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Generalization of Renyi's Entropy and its Application in Source Coding

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Abstract: In this paper, we introduce a new generalization of Renyi's entropy $R^{\alpha}_{\beta}(P)$ and the most important feature of this generalized entropy $R^{\alpha}_{\beta}(P)$ is that it derives most important entropies that are well known and influence information theory and applied mathematics. Some significant properties of $R^{\alpha}_{\beta}(P)$ has been undertaken in this article. In addition, we introduce a new generalized exponentiated mean codeword length $L^{\alpha}_{\beta}(P)$ in this article then determine how $R^{\alpha}_{\beta}(P)$ and $L^{\alpha}_{\beta}(P)$ are related in terms of source coding theorem.

Keywords: Shannon's entropy, Renyi's entropy, L'Hopitals rule, Code-word length, Exponential Inequalities, Logarithm Inequalities, Kraft's inequality, Reverse Holder's inequality

1 Introduction

Consider a discrete random variable X having values $X = \{x_1, x_2, x_3, ..., x_n\}$ with respective probabilities $P = \{p_1, p_2, p_3, ..., p_n\}$. Claude Shannon [1] defined the entropy $H(P)$ for a discrete random variable X as:

$$H(P) = -\sum_{i=1}^{n} p_i \log_D p_i \tag{1}$$

The unit of entropy measure is determined by the base of the logarithm D, if $D = 2$, then the entropy measure is known as bit, if $D = e$, then the entropy measure is known as nat and if $D = 10$, then the entropy measure is known as hartley. Various generalized versions of Shannon's entropy under discrete random variable have been introduced in the literature of information theory. These generalized entropies are classified among parametric, trigonometric and weighted entropies. Firstly Renyi [2] gave the idea of parametric entropy and defined the entropy of order α as:

$$R^{\alpha}(P) = \frac{1}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^{\alpha}\right], \alpha \neq 1, \alpha > 0 \tag{2}$$

After Renyi [2], other researchers viz., Havrda and Charvat [3], Sharma and Mittal [4], Bhat and Baig[5,6,7,8,9,10,11], Bhat et.al [12] etc., developed various generalized entropy measures to the literature of information theory and the application of entropy measures have been discussed in different aspects in statistics and mathematics see papers [13,14,15,16,17].

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2 Generalization of Renyi's entropy

In this article we define we define a new generalization of Renyi's entropy $R^{\alpha}_{\beta}(P)$, for a random variable $X = \{x_1, x_2, x_3, ..., x_n\}$ with respective probabilities $P = \{p_1,\ p_2,\ p_3,\ \dots,\ p_n\}$ as:

$$R^{\alpha}_{\beta}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right], \alpha > 0, \beta > 0, \alpha \neq \beta. \tag{3}$$

The following interpretation of α and β is interesting from an application point of view. Considering a cybernetic system $[x_i,\ p_i]$, where x_i are the events and p_i be the corresponding probabilities, then α and β can be taken as flexibility parameters or as a pre-determined numbers linked to several cybernetic systems. Suppose two cybernetic systems, having same set of x_i, p_i, but may have different informations (with regard to the same aim) for different values of α and β. The parameters α and β can be considered as the environment factors, such as temperature, humidity, etc. Moreover, a variety a factors affects the diversity in cost. Let α and β are such factors upon which the information regarding a cybernetic system $[x_i,\ p_i]$ depends.

Particular cases (c.f., (3))

I. For $\beta = 1$, equation (3) reduces to Renyi's [2] entropy of order α given in equation (2) i.e.,

$$R^{\alpha}_{\beta=1}(P) = R^{\alpha}(P) = \frac{1}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^{\alpha} \right]$$

II. For $\alpha = 1$, equation (3) reduces to Renyi's [2] entropy of order $\frac{1}{\beta}$ i.e.,

$$R^{\alpha=1}_{\beta}(P) = R^{\frac{1}{\beta}}(P) = \frac{1}{1-\frac{1}{\beta}} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{1}{\beta}} \right]$$

III. For $\alpha = 2\beta$, equation (3) reduces to Collision entropy (Also called Renyi's quadratic entropy) i.e.,

$$R^{\alpha=2\beta}(P) = R^{2}(P) = -\log_D \left[\sum_{i=1}^{n} p_i^{2} \right]$$

IV. For $\beta = 1$ and $\alpha \to 1$, then by applying L' Hopital's rule, equation (3) reduces to entropy given by Shannon [1] i.e.,

$$R^{\alpha\to1}_{\beta=1}(P) = H(P) = -\sum_{i=1}^{n} p_i \log_D p_i$$

V. For $\alpha = 1$, $\beta \to 1$, then by applying L'Hopital's rule, equation (3) reduces to entropy given by Shannon[1] i.e.,

$$R^{\alpha=1}_{\beta\to1}(P) = H(P) = -\sum_{i=1}^{n} p_i \log_D p_i$$

VI. For $\alpha \to \beta$, then by applying L'Hopital's rule equation (3), reduces to entropy given by Shannon [1] i.e.,

$$R^{\alpha\to\beta}(P) = H(P) = -\sum_{i=1}^{n} p_i \log_D p_i$$

VII. For $\alpha > 0, \beta > 0$ *and* $\alpha \neq \beta$, if all the events are equally likely, i.e., $p_i = \frac{1}{n} \forall i = 1, 2, \dots, n$, then we have

$$R^{\alpha}_{\beta}\left(\frac{1}{n}\right) = H\left(\frac{1}{n}\right) = \log_D n.$$

Which is maximum entropy.

3 Properties of our proposed measure

Some significant aspects of our generalized entropy measure $R^{\alpha}_{\beta}(P)$ have been investigated in this section:
Property 1: $R^{\alpha}_{\beta}(P) > 0$ for α and β (c.f., (3)).

Proof: We have

$$R^{\alpha}_{\beta}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right], \alpha > 0, \beta > 0, \alpha \neq \beta.$$

Case-I: For $\alpha > \beta$.
For $\alpha > \beta$, we have $\frac{\alpha}{\beta} > 1$. Since, $0 \leq p_i \leq 1 \forall\ i = 1, 2, \dots, n$ and $\sum_{i=1}^{n} p_i = 1$, which implies that

$$p_i^{\frac{\alpha}{\beta}} < p_i$$

After some mathematical manipulation, it follows that:

$$\log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right] < 0 \tag{4}$$

As we have $\alpha > \beta$, which implies that $\beta - \alpha < 0$. Also for $\beta > 0$, so we have

$$\frac{\beta}{(\beta-\alpha)} < 0 \tag{5}$$

Combining equation (4) and (5), we have
For $\alpha > \beta$

$$R^{\alpha}_{\beta}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right] > 0 \tag{6}$$

Case-II: For $\alpha < \beta$.
For $\alpha < \beta$, we have $\frac{\alpha}{\beta} < 1$. Since, $0 \leq p_i \leq 1 \forall\ i = 1, 2, \dots, n$ and $\sum_{i=1}^{n} p_i = 1$, which implies that

$$p_i^{\frac{\alpha}{\beta}} > p_i$$

After some mathematical manipulation, it follows that:

$$\log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right] > 0 \tag{7}$$

As we have $\alpha < \beta$, which implies that $\beta - \alpha > 0$. Also for $\beta > 0$, so we have

$$\frac{\beta}{(\beta-\alpha)} > 0 \tag{8}$$

Combining equation (7) and (8), we get
For $\alpha < \beta$

$$R_{\beta}^{\alpha}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right] > 0 \tag{9}$$

From equation (6) and (9), we observe that $R_{\beta}^{\alpha}(P)$ is positive for the defined values of the parameters α and β i.e.,

$$R_{\beta}^{\alpha}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right] > 0.$$

For $\alpha > 0, \beta > 0, \alpha \neq \beta$.

Property 2: $R_{\beta}^{\alpha}(P)$ is a symmetric function on every $p_i,\ i = 1,\ 2,\ 3,\ \dots,n$.

Proof: This property is trivially true, i.e.,

$$R_{\alpha}^{\beta}(p_1, p_2,\ \dots, p_{n-1}, p_n) = P_{\alpha}^{\beta}(p_n, p_1, p_2,\ \dots, p_{n-1})$$

Property 3: The maximum value of $R_{\beta}^{\alpha}(P)$ is attained when the chance of happening of all the events are equal.

Proof: We have

$$R_{\beta}^{\alpha}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}} \right], \alpha > 0, \beta > 0, \alpha \neq \beta.$$

Suppose the chance of happening of all the events are equal i.e., $p_i = \frac{1}{n} \forall i = 1, 2, \dots, n$, then we have

$$R_{\beta}^{\alpha}(P) = \frac{\beta}{(\beta-\alpha)} \log_D \left[\sum_{i=1}^{n} \left(\frac{1}{n}\right)^{\frac{\alpha}{\beta}} \right]$$

After some mathematical manipulation, it follows that:

$$R_{\beta}^{\alpha}(P) = \log_D n.$$

Which is the maximum entropy.

Property 4: The additive property is satisfied by $R_{\beta}^{\alpha}(P)$ in the following mathematical context:

$$R_{\alpha}^{\beta}(P*Q) = R_{\alpha}^{\beta}(P) + R_{\alpha}^{\beta}(Q)$$

Where

$$(P*Q) = (p_1q_1,\ \dots, p_1q_m,\ p_2q_1,\ \dots, p_nq_1,\ \dots, p_nq_m)$$

is the joint probability mass function of two independent discrete random variables.

Proof:

Let $(P*Q) = (p_1q_1,\ \dots, p_1q_m,\ p_2q_1,\ \dots, p_nq_1,\ \dots, p_nq_m)$, be the joint probability mass function of two independent discrete random variables, then we have

$$\begin{aligned}
R_{\alpha}^{\beta}(P*Q) &= \frac{\beta}{\beta-\alpha}\left[\log_D\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(p_iq_j)^{\frac{\alpha}{\beta}}\right)\right] \\
&= \frac{\beta}{\beta-\alpha}\left[\log_D\left(\sum_{i=1}^{n}\sum_{j=1}^{m}p_i^{\frac{\alpha}{\beta}}q_j^{\frac{\alpha}{\beta}}\right)\right] \\
&= \frac{\beta}{\beta-\alpha}\left[\log_D\left(\left(\sum_{i=1}^{n}p_i^{\frac{\alpha}{\beta}}\right)\left(\sum_{j=1}^{m}q_j^{\frac{\alpha}{\beta}}\right)\right)\right] \\
&= \frac{\beta}{\beta-\alpha}\left[\log_D\left(\sum_{i=1}^{n}p_i^{\frac{\alpha}{\beta}}\right) + \log_D\left(\sum_{j=1}^{m}q_j^{\frac{\alpha}{\beta}}\right)\right] \\
&= \frac{\beta}{\beta-\alpha}\log_D\left(\sum_{i=1}^{n}p_i^{\frac{\alpha}{\beta}}\right) + \frac{\beta}{\beta-\alpha}\log_D\left(\sum_{j=1}^{m}q_j^{\frac{\alpha}{\beta}}\right) \\
&= R_{\alpha}^{\beta}(P) + R_{\alpha}^{\beta}(Q).
\end{aligned}$$

This completes the proof.

4 Source Coding theorems

Consider a finite input source symbols $X = \{x_1,\ x_2,\ x_3,\ \dots,\ x_n\}$ with respective probabilities of transmission as $P = \{p_1,\ p_2,\ p_3,\ \dots,\ p_n\}$ and suppose these input source symbols have to be transmitted to the receiver, before communicating these input source symbols to the receiver, the sender first encode these input source symbols by using any encoding procedure. Let us suppose that these each input source symbol have been encoded using alphabet of D symbols by any encoding procedure. Let $L = \{l_1,\ l_2,\ l_3,\ \dots,\ l_n\}$ be the code-word lengths corresponding to encoded symbols, then Shannon [1] defined the mean code-word length of the source encoder as:

$$L(P) = \sum_{i=1}^{n} p_i l_i \tag{10}$$

A code is said to be a uniquely decipherable code over an alphabet of D symbols with lengths $L = \{l_1,\ l_2,\ l_3,\ \dots,\ l_n\}$ iff the Kraft's inequality holds i.e.,

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{11}$$

For all codes satisfying the inequality (11), then the mean code-word length $L(P)$ defined at (10), lies between $H(P)$ and $H(P)+1$ i.e.,

$$H(P) < L(P) < H(P)+1 \tag{12}$$

Shannon's noiseless coding theorem is another name of this result.

Campbell [18] defined the exponentiated mean codeword length for a discrete channel as:

$$L^{\alpha}(P) = \frac{\alpha}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right], \alpha>0, \alpha\neq 1 \tag{13}$$

Campbell [18] generalizes the Shannon's source coding theorem and showed that $L^{\alpha}(P)$ lies between $R^{\alpha}(P)$ and $R^{\alpha}(P)+1$ under the condition that if the codes satisfy inequality (11) , i.e.,

$$R^{\alpha}(P) < L^{\alpha}(P) < R^{\alpha}(P)+1$$

Various generalized source coding theorems under the condition of unique decipherability have been developed by various scholars over the last few decades; see, for example, publications [19,20,21,22,23,24,25,26,27,28,29].

We introduce a new generalized exponentiated mean codeword length $L_{\beta}^{\alpha}(P)$ in this article as:

$$L_{\beta}^{\alpha}(P) = \frac{\alpha}{\beta-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right], \alpha>0, \beta>0, \alpha\neq\beta. \tag{14}$$

Where, D is the number of alphabets used to code the input source symbols.

Particular cases (c.f., (14))

I. For $\beta=1$, (14) reduces to Campbell [18] mean code-word length i.e.,

$$L_{\beta=1}^{\alpha}(P) = L^{\alpha}(P) = \frac{\alpha}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]$$

II. For $\alpha=1$, (14) reduces to Campbell [18] mean codeword length with parameter $\frac{1}{\beta}$ i.e.,

$$L_{\beta}^{\alpha=1}(P) = L^{\frac{1}{\beta}}(P) = \frac{\frac{1}{\beta}}{1-\frac{1}{\beta}}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\frac{1}{\beta}-1}{\frac{1}{\beta}}\right)}\right]$$

III. For $\beta=1$ and $\alpha\to 1$, then by applying L'Hopital's rule (14) reduces to optimum mean code-word length given by Shannon [1] i.e.,

$$L_{\beta=1}^{\alpha\to 1}(P) = L(P) = \sum_{i=1}^{n} p_i l_i$$

IV. For $\alpha=1$ and $\beta\to 1$, then by applying L'Hopital's rule (14) reduces to optimum mean codeword length given by Shannon [1] i.e.,

$$L_{\beta\to 1}^{\alpha=1}(P) = L(P) = \sum_{i=1}^{n} p_i l_i$$

V. For $\alpha\to\beta$, then by applying L'Hopital's rule (14) reduces to optimum mean codeword length given by Shannon [1] i.e.,

$$L^{\alpha\to\beta}(P) = L(P) = \sum_{i=1}^{n} p_i l_i$$

Now we derive the relationship between (3) and (14) in terms of source coding theorem.

Theorem 1: For all alphabets of $D>1$ symbols, if the codeword lengths $L=\{l_1, l_2, l_3, \dots, l_n\}$ satisfy the Kraft's inequality, then the relationship between $R_{\beta}^{\alpha}(P)$ and $L_{\beta}^{\alpha}(P)$ is as follows:

$$R_{\alpha}^{\beta}(P) \leq L_{\alpha}^{\beta}(P)$$

The equality i.e., $R_{\alpha}^{\beta}(P) = L_{\alpha}^{\beta}(P)$ holds iff

$$l_i = -\log_D\left[\frac{p_i^{\frac{\alpha}{\beta}}}{\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}}\right] \tag{15}$$

Proof:

For all $a_i, b_i > 0$, $i=1, 2, 3, \dots, n$ and $\frac{1}{\gamma}+\frac{1}{\delta}=1$, $\gamma<1(\neq 0)$, $\delta<0$ or $\delta<1(\neq 0)$, $\gamma<0$, then by reverse of Holder's inequality we have

$$\left(\sum_{i=1}^{n} a_i^{\gamma}\right)^{\frac{1}{\gamma}}\left(\sum_{i=1}^{n} b_i^{\delta}\right)^{\frac{1}{\delta}} \leq \sum_{i=1}^{n} a_i b_i \tag{16}$$

The equality of (16) holds if $\exists\, c>0$, such that:

$$a_i^{\gamma} = c b_i^{\delta} \tag{17}$$

Let

$$a_i = p_i^{\frac{\alpha}{\alpha-\beta}} D^{-l_i},\ b_i = p_i^{\frac{\alpha}{\beta-\alpha}}$$

$$\gamma = \frac{\alpha-\beta}{\alpha} \ and\ \delta = \frac{\beta-\alpha}{\beta}$$

Substitute these values into (16), and after some mathematical manipulation, we get

$$\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right]^{\frac{\alpha}{\alpha-\beta}}\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\beta-\alpha}} \leq \sum_{i=1}^{n} D^{-l_i}$$

By using the inequality (16), and after some mathematical manipulations, we get

$$\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\beta-\alpha}} \leq \left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right]^{\frac{\alpha}{\beta-\alpha}}$$

By applying logarithms on both sides with base D to the above inequality and after performing the necessary mathematical manipulations, it follows that

$$\frac{\beta}{(\beta-\alpha)}\log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] \le \frac{\alpha}{\beta-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right]$$

Or we can write the above inequality as:

$$R_\alpha^\beta(P) \le L_\alpha^\beta(P).$$

Now we will show the equality i.e., $R_\alpha^\beta(P) = L_\alpha^\beta(P)$ holds if and only if

$$l_i = -\log_D\left[\frac{p_i^{\frac{\alpha}{\beta}}}{\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}}\right]$$

After suitable mathematical manipulations one gets

$$D^{-l_i} = \left[\frac{p_i^{\frac{\alpha}{\beta}}}{\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}}\right]$$

By applying the appropriate mathematical operations one gets

$$D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)} = p_i^{\frac{\alpha}{\beta}-1}\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta-\alpha}{\alpha}}$$

Multiply above equation throughout by p_i, and after appropriate mathematical operations it follows that:

$$\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)} = \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\alpha}}$$

More interestingly, after some mathematical steps, it is implied that

$$\frac{\beta}{(\beta-\alpha)}\log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] = \frac{\alpha}{\beta-\alpha}\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right]$$

Or we can write the above equality as:

$$R_\alpha^\beta(P) = L_\alpha^\beta(P).$$

Theorem 2: For every codeword with lengths $L = \{l_1,\, l_2,\, l_3,\, \ldots,\, l_n\}$ satisfy the Kraft's inequality, then $R_\beta^\alpha(P)$ and $L_\beta^\alpha(P)$ are related as follows:

$$L_\beta^\alpha(P) < R_\alpha^\beta(P) + 1. \text{ For, } \alpha > 0, \beta > 0, \alpha \ne \beta$$

Proof: From the theorem 1 we see that $R_\alpha^\beta(P) = L_\alpha^\beta(P)$ is satisfied iff

$$l_i = -\log_D\left[\frac{p_i^{\frac{\alpha}{\beta}}}{\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}}\right]$$

The above expression can also be written as:

$$l_i = -\log_D p_i^{\frac{\alpha}{\beta}} + \log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]$$

Consider the codeword lengths $L = \{l_1,\, l_2,\, l_3,\, \ldots,\, l_n\}$ in such a manner that the following inequalities hold:

$$-\log_D p_i^{\frac{\alpha}{\beta}} + \log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] \le l_i < -\log_D p_i^{\frac{\alpha}{\beta}} + \log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] + 1 \tag{18}$$

From the left of inequality (18), it is easy to see that the code-word length $L = \{l_1,\, l_2,\, l_3,\, \ldots,\, l_n\}$ satisfies the Kraft's inequality.

From R.H.S of inequality (18), we have

$$l_i < -\log_D p_i^{\frac{\alpha}{\beta}} + \log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] + 1$$

After suitable mathematical manipulation above inequality can be written as:

$$D^{l_i} < p_i^{-\frac{\alpha}{\beta}}\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] D \tag{19}$$

Consider the following two cases:

Case-I: For $\alpha > \beta$.

For given values of α and β and for $\alpha > \beta$, we have $\frac{\beta-\alpha}{\alpha} < 0$, raising power $\frac{\beta-\alpha}{\alpha} < 0$ on both sides to the inequality (19) and after suitable mathematical manipulations, we get

$$D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)} > p_i^{\frac{\alpha}{\beta}-1}\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta-\alpha}{\alpha}} D^{\frac{\beta-\alpha}{\alpha}} \tag{20}$$

Multiply $p_i > 0$, on both sides to the inequality (20), and then by applying suitable mathematical operations we get the following inequality:

$$\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)} > \left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\alpha}} D^{\frac{\beta-\alpha}{\alpha}} \tag{21}$$

By the inequality (21), combined with the increasability property of the logarithmic function and applying suitable mathematical operations, we get

$$\log_D\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right] > \frac{\beta}{\alpha}\log_D\left[\sum_{i=1}^{n} p_i^{\frac{\alpha}{\beta}}\right] + \frac{\beta-\alpha}{\alpha} \tag{22}$$

Since $\frac{\beta-\alpha}{\alpha}<0$, then $\frac{\alpha}{\beta-\alpha}<0$, multiply $\frac{\alpha}{\beta-\alpha}<0$ both sides to the inequality (22), and after suitable mathematical manipulations it follows that:

$$L_{\beta}^{\alpha}(P)<R_{\alpha}^{\beta}(P)+1.$$

Case-II: For $\alpha<\beta$.

For given values of α and β and for $\alpha<\beta$, we have $\frac{\beta-\alpha}{\alpha}>0$, raising power $\frac{\beta-\alpha}{\alpha}>0$, on both sides to the inequality (19) and after suitable mathematical manipulations, we get

$$D^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}<p_i^{\frac{\alpha}{\beta}-1}\left[\sum_{i=1}^{n}p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta-\alpha}{\alpha}}D^{\frac{\beta-\alpha}{\alpha}} \tag{23}$$

Multiply $p_i>0$, on both sides to the inequality (23), and then by applying suitable mathematical operations we get, the following inequality:

$$\sum_{i=1}^{n}p_iD^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}<\left[\sum_{i=1}^{n}p_i^{\frac{\alpha}{\beta}}\right]^{\frac{\beta}{\alpha}}D^{\frac{\beta-\alpha}{\alpha}} \tag{24}$$

By the inequality (24) combined with the increasability property of the logarithmic function and after suitable mathematical operations, we get

$$\log_D\left[\sum_{i=1}^{n}p_iD^{-l_i\left(\frac{\alpha-\beta}{\alpha}\right)}\right]<\frac{\beta}{\alpha}\log_D\left[\sum_{i=1}^{n}p_i^{\frac{\alpha}{\beta}}\right]+\frac{\beta-\alpha}{\alpha} \tag{25}$$

Since $\frac{\beta-\alpha}{\alpha}>0$, then $\frac{\alpha}{\beta-\alpha}>0$, multiply $\frac{\alpha}{\beta-\alpha}>0$ both sides to the inequality (25) and after suitable mathematical manipulations, the following inequality holds

$$L_{\beta}^{\alpha}(P)<R_{\alpha}^{\beta}(P)+1.$$

Thus based on the above two source coding theorems, $R_{\beta}^{\alpha}(P)$ and $L_{\beta}^{\alpha}(P)$ are related as follows:

$$R_{\alpha}^{\beta}(P)\le L_{\alpha}^{\beta}(P)<R_{\alpha}^{\beta}(P)+1.\text{For, }\alpha>0,\beta>0,\alpha\neq\beta.$$

5 Conclusion and Future Research

The current study introduces a novel generalization of Renyi's entropy and the most important feature of this generalized entropy is that it generalizes most important entropies that are well known and influences information theory and applied mathematics. The study of some significant properties of this novel generalization of Renyi's entropy has been undertaken in this paper. Additionally, we introduced a new generalized exponentiated mean code-word length in this article then determine the relation between $R_{\beta}^{\alpha}(P)$ and $L_{\beta}^{\alpha}(P)$ in terms of source coding theorem. The next phase of this research includes replacing expression by other higher-level entropy functionals, for example, Ismail's entropy, namely [13]. Also the results presented in this paper can be used to discuss more insights on quantum algorithms [30,31,32,33,34,35].

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