survival function of the improved system that is obtained by reduction method with duplication method at the level γ. $\rho_{r,\ell}^D(\gamma)$, can be obtained by solving the following system:

$$S_{r,\rho}(t) = \gamma, \quad S_\ell^D(t) = \gamma, \; \gamma \in (0,1). \quad (29)$$

1.Using equation (29) together with equations (9) and (12), the HREF, $\rho = \rho_{r,\ell}^H(\gamma)$, can be derived by solving the following system

$$\left.\begin{aligned} &\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)e^{-\rho\theta t}\right]^{r_i} \times \left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-r_i}\right\} - \ln(\gamma) = 0 \\ &\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i+\ell_i}\right\} - \ln(\gamma) = 0 \end{aligned}\right\} \quad (30)$$

2.The cold REF, $\rho = \rho_{r,\ell}^C(\gamma)$, can be obtained by substituting from (9) and (16) into (29), and solve the following system with respect to ρ.

$$\left.\begin{aligned} &\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)e^{-\rho\theta t}\right]^{r_i} \times \left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-r_i}\right\} - \ln(\gamma) = 0 \\ &\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\theta t}{\beta+\alpha\theta}+\frac{\theta^2 t}{6(\beta+\alpha\theta)^2}\times [6\alpha(\beta+\alpha\theta)+3\beta(\beta+2\alpha\theta)t+\theta\beta^2t^2]\right)e^{-\theta t}\right]^{\ell_i} \times \left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-\ell_i}\right\} - \ln(\gamma) = 0 \end{aligned}\right\} \quad (31)$$

3. Using (9) and (20) together with (29), the imperfect REF, $\rho = \rho^{I}_{r,\ell}(\gamma)$, satisfies the following system

$$\left.\begin{aligned}
&\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\rho\theta t}{\alpha\rho\theta+\beta}\right)e^{-\rho\theta t}\right]^{r_i}\times\right.\\
&\qquad\left.\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-r_i}\right\}-\ln(\gamma)=0\\
&\sum_{i=1}^{n} \ln\left\{1-\left[1-\left(1+\frac{\beta\theta t}{\alpha\theta+\beta}\right)e^{-\theta t}\right]^{m_i-\ell_i}\times\right.\\
&\left[1-\left(1+\frac{\beta\theta t}{\beta+\alpha\theta}\right)e^{-\theta t}-\frac{\theta^2 e^{-(\theta+\nu)t}}{(\beta+\alpha\theta)^2(\beta+\alpha\nu)\nu^3}\times\right.\\
&\Big\{(-1+e^{\nu t})\alpha^3\theta\nu^3+\alpha\beta^2\nu\Big[-\theta(-1+\nu t)(2+\nu t)-\\
&\qquad\nu(3+2\nu t-3e^{\nu t})+\theta(-2+3\nu t)e^{\nu t}\Big]+\\
&\alpha^2\beta\nu^2\left[-\nu-2\theta(1+\nu t)+(\nu+\theta(2+\nu t))e^{\nu t}\right]\\
&+\beta^3\Big[-\nu(3-3e^{\nu t}+(3+\nu t)\nu t)+\theta(8+(5+\nu t)\nu t\\
&\qquad\left.\left.+(-8+3\nu t)e^{\nu t})\Big]\Big\}\right]^{\ell_i}\right\}-\ln(\gamma)=0
\end{aligned}\right\}. \tag{32}$$

By using some numerical techniques $\rho = \rho^{D}_{r,\ell}(\gamma)$ can be obtained from the systems (30)-(32).

5.2 The MREF

The MREF, $\xi^{D}_{r,\ell}$, can be derived by equating the MTTF of the improved system that obtained by improving the system according to RM with the duplication method. The $\xi = \xi^{D}_{r,\ell}$ is the solution of the following equation:

$$m_{r,\xi} = m^{D}_{\ell}. \tag{33}$$

By substituting from (10), (13), (17) and (21) into (33), the $\xi = \xi^{D}_{r,\ell}$ can be obtained for $D = H, C$ and I, respectively.

6 Numerical Results

Consider the following assumptions:

1. Let $n = 2$, and $m_1 = 1, m_2 = 2$, so $M = \sum_{i=1}^{n} m_i = 3$. The SPS has the following structure (Radar system), see Figure 2.
2. The lifetime of the components is TPLD, with $\alpha = 0.1, \beta = 0.2, \theta = 0.7$ and $\nu = 0.3$.
3. The system will be improved by improving ℓ components according to HDM, CDM and IDM.
4. In the reduction method, r_1 components from subsystem 1, and r_2 components from subsystem 2 are improved by reducing their failure rates by the factor ρ.

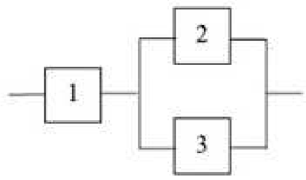

Fig. 2: The radar system.

Therefore, the MTTF of the system is 1.83258. The values of m^{D}_{ℓ} for $D = H, I$ and C are displayed is Table 1.

Table 1: The values of m^{D}_{ℓ} for $D = H, I, C$ and $\ell = (\ell_1, \ell_2)$.

(ℓ_1, ℓ_2)	m^{H}_{ℓ}	m^{I}_{ℓ}	m^{C}_{ℓ}
(1,0)	2.40608	2.68710	2.80534
(0,1)	2.01286	2.12252	2.17095
(1,1)	2.68941	3.31509	3.66955
(0,2)	2.11590	2.25021	2.30204
(1,2)	2.85903	3.63069	4.07473

Figure 3–5 displays the comparison among original and improved systems for each $\ell = (\ell_1, \ell_2)$.

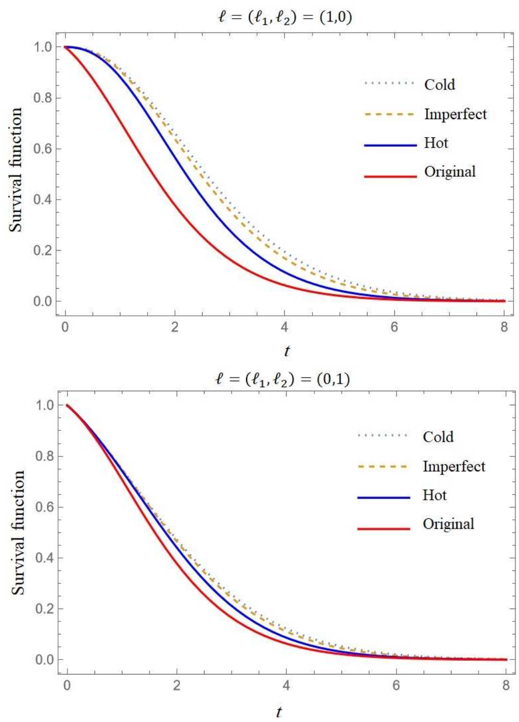

Fig. 3: The $S(t)$, $S^{D}_{\ell}(t)$, when $\ell = 1$.

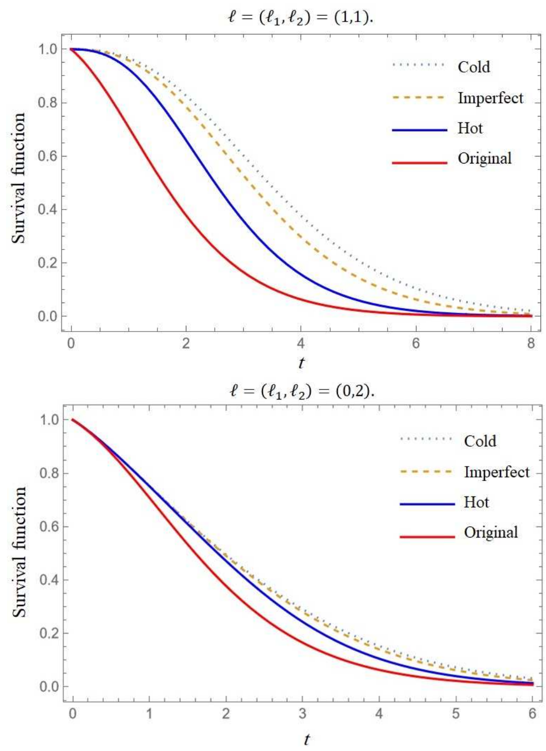

Fig. 4: The $S(t)$, $S^D_\ell(t)$, for $\ell = 2$.

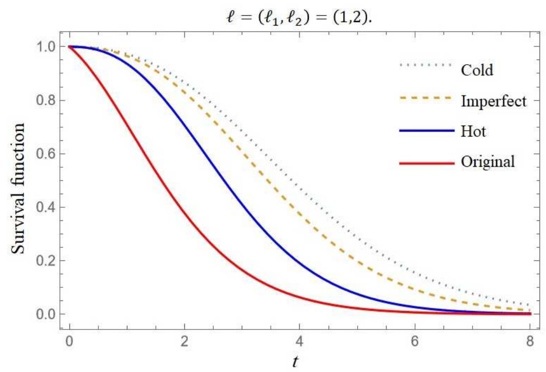

Fig. 5: The $S(t)$, $S^D_\ell(t)$, for $\ell = 3$.

Figures 6-8 compare the SF of the original system with each improved system separately for different values of $\ell = (\ell_1, \ell_2)$.

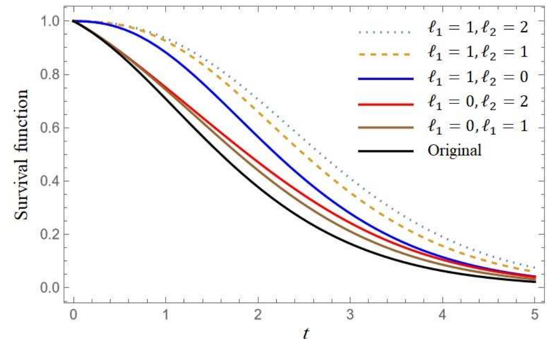

Fig. 6: The $S(t)$, $S^H_\ell(t)$, for different values of $\ell = (\ell_1, \ell_2)$.

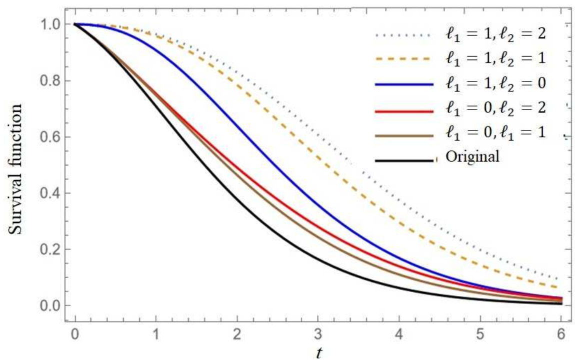

Fig. 7: The $S(t)$, $S^I_\ell(t)$, for different values of $\ell = (\ell_1, \ell_2)$.

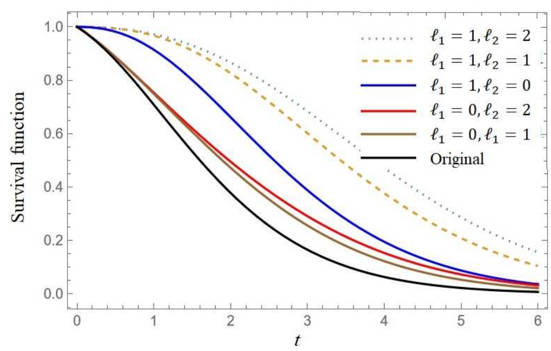

Fig. 8: The $S(t)$, $S^C_\ell(t)$, for different values of $\ell = (\ell_1, \ell_2)$.

The Mathematica Program System are used to calculate the values of γ-fractiles, $\mathscr{F}(\gamma)$, $\mathscr{F}^D_\ell(\gamma)$ and REFs, $\rho^D_{r,\ell}(\gamma)$, $D = H, I$ and C. The γ is chosen to be $0.1, 0.2, \cdots, 0.9$. Tables 2 and 3 introduce the values of $\mathscr{F}(\gamma)$, $\mathscr{F}^D_\ell(\gamma)$, $D = H, I, C$ for different values of $\ell = (\ell_1, \ell_2)$.
From Figures 3-8 and Tables 2 - 3, we can conclude that:

Table 2: The values of $\mathscr{F}(\gamma)$, $\mathscr{F}_\ell^D(\gamma)$, $D=H,I,C$ for $(\ell_1,\ell_2)=\{(1,0),(0,1),(1,1)\}$.

		$\ell=(1,0)$			$\ell=(0,1)$			$\ell=(1,1)$		
γ	$\mathscr{F}$	$\mathscr{F}^H$	$\mathscr{F}^I$	$\mathscr{F}^C$	$\mathscr{F}^H$	$\mathscr{F}^I$	$\mathscr{F}^C$	$\mathscr{F}^H$	$\mathscr{F}^I$	$\mathscr{F}^C$
0.1	7.4287	8.7029	9.7042	10.1684	8.0779	8.6552	8.9287	9.4107	11.4680	12.7342
0.2	5.8481	7.1274	7.9652	8.3297	6.4411	6.8596	7.0395	7.8202	9.5974	10.6384
0.3	4.8298	6.1074	6.8328	7.1346	5.3650	5.6753	5.7987	6.7824	8.3632	9.2583
0.4	4.0374	5.3090	5.9432	6.1973	4.5106	4.7359	4.8192	5.9633	7.3816	8.1626
0.5	3.3600	4.6213	5.1749	5.3892	3.7649	3.9202	3.9734	5.2513	6.5230	7.2059
0.6	2.7431	3.9882	4.4662	4.6452	3.0719	3.1692	3.2001	4.5888	5.7197	6.3126
0.7	2.1502	3.3695	3.7727	3.9187	2.3941	2.4455	2.4606	3.9329	4.9201	5.4252
0.8	1.5468	2.7202	3.0444	3.1575	1.6976	1.7171	1.7224	3.2327	4.0615	4.4744
0.9	0.8808	1.9495	2.1798	2.2568	0.9383	0.9415	0.9423	2.3795	3.0074	3.3103

Table 3: The values of $\mathscr{F}(\gamma)$, $\mathscr{F}_\ell^D(\gamma)$, $D=H,I,C$ for $(\ell_1,\ell_2)=\{(0,2),(1,2)\}$.

		$\ell=(0,2)$			$\ell=(1,2)$		
γ	$\mathscr{F}$	$\mathscr{F}^H$	$\mathscr{F}^I$	$\mathscr{F}^C$	$\mathscr{F}^H$	$\mathscr{F}^I$	$\mathscr{F}^C$
0.1	7.4287	8.5173	9.3072	9.6381	9.8948	12.4055	13.9527
0.2	5.8481	6.8276	7.3722	7.5702	8.2839	10.4793	11.7804
0.3	4.8298	5.6969	6.0712	6.1904	7.2239	9.1912	10.3262
0.4	4.0374	4.7844	5.0280	5.0959	6.3799	8.1535	9.1543
0.5	3.3600	3.9769	4.1206	4.1555	5.6392	7.2337	8.1153
0.6	2.7431	3.2200	3.2922	3.3075	4.94276	6.3609	7.1296
0.7	2.1502	2.4807	2.5086	2.5138	4.2443	5.4785	6.1334
0.8	1.5468	1.7338	1.7404	1.7416	3.4879	4.5146	5.0462
0.9	0.8808	0.9449	0.9453	0.9455	2.5523	3.3112	3.6913

1. $S(t) < S_\ell^H(t) < S_\ell^I(t) < S_\ell^C(t)$, in all studied cases.
2. $m < m_\ell^H < m_\ell^I < m_\ell^C$, in all studied cases.
3. $\mathscr{F}(\gamma) < \mathscr{F}_\ell^H(\gamma) < \mathscr{F}_\ell^I(\gamma) < \mathscr{F}_\ell^C(\gamma)$, in all studied cases.
4. Improving one component from the subsystem 1, $\ell_1 = 1$, produces a better design than improving one component from the subsystem 2, $\ell_2 = 1$, according to the same method.
5. Improving two components, one from each subsystem, $\ell = (1,1)$, gives a better design than improving two components from the subsystem 2, $\ell = (0,2)$.
6. Improving all system components, $\ell = (1,2)$, gives the best design.
7. CDM gives the best improvement than other duplication methods.

Tables 4 and 5 contain the values of the SREF for different values of r,ℓ.
According to the results presented in Tables 4 and 5:

1. Improving one component, $\ell_1 = 1$, by HDM, the $\mathscr{F}(0.1)$ will be increased from $\frac{7.4287}{\Theta}$ to $\frac{8.7029}{\Theta}$, see Table 2. The same effect can be obtained by reducing the failure rates of (i) one component, $r_1 = 1$, by $\rho^H = 0.73987$, (ii) one component, $r_2 = 1$, by $\rho^H = 0.53439$, (iii) two components, $r_1 = r_2 = 1$, by $\rho^H = 0.82909$, (iv) two components, $r_2 = 2$, by $\rho^H = 0.69363$, (v) three components, $r_1 = 1, r_2 = 2$, by $\rho^H = 0.86449$, see Table 4.
2. Improving one component, $\ell_1 = 1$, by IDM, the $\mathscr{F}(0.1)$ will be increased from $\frac{7.4287}{\Theta}$ to $\frac{9.7042}{\Theta}$, see Table 2. The same effect can be occurred by reducing the failure rates of (i) one component, $r_1 = 1$, by $\rho^I = 0.57072$, (ii) one component, $r_2 = 1$, by $\rho^I = 0.30209$, (iii) two components, $r_1 = r_2 = 1$, by $\rho^I = 0.73059$, (iv) two components, $r_2 = 2$, by $\rho^I = 0.47560$, (v) three components, $r_1 = 1, r_2 = 2$, by $\rho^I = 0.78154$, see Table 4.
3. Improving one component, $\ell_1 = 1$, by using CDM, the $\mathscr{F}(0.1)$ can be increased from $\frac{7.4287}{\Theta}$ to $\frac{10.1684}{\Theta}$, see Table 2. The same effect can be obtained by reducing the failure rates of (i) one component, $r_1 = 1$, by $\rho^C = 0.49970$, (ii) one component, $r_2 = 1$, by $\rho^C = 0.20430$, (iii) two components, $r_1 = r_2 = 1$, by $\rho^C = 0.69257$, (iv) two components, $r_2 = 2$, by $\rho^C = 0.18714$, (v) three components, $r_1 = 1, r_2 = 2$, by $\rho^C = 0.74831$, see Table 4.
4. The rest of the results in Tables 4 and 5 can be interpreted in the same way.
5. The symbol – means that there is no equivalence between the two optimized systems: the one obtained by reducing the failure rates of r components and the one obtained by optimizing the ℓ components according to duplication methods.

Table 6 displays the values of MREF for different value of $r,\ell \in \{(1,0),(0,1),(1,1),(0,2),(1,2)\}$.

From Table 6, one can conclude that:

Table 4: The values of, $\rho^{D}_{r,\ell}(\gamma)$, $D = H, I, C$ for different values of r and $\ell \in \{(1,0),(0,1),(1,1)\}$.

		$\ell=(1,0)$			$\ell=(0,1)$			$\ell=(1,1)$		
γ	(r_1,r_2)	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C
0.1	(1,0)	0.73987	0.57072	0.49970	0.85990	0.74859	0.69949	0.61780	0.31670	0.09668
	(0,1)	0.53439	0.30209	0.20430	0.72316	0.54715	0.47670	0.36509	–	–
	(1,1)	0.82909	0.73059	0.69257	0.90521	0.83445	0.80461	0.75691	0.60516	0.53984
	(0,2)	0.69363	0.47560	0.18714	0.83686	0.70421	0.64403	0.53972	0.08000	0.01599
	(1,2)	0.86449	0.78154	0.74831	0.92597	0.86889	0.84426	0.80413	0.66882	0.60620
0.2	(1,0)	0.69419	0.53012	0.46452	0.84783	0.74170	0.61275	0.55705	0.24912	–
	(0,1)	0.40532	0.12170	0.19223	0.67012	0.49981	0.43544	0.17707	–	–
	(1,1)	0.79601	0.70093	0.66620	0.89452	0.78319	0.70346	0.71575	0.56843	0.50773
	(0,2)	0.59177	0.37985	0.00445	0.80407	0.67421	0.61894	0.35182	0.01100	0.00092
	(1,2)	0.83621	0.75555	0.72518	0.91662	0.86582	0.83457	0.76836	0.63641	0.57841
0.3	(1,0)	0.65730	0.49902	0.43774	0.84662	0.63421	0.57311	0.50948	0.19337	–
	(0,1)	0.25944	0.07391	0.05570	0.62574	0.46224	0.40289	–	–	–
	(1,1)	0.76804	0.67678	0.64470	0.88885	0.73432	0.68143	0.68245	0.54010	0.48308
	(0,2)	0.45754	0.23817	0.00078	0.77590	0.64934	0.59780	0.10822	0.00911	NA
	(1,2)	0.81143	0.73341	0.70526	0.91106	0.76653	0.76500	0.73834	0.61019	0.55578
0.4	(1,0)	0.62317	0.47128	0.41392	0.84441	0.57752	0.53787	0.46669	0.13694	–
	(0,1)	0.16358	0.00632	0.00477	0.58322	0.42761	0.37301	–	–	–
	(1,1)	0.74100	0.65395	0.62430	0.88654	0.64044	0.52449	0.65151	0.51463	0.46099
	(0,2)	0.30128	0.20225	–	0.74818	0.63752	0.57725	0.00226	0.00217	–
	(1,2)	0.78667	0.61158	0.60854	0.87993	0.68700	0.68568	0.70944	0.58552	0.53435
0.5	(1,0)	0.58916	0.44441	0.39091	–	0.49823	0.47809	0.42524	0.06213	–
	(0,1)	–	–	–	0.53932	0.39302	0.34328	–	–	–
	(1,1)	0.71281	0.63047	0.60321	0.87533	0.58503	0.48382	0.62049	0.48973	0.43944
	(0,2)	0.23750	0.17307	–	0.71866	0.59525	0.55562	0.00101	–	–
	(1,2)	0.69643	0.58822	0.56640	0.70739	0.57644	0.58630	0.67939	0.56029	0.45685
0.6	(1,0)	0.55323	0.41670	0.36723	–	0.42459	0.38130	0.38278	0.00599	–
	(0,1)	–	–	–	0.43223	0.35632	0.21573	–	–	–
	(1,1)	0.68155	0.60463	0.57984	0.82540	0.46474	0.39794	0.58744	0.46376	0.41700
	(0,2)	0.10058	0.09833	–	0.68511	0.52643	0.34646	–	–	–
	(1,2)	0.48601	0.46146	0.43929	0.65717	0.48864	0.45792	0.53058	0.52368	0.38801
0.7	(1,0)	0.51284	0.38622	0.24475	–	0.38574	0.24937	0.33677	–	–
	(0,1)	–	–	–	0.39283	0.31504	0.20766	–	–	–
	(1,1)	0.64442	0.57402	0.55193	0.80306	0.38848	0.35605	0.54985	0.43480	0.39200
	(0,2)	0.01476	–	–	0.64418	0.43715	0.33505	–	–	–
	(1,2)	0.37877	0.36284	0.29276	0.55684	0.39011	0.37658	0.49063	0.49002	0.29459
0.8	(1,0)	0.46342	0.34967	0.14628	–	0.29744	0.15536	0.28312	–	–
	(0,1)	–	–	–	0.28110	0.22458	0.15323	–	–	–
	(1,1)	0.59591	0.53391	0.47723	0.72166	0.25607	0.15511	0.50309	0.39937	0.36139
	(0,2)	–	–	–	0.48864	0.37015	0.25196	–	–	–
	(1,2)	0.19333	0.18334	0.10333	0.46100	0.22940	0.20328	0.43539	0.41583	0.27422
0.9	(1,0)	0.39230	0.29798	0.11945	–	0.14521	0.09628	0.21153	–	–
	(0,1)	–	–	–	–	–	–	–	–	–
	(1,1)	0.51986	0.47023	0.33435	0.67153	0.19662	0.09714	0.43391	0.34767	0.31662
	(0,2)	–	–	–	0.29524	0.10914	0.09402	–	–	–
	(1,2)	0.09833	0.08952	0.00445	0.37363	0.19687	0.14685	0.36440	0.29352	0.19685

1. Improving one component, $\ell_1 = 1$, by HDM, has the same MTTF of the system which can be obtained by reducing the failure rate of (i) one component, $r_1 = 1$, by $\xi^H = 0.62457$, (ii) one component, $r_2 = 1$, by $\xi^H = 0.16540$, (iii) two components, $r_1 = r_2 = 1$, by $\xi^H = 0.32308$, (iv) two components, $r_2 = 2$, by $\xi^H = 0.32308$, (v) three components, $r_1 = 1, r_2 = 2$, by $\xi^H = 0.78642$, see Table 6.
2. Improving one component, $\ell_1 = 1$, by IDM, has the same MTTF of the system which can be obtained by reducing the failure rate of (i) one component, $r_1 = 1$, by $\xi^I = 0.47935$, (ii) two components, $r_1 = r_2 = 1$, by $\xi^I = 0.65853$, (iii) Three components, $r_1 = 1, r_2 = 2$, by $\xi^I = 0.71251$, see Table 6.
3. Improving one component, $\ell_1 = 1$, by CDM, has the same MTTF of the system which can be obtained by

Table 5: The values of, $\rho^D_{r,\ell}(\gamma)$, $D = H, I, C$ for different values of r and $\ell \in \{(0,2),(1,2)\}$.

		$\ell=(0,2)$			$\ell=(1,2)$		
γ	(r_1,r_2)	ρ^H	ρ^I	ρ^C	ρ^H	ρ^I	ρ^C
0.1	(1,0)	0.77418	0.63487	0.58116	0.54108	0.18792	–
	(0,1)	0.58533	0.38803	0.31608	0.26207	–	–
	(1,1)	0.85034	0.76667	0.73635	0.71447	0.55530	0.48984
	(0,2)	0.73509	0.56210	0.49019	0.20440	0.11665	NA
	(1,2)	0.88186	0.81243	0.78652	0.76754	0.62129	0.55616
0.2	(1,0)	0.75876	0.62802	0.46048	0.47261	0.07046	–
	(0,1)	0.51170	0.32450	0.26066	–	–	–
	(1,1)	0.83637	0.76572	0.72482	0.67037	0.51612	0.45505
	(0,2)	0.68403	0.51437	0.44792	0.09240	0.02752	–
	(1,2)	0.79496	0.81088	0.69151	0.72886	0.58658	0.52593
0.3	(1,0)	0.75651	0.57357	0.43819	0.41996	–	–
	(0,1)	0.45167	0.27639	0.21975	–	–	–
	(1,1)	0.83075	0.67322	0.66156	0.63577	0.48693	0.42936
	(0,2)	0.31660	0.24391	0.21373	0.01979	0.00193	–
	(1,2)	0.76359	0.78158	0.58017	0.69735	0.55952	0.50243
0.4	(1,0)	–	0.49633	0.36783	0.37337	–	–
	(0,1)	0.39572	0.23445	0.18495	–	–	–
	(1,1)	0.83108	0.58686	0.57531	0.60453	0.46155	0.40721
	(0,2)	0.30833	0.23131	0.21746	0.01690	–	–
	(1,2)	0.62276	0.62539	0.51569	0.66782	0.53490	0.48109
0.5	(1,0)	–	0.37345	0.22378	0.32903	–	–
	(0,1)	0.33999	0.19548	0.15337	–	–	–
	(1,1)	0.76730	0.50633	0.49911	0.57409	0.43761	0.38648
	(0,2)	0.23437	0.17560	0.14601	0.00289	–	–
	(1,2)	0.56565	0.53953	0.48334	0.63792	0.51052	0.45997
0.6	(1,0)	–	0.28478	0.17381	0.28451	–	–
	(0,1)	0.20738	0.15796	0.12361	–	–	–
	(1,1)	0.63432	0.48317	0.38278	0.76730	0.41360	0.36586
	(0,2)	0.13347	0.13196	0.12393	0.00144	–	–
	(1,2)	0.47403	0.45843	0.28946	0.61148	0.48471	0.43761
0.7	(1,0)	–	–	–	0.23751	–	–
	(0,1)	0.18595	0.10975	0.08772	–	–	–
	(1,1)	0.58591	0.36337	0.28617	0.72314	0.38794	0.34402
	(0,2)	0.04297	0.09833	0.07259	–	–	–
	(1,2)	0.35146	0.38257	0.19371	0.57009	0.45538	0.41219
0.8	(1,0)	–	–	–	0.18478	–	–
	(0,1)	0.16538	0.07914	0.06306	–	–	–
	(1,1)	0.43023	0.25511	0.19010	0.66695	0.35805	0.31884
	(0,2)	–	–	–	–	–	–
	(1,2)	0.29524	0.29128	0.15032	0.47143	0.41869	0.38033
0.9	(1,0)	–	–	–	0.11947	–	–
	(0,1)	–	–	–	–	–	–
	(1,1)	0.31848	0.16815	0.09467	0.58300	0.31654	0.28424
	(0,2)	–	–	–	–	–	–
	(1,2)	0.19707	0.15065	0.09705	0.36248	0.36368	0.33235

reducing the failure rate of (i) one component, $r_1 = 1$, by $\xi^C = 0.42250$, (ii) two components, $r_1 = r_2 = 1$, by $\xi^C = 0.62863$, (iii) three components, $r_1 = 1, r_2 = 2$, by $\xi^C = 0.68550$, see Table 6.

4. The rest results in Table 6, can be explained in the same manner.

7 Conclusion

The performance of SPS based on TPLD was improved. The lifetime of the components assumed to be independently and identically TPLD. Four methods were used to improve the performance of the system, RM, HDM, CDM and IDM. The survival function and mean time to failure for each method was derived. Two

Table 6: The values of $\xi_{r,\ell}^{D}$ for $D = H, I, C$, $r = (r_1, r_2)$ and $\ell = (\ell_1, \ell_2)$.

	$\ell=(1,0)$			$\ell=(0,1)$			$\ell=(1,1)$		
(r_1,r_2)	H	I	C	H	I	C	H	I	C
(1,0)	0.62457	0.47935	0.42250	0.86619	0.79286	0.76211	0.47825	0.18134	–
(0,1)	0.16540	–	–	0.65761	0.50309	0.44202	–	–	–
(1,1)	0.74248	0.65853	0.62863	0.90241	0.85149	0.83073	0.65794	0.52586	0.47245
(0,2)	0.32308	–	–	0.79331	0.67238	0.61883	–	–	–
(1,2)	0.78642	0.71251	0.68550	0.92097	0.87890	0.86155	0.71197	0.58951	0.53740

	$\ell=(0,2)$			$\ell=(1,2)$		
(r_1,r_2)	H	I	C	H	I	C
(1,0)	0.79714	0.71371	0.68324	0.39721	–	–
(0,1)	0.51171	0.34795	0.28853	–	–	–
(1,1)	0.85441	0.79879	0.77916	0.61593	0.47775	0.42357
(0,2)	0.67966	0.52908	0.46728	–	–	–
(1,2)	0.88132	0.83463	0.81794	0.67392	0.54265	0.48821

reliability equivalence factors, (SREF, MREF) and γ-fractiles were established. To interpret the theoretical results obtained in this work numerical example was introduced. Cold duplication method gives the best improvement than other methods.

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A. Mustafa received the Ph. D. degree in Statistics and Computer Science at Mansoura University. Currently, he is working as an associate professor at Islamic University of Madinah, Kingdom of Saudi Arabia, since 2019. His research interests are in the areas of reliability engineering, lifetime distributions and mathematical statistics. He has published research articles in reputed international journals of mathematics and statistics. He is a referee of several international journals in the frame of statistics, probability and reliability engineering.

M. I. Khan received his M. Sc. and Ph.D. from Aligarh Muslim University, India. Currently, he is working as an assistant professor at Islamic University of Madinah, Kingdom of Saudi Arabia, since 2014. His research has been focused in the area of mathematical statistics and ordered random variables.

Maher A. Alraddadi received the B.Sc. degree in Mathematics from the Department of Mathematics at Taibah University, Faculty of Science, Saudi Arabia. Currently, he is pursuing master degree in Mathematics at Islamic University of Madinah, Kingdom of Saudi Arabia. His research interests include reliability engineering and mathematical statistics.