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Abstract: In this work, we studied the controllability results for neutral differential time-varying equation with impulses on time scales & extend these results into nonlocal controllability of neutral functional integro-differential time-varying equation with impulses on time scales. The solutions are obtained employing standard fixed point theorems.

Keywords: Control, Integro-differential equation, Impulsive, Time scales.

1 Introduction

Newton & Leibniz introduced calculus invention of differential equation which was brought into the reality. Bernoulli differential equation put forward by Jacob Bernoulli. Leibniz obtained with simplification of ordinary differential equation form for which the year happening next. Joseph Louis Lagrange succeeded in addressing the issue of string vibration in musical instruments in previous times. d'Alembert established the one-dimensional wave equation in the year 1746. An integrodifferential equations has appeared in having happened years as an self-sufficient part of modern research since of its relationship to a large number of fields for example Ecology, continuum mechanics, System theory, population dynamics, Biology, Viscoelasticity, Epidemics & other branches of Engineering & Science [25,33]. Integrodifferential equations arise in an interim state in the change of a differential equations to an integral equations [9].

Neutral integrodifferential equations takes place in the study of compartmental systems, viscoelasticity, population dynamics, & more than two fields of science [22,23]. In applied mathematics, neutral differential equations used widely in several areas. Cause of this purpose, these equations have gained more observation for the past few decades. Neutral delay differential equations include the derivative of the unfamiliar function, with both accompanied and unaccompanied delays.Few additional instances can be observed, leading to the conclusion that the theory of neutral delay differential equations is more complex than that of non-neutral delay equations. The swinging behaviour of the neutral systems solutions that is of significance in both the applications & theory, for example population growth, the radiating electrons motion, in networks containing lossless transmission lines, & the epidemics spread [34].

The impulsive differential equations theory plays a significant role in technology development & social sciences development & science development. More than two occurrences in these branches have mathematical models with regard to a few impulsive differential equations part [8,31].It is widely recognized that various biological phenomena such as optimal control models in economics, thresholds & biology exhibit impulse effects, & also impact the modulation of system frequencies [16,17,18,19,20]. There are several development processes that stand out because they have the ability to undergo a sudden change of state at a specific time period. We are referring to temporary disruptions that have a relatively short duration, which do not affect the overall process that is being compared. Hence, the form of impulses where perturbations act instantaneously that is natural to presume. Involving impulse effects of differential equations, appear as several real world problems such as natural description of observed evolution occurrence [29].

In existence, one of the most significant interdisciplinary research area & emerges in the very first technological inventories of the industrial uprising as well

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as in the most recent technological applications.Deterministic & non-deterministic partial differential equations, integro-differential equations, or a combination of ordinary & partial differential equations with or without delay in finite or infinite dimensional spaces using semigroup & cosine family can be used to represent numerous scientific & engineering problems. The majority of the systems that emerges in application are nonlinear to few area, at least over part of their functioning range. The primary step in dealing with a nonlinear system is usually, if possible, to represent in it around few formal utilizing spot is usual as long as, linear systems are more familiar to manipulate mathematically. A superior estimation to nonlinear system is the semilinear system that is by making a local approximation about few formal trajectories to a system with a linear part in addition a nonlinear part & can be obtained from a common nonlinear system. Some qualities of this system include regularity of the solutions, existence, stability of equilibrium points, & uniqueness, among others. An important area of study in control theory is Controllability. In many applications the aim of the control undertaking is to operate the system from one state to another in an optimal fashion.Nevertheless, prior to addressing the question concerning optimality, it is essential to consider the more fundamental inquiry of whether attaining a desired state from an initial state is feasible or not. The basic definitions & general theory of controllability & controllability with nonlocal conditions are referred in [1,6,21,24,30,12,28,26,32].

A time scale can be said to be time replica [11]. Time can exist in a continuous or discrete form, or it may possess characteristics of both continuousness & discreteness to varying degrees [14,15]. After 1980s, continuous-time Controllability & discrete-time Controllability linear positive systems has been subject of analysis [27]. After 1980s, S Hilger initiated this idea in his PhD thesis of time scales calculus & also establish how to fuse discrete- & continuous- time dynamical systems. Presently more than many analyser are acquiring curious in the time scale calculus, put up to its evolution & exhibiting theory applications & methods applications in more than two field. M Bohner introduced calculus of variations on time scales in 2004 by using delta integral & delta derivative, & it has been additionally evaluated by more than many distinct writers in more than many different views. Applying those results to profitable models & to display advantages of using time scales device that is the main aim of present work.

Time models has numerous examples which may be to some extent continuous & to some extent discrete, in addition to the level of attainment the whole line cases & the set of integers - continuous time, discrete time. The true delta derivative function is defined as the discrepancy between the classical derivative in continuous time and the discrepancy in discrete time, with their similarities limited. R. H. Middleton & G. C. Goodwin were responsible for the main attempt to combine the continuous-time theories & discrete-time control systems. Both considered realizations of linear systems, controllability & observability. For more details, refer [2,3,4,5].

We examine the specific results of controllability for the neutral integro-differential equation with impulsive events on time scales, as well as the associated nonlocal problem. Many classical problems can be solved by the method of modern analysis. The fixed point method is a valuable approach primarily utilized for demonstrating the existence & uniqueness of solutions in differential equations. In this study, our primary approach involves utilizing the Banach fixed point theorem(BFT) to examine the existence of controllability results. Motivation is to discuss neutral integrodifferential equations controllability within finite dimensional spaces along impulses on time scales. In this scenario, a non-linear control system is being examined. The system is time-varying & includes impulses on a specific time scale in $\mathscr{R}^n$.

$$\begin{aligned}
&\left[z(\zeta)-\omega\left(\zeta,z_a(\zeta),\int_{\zeta_0}^{\zeta} l(\zeta,s)E_3(\zeta,z(s))\Delta s\right)\right]^{\Delta} = \\
&A(\zeta)z(\zeta)+B(\zeta)u(\zeta)+f\left(\zeta,z(\zeta),\int_{\zeta_0}^{\zeta} v(\zeta,s)G(s,z(s))\Delta s\right), \\
&\zeta\in I, \zeta\neq\zeta_k,\ z(\zeta_k^+)-z(\zeta_k^-)=V_{1k}(\zeta_k,z_(\zeta_k^-)), \\
&k=1,2,...,mz(\zeta_0)-\varepsilon(z)=z_0. \qquad (1.1)
\end{aligned}$$

Here state function is $z(\zeta)\in\mathscr{R}^n$ & time scale be $\mathscr{T}$ with $\zeta_0,\zeta_k,\mathfrak{b}\in\mathscr{T}$. $z_{\mathfrak{a}}(\zeta)=z(\mathfrak{a}(\zeta))$, where $\mathfrak{a}:I=[\zeta_0,\mathfrak{b}]_{\mathscr{T}}\to I$ is a delay function which satisfies $\mathfrak{a}(\zeta)\leq\zeta$. $A(\zeta)\in C_{rd}R(\mathscr{T},M_n(\mathscr{R}))$, $B(\zeta)\in C_{rd}R(\mathscr{T},M_{n\times m}(\mathscr{R}))$. Throughout, for $k=1,2,...,m$, the points of impulses ζ_k satisfy $0\leq\zeta_0<\zeta_1<\zeta_2<...<\zeta_m<\zeta_{m+1}=b, z(\zeta_k^+)=\lim_{h\to0^+}z(\zeta_k+h), z(\zeta_k^-)=\lim_{h\to0^+}z(\zeta_k-h)$ denotes left & right limits of $z(\zeta)$ at in case of time scale $\zeta=\zeta_k$. The control function is $u(\cdot)\in\mathscr{R}^m$. $\mathfrak{f}:\mathfrak{I}\times\mathscr{R}^n\to\mathscr{R}^n$, $V_{1k}:\mathfrak{I}\times\mathscr{R}^n\to\mathscr{R}^n$ & $\omega:\mathfrak{I}\times\mathscr{R}^n\to\mathscr{R}^n$ are the functions defined which satisfies some contexts that will be stated thereafter. The obtained results are entirely novel, indeed the context of differential equations ($\mathscr{T}=\mathscr{R}$) & in context of difference equation ($\mathscr{T}=\mathfrak{Z}$).

2 Preliminaries

Assume n-dimensional vectors space along $\|\cdot\|$ is $\mathscr{R}^n$. Banach spaces(BS) of $\tilde{f}:\mathfrak{I}\to\mathscr{R}^n$ is $C(\mathfrak{I},\mathscr{R}^n)$, i.e continuous & provided with $\|\tilde{f}\|_C=\sup_{t\in\mathfrak{I}}\|\tilde{f}(t)\|$. $L^1(\mathfrak{I},\mathscr{R}^n)$ is Lebesgue integrable in terms of time scales which denotes the space of functions from I into $\mathscr{R}^n$.
Assume $PC(I,\mathscr{R}^n)$ as $PC(\mathfrak{I},\mathscr{R}^n)=\{x:\mathfrak{I}\to\mathscr{R}^n : x\in C((t_k,t_{k+1}],\mathscr{R}^n)$, $k=0,1,...,m$ & $\exists\ x(t_k^-)$ & $x(t_k^+)$, $k=1,2,...,m$ with $x(t_k^-)=x(t_k)\}$. $PC(I,\mathscr{R}^n)$ forms a BS provided along sup norm can be easily proven, $\|x\|_{PC}=\sup_{t\in I}\|x(t)\|$. Denote PC for $PC(I,\mathscr{R}^n)$ for the notional convenience.

A non empty closed subset of the real no.$\mathscr{R}$ is referred to as a time scale $\mathscr{T}$. If max $\mathscr{T}$ appears, we get $\mathscr{T}^k = \mathscr{T}/\{\max \mathscr{T}\}$ else $\mathscr{T}^k = \mathscr{T}$. Some of the most frequently encountered instances are $\mathscr{T} = \mathscr{R}, \mathscr{T} = \mathfrak{N}, \mathscr{T} = \mathfrak{h}\mathfrak{Z}$, where $\mathfrak{h} > 0$. $[\mathfrak{a},\mathfrak{b}] = \{t \in \mathscr{T} : \mathfrak{a} \le t \le \mathfrak{b}\}$ be a time scale interval, also defining $(\mathfrak{a},\mathfrak{b}), (\mathfrak{a},\mathfrak{b}], [\mathfrak{a},\mathfrak{b})$ & so on. From now on time scale interval will be used.
$\rho(t) := \inf\{s \in \mathscr{T} : s > t\}$ has $\inf\{\phi\} = \sup \mathscr{T}$ is defined as forward jump operator $\rho(\mathscr{T}^k, \mathscr{T})$. Also, $\mu(t) := \rho(t) - t,\ \forall\ t \in \mathscr{T}^k$ is defined as the graininess function $\mu(\mathscr{T}^k, [0,\infty))$.

Definition 1.[10] *Let* $\mathfrak{Z} : \mathscr{T} \to \mathscr{R}^n$ *&* $t \in \mathscr{T}^k$. *The* Δ *derivative* $\mathfrak{Z}^\Delta(t)$ *is the no. s.t.,* $\forall\ \varepsilon > 0$, $\exists$ *a nbd* $U(t)$ *s.t.,*

$$|[\mathfrak{Z}(\rho(t)) - \mathfrak{Z}(s)] - \mathfrak{Z}^\Delta(t)[\rho(t) - s]| \le \varepsilon|\rho(t) - s|, \forall\ s \in U.$$

Definition 2.[10] $\mathfrak{F}$ *is called antiderivative of* $\mathfrak{Z} : T \to X$ *endowed* $D^\Delta(t) = \mathfrak{Z}(t)\ \forall\ t \in T^k$, *we have:*

$$\int_{t_0}^{t} \mathfrak{Z}(s)\Delta s = D(t) - D(t_0).$$

Definition 3.[10] $p(T, \mathscr{R})$ *known as regressive,* $\forall\ t \in T, 1 + \eta(t)p(t) \neq 0$. *R denotes for every regressive functions.*

Definition 4.[10] *The generalized exp function for* $p \in R$ *is given as:*

$$e_p(t,s) = \exp\left(\int_s^t \xi\eta(\varsigma)(p(\varsigma))\Delta\varsigma\right),\ t,s \in \mathscr{T},$$

here $\xi_{\eta(\varsigma)}(p(\varsigma))$ *be cylinder transformation defined as:*

$$\xi_{\eta(\varsigma)}(q) = \begin{cases} \frac{1}{\eta(\varsigma)} log(1 + \eta(\varsigma)q), & if\ \eta(\varsigma) \neq 0. \\ q, & if\ \eta(\varsigma) = 0. \end{cases}$$

Theorem 1.[3] *For* $A \in C_{rd}R(\mathscr{T}, M_n(\mathscr{R}))$ *&* $w \in C_{rd}(\mathscr{T}, \mathscr{R}^n)$, $\forall\ x_0 \in \mathscr{R}^n$

$$x^\Delta(t) = A(t)x(t) + w(t), x(t_0) = x_0,\ t_0 \in \mathscr{T}, \tag{1.2}$$

has unique solution as:

$$x(t) = \Phi_A(t, t0) + \int_{t_0}^{t} \Phi_A(t, \rho(\varsigma))w(\varsigma)\Delta\varsigma,\ t \ge t_0, \tag{1.3}$$

here $\Phi_A(\cdot,t)$ *is the transition matrix of homogeneous eqn corresponding to eqn* (1.2). *Assume,*

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + B(t)u(t),\ t \in I',\ t \neq t_k, \\ x(t_k^+) - x(t_k^-) &= J_k(t_k, x(t_k^-)),\ t = t_k,\ k = 1,2,\ldots,m, \\ x(t_0) &= x_0. \end{aligned} \tag{1.4}$$

Lemma 1. $x \in PC$ *is known as solution of* (1.4), *if it satisfies* $x(t_0) = x_0$, $x(t_k^+) - x(t_k^-) = J_k(t_k, x(t_k^-))$, $k = 1,2,\ldots,m$ *&* $x(t)$ *known as solution of eqn*

$$\begin{aligned} x(t) = \Phi_A(t,t_0)x_0 &+ \int_{t_0}^{t} \Phi_A(t,\rho(s))B(s)u(s)\Delta s \\ &+ \sum_{t_0 < t_k < t} \Phi_A(t,t_k)J_k(t_k, x(t_k^-)). \end{aligned} \tag{1.5}$$

Proof: The proof is obtained using mathematical induction.
Remark: If the impulses $x(t_k^+) - x(t_k^-) = 0$, $k = 1,2,\ldots,m$, then (1.4) is given by

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + B(t)u(t),\ t \in I', \\ x(t_0) &= x_0, \end{aligned} \tag{1.6}$$

& eqn (1.5) becomes

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\rho(s))B(s)u(s)\Delta s.$$

The result of controllability for (1.5) is explained in [7].

Theorem 2. *The controllability Gramian matrix* $W_{t_0}^b$ *is reversible iff* (1.5) *is controllable on* $[t_0, b]$, *where*

$$W_{t_0}^b = \int_{t_0}^{b} \Phi_A(b,\rho(s))B(s)B^T(s)\Phi_A^T(b,\rho(s))\Delta s.$$

The following assumptions are needed based on [13]:

$(\mathbf{H_1})$: $\omega : \mathfrak{I} \times \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^n$ is Δ differentiable & $\exists$ $\mathfrak{L}_{\omega_1}, \mathfrak{L}_{\omega_2} > 0$ s.t.,

$$\|\omega(\zeta,z,y) - \omega(\zeta,\bar{z},\bar{y})\| \le \mathfrak{L}_{\omega_1}\|z - \bar{z}\| + \mathfrak{L}_{\omega_2}\|y - \bar{y}\|,$$
$$\forall z,y,\bar{z},\bar{y} \in \mathscr{R}^n, \zeta \in \mathfrak{I}.$$

&, $\exists$ $\mathfrak{C}_\omega, \mathfrak{M}_\omega, \mathfrak{N}_\omega > 0$ such that $\|\omega(\zeta,z,y)\| \le \mathfrak{C}_\omega + \mathfrak{M}_\omega\|z\| + \mathfrak{N}_\omega\|y\|$, $\forall z,y \in \mathscr{R}^n$ & $\zeta \in I$.

$(\mathbf{H_2})$: $f : \mathfrak{I} \times \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^n$ is continuous & $\exists$ $\mathfrak{L}_{\mathfrak{f}_1}, \mathfrak{L}_{\mathfrak{f}_2} > 0$ s.t.,

$$\|f(\zeta,z,y) - f(\zeta,\bar{z},\bar{y})\| \le \mathfrak{L}_{\mathfrak{f}_1}\|z - \bar{z}\| + \mathfrak{L}_{\mathfrak{f}_2}\|y - \bar{y}\|,$$
$$\forall z,y,\bar{z},\bar{y} \in \mathscr{R}^n, \zeta \in \mathfrak{I}.$$

Also, $\exists$ $\mathfrak{C}_\mathfrak{f}, \mathfrak{M}_\mathfrak{f}, \mathfrak{N}_\mathfrak{f} > 0$ s.t., $\|f(\zeta,z,y)\| \le \mathfrak{C}_\mathfrak{f} + \mathfrak{M}_\mathfrak{f}\|z\| + \mathfrak{N}_\mathfrak{f}\|y\|$, $\forall z,y \in \mathscr{R}^n$ & $\zeta \in \mathfrak{I}$.

$(\mathbf{H_3})$: $V_{1k}(\zeta_k, z(\zeta_k)) \in C(I, \mathscr{R}^n)$, $k = 1,2,\ldots,m$ & $\exists$ '+'ve constant $\mathfrak{L}_{\mathfrak{V}_1} > 0$ such that

$$\|V_{1k}(\zeta,z) - V_{1k}(\zeta,y)\| \le \mathfrak{L}_{\mathfrak{V}_1}\|z - y\|, \forall z,y \in \mathscr{R}^n, \zeta \in \mathfrak{I}.$$

Also, $\exists$ $\mathfrak{M}_{\mathfrak{V}_1}, \mathfrak{N}_{\mathfrak{V}_1} > 0$ s.t., $\|V_1(\zeta,z)\| \le \mathfrak{M}_{\mathfrak{V}_1} + \mathfrak{N}_{\mathfrak{V}_1}\|z\|, \forall z \in \mathscr{R}^n$ & $\zeta \in \mathfrak{I}$.

$(\mathbf{H_4})$: l is a continuous function & $\exists$ a no. $\mathfrak{M}_\mathfrak{l} > 0$ s.t.,

$$\mathfrak{M}_\mathfrak{l} = \int_{\zeta_0}^{\zeta} |l(\zeta,s)|\Delta s.$$

$(\mathbf{H_5})$: $E_3 : I \times \mathscr{R}^n \to \mathscr{R}^n$ is continuous function & $\exists$ $\mathfrak{L}_{\mathfrak{E}_3} > 0$ s.t.,

$$\|E_3(\zeta,z) - E_3(\zeta,y)\| \le \mathfrak{L}_{\mathfrak{E}_3}\|z - y\|, \forall z,y \in \mathscr{R}^n, \zeta \in \mathfrak{I}.$$

Also, $\exists\, \mathfrak{M}_{\mathfrak{E}_3}, \mathfrak{N}_{\mathfrak{E}_3} > 0$ s.t., $\|E_3(\zeta,z)\| \leq \mathfrak{M}_{\mathfrak{E}_3} + \mathfrak{N}_{\mathfrak{E}_3}\|z\|$, $\forall z \in \mathscr{R}^n$ & $\zeta \in \mathfrak{I}$.

$(\mathbf{H_6})$: v is a continuous function & $\exists$ a no. $\mathfrak{M}_{\mathfrak{v}} > 0$ s.t.,

$$\mathfrak{M}_{\mathfrak{v}} = \int_{\zeta_0}^{\zeta} |v(\zeta,s)|\Delta s.$$

$(\mathbf{H_7})$: $G : I \times \mathscr{R}^n \to \mathscr{R}^n$ is continuous function & $\exists$ $\mathfrak{L}_{\mathfrak{G}} > 0$ s.t.,

$$\|G(\zeta,z) - G(\zeta,y)\| \leq \mathfrak{L}_{\mathfrak{G}}\|z-y\|, \forall z,y \in \mathscr{R}^n, \zeta \in \mathfrak{I}.$$

Also, $\exists\ \mathfrak{M}_{\mathfrak{g}}, \mathfrak{N}_{\mathfrak{g}} > 0$ s.t., $\|G(\zeta,z)\| \leq \mathfrak{M}_{\mathfrak{g}} + \mathfrak{N}_{\mathfrak{g}}\|z\|$, $\forall z \in \mathscr{R}^n$ & $\zeta \in \mathfrak{I}$.

$(\mathbf{H_8})$: $\varepsilon : C(I,\mathscr{R}^n) \to \mathscr{R}^n$ is continuous & $\exists$ a constant $\mathfrak{L}_{\varepsilon} > 0$ s.t.,

$$\|\varepsilon(z) - \varepsilon(y)\| \leq \mathfrak{L}_{\varepsilon}\|z-y\|, \forall z,y \in \mathscr{R}^n.$$

For the notional convenience, consider

$$M_{\Psi} = \max\{\Psi_A(\zeta,s),\ \zeta_0 \leq s \leq \zeta \leq b\},\ \mathfrak{M}_{\mathfrak{A}} = \sup_{\zeta \in I}\|A(\zeta)\|.$$

$$\mathfrak{M}_{\mathfrak{B}} = \sup_{\zeta \in I}\|B(\zeta)\|,\ \mathfrak{M}_{\mathfrak{W}} = \|(W_0^b)^{-1}\|.$$

$$\begin{aligned}
\mathfrak{M}^* &= (1 + M_{\Psi} + \mathfrak{M}_{\mathfrak{A}} M_{\Psi} b)(\mathfrak{C}_{\omega} + \mathfrak{N}_{\omega}\mathfrak{M}_{\mathfrak{l}}\mathfrak{M}_{\mathfrak{E}_3}) \\
&\quad + M_{\Psi} b(\mathfrak{C}_{\mathfrak{f}} + \mathfrak{N}_{\mathfrak{f}}\mathfrak{M}_{\mathfrak{v}}\mathfrak{M}_{\mathfrak{G}}) + m M_{\Psi}\mathfrak{M}_{\mathfrak{V}_1}. \\
\mathfrak{M}^{**} &= (1 + M_{\Psi} + \mathfrak{M}_{\mathfrak{A}} M_{\Psi} b)(\mathfrak{M}_{\omega} + \mathfrak{N}_{\omega}\mathfrak{M}_{\mathfrak{l}}\mathfrak{N}_{\mathfrak{E}_3}) \\
&\quad + M_{\Psi} b(\mathfrak{M}_{\mathfrak{f}} + \mathfrak{N}_{\mathfrak{f}}\mathfrak{M}_{\mathfrak{v}}\mathfrak{N}_{\mathfrak{G}}) + m M_{\Psi}\mathfrak{N}_{\mathfrak{V}_1}. \\
\mathfrak{K}_1 &= (1 + M_{\Psi}^2\mathfrak{M}_{\mathfrak{B}}{}^2\mathfrak{M}_{\mathfrak{W}} b)(M_{\Psi}(\|z_0 + \varepsilon(z)\|) + \mathfrak{M}^*) \\
&\quad + M_{\Psi}^2\mathfrak{M}_{\mathfrak{B}}{}^2\mathfrak{M}_{\mathfrak{W}} b\|z_b\|. \\
\mathfrak{K}_2 &= (1 + M_{\Psi}^2\mathfrak{M}_{\mathfrak{B}}{}^2\mathfrak{M}_{\mathfrak{W}} b)\mathfrak{M}^{**}.
\end{aligned}$$

3 Neutral Integro-Differential Equations Controllability Results

Definition 5. *$z \in PC$ is called solution of (1.1), if it satisfies $z(\zeta_0) = z_0, z(\zeta_k^+) - z(\zeta_k^-) = V_{1k}(\zeta_k, z(\zeta_k^-))$, $k = 1,2,\ldots,m$ & $z(\zeta)$ is the solution of the following equation*

$$\begin{aligned}
z(\zeta) &= \omega\left(\zeta, z_a(\zeta), \int_{\zeta_0}^{\zeta} l(\zeta,s)E_3(\zeta,z(s))\Delta s\right) \\
&\quad + \Psi_A(\zeta,\zeta_0)\Big[z_0 + \varepsilon(z) - \omega\Big(\zeta_0, z_a(\zeta_0), \\
&\qquad \int_{\zeta_0}^{\zeta} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big] \\
&\quad + \int_{\zeta_0}^{b} A(r)\Psi_A(\zeta,\rho(r))\omega\left(r, z_a(r), \int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\right)\Delta r \\
&\quad + \int_{\zeta_0}^{b}\Psi_A(\zeta,\rho(r))\left(B(r)u(r) + f\left(r, z(r), \int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\right)\right)\Delta r \\
&\quad + \sum_{\zeta_0<\zeta_k<\zeta}\Psi_A(\zeta,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k))
\end{aligned} \tag{1.8}$$

Lemma 2. *Consider $(\mathbf{H_1}) - (\mathbf{H_7})$ holds, control input*

$$u(\zeta) = B^T(t)\Psi_A^T(b,\rho(\zeta))(W_{\zeta_0}^b)^{-1}P(z), \tag{1.9}$$

transfer the system

$$\begin{aligned}
z(\zeta) &= \omega\left(\zeta, z_a(\zeta), \int_{\zeta_0}^{\zeta} l(\zeta,s)E_3(\zeta,z(s))\Delta s\right) \\
&\quad + \Psi_A(\zeta,\zeta_0)\Big[z_0 + \varepsilon(z) \\
&\quad - \omega\left(\zeta_0, z_a(\zeta_0), \int_{\zeta_0}^{\zeta} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\right)\Big] \\
&\quad + \int_{\zeta_0}^{b} A(r)\Psi_A(\zeta,\rho(r))\omega\left(r, z_a(r), \int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\right)\Delta r \\
&\quad + \int_{\zeta_0}^{b}\Psi_A(\zeta,\rho(r))\Big[B(r)u(r) \\
&\quad + f\left(r, z(r), \int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\right)\Big]\Delta r \\
&\quad + \sum_{k=1}^{m}\Psi_A(\zeta,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k))
\end{aligned}$$

from z_0 to z_b in b &, M_u is approximate for the control input $u(\zeta)$,

$$\begin{aligned}
P(z) &= z_b - \Psi_A(b,\zeta_0)\Big[z_0 + \varepsilon(z) \\
&\quad - \omega\left(\zeta_0, z_a(\zeta_0), \int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\right)\Big] \\
&\quad - \omega\left(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s\right) \\
&\quad - \int_{\zeta_0}^{b}\Psi_A(b,\rho(r))f\left(r, z(r), \int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\right)\Delta r \\
&\quad - \int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\left(r, z_a(r), \int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\right)\Delta r \\
&\quad - \sum_{k=1}^{m}\Psi_A(b,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k^-))
\end{aligned}$$

&

$$\begin{aligned}
M_u &= \mathfrak{M}_{\mathfrak{B}} M_{\Psi}\mathfrak{M}_{\mathfrak{W}}(\|z_b\| + M_{\Psi}\|z_0 + \varepsilon(z)\| + \mathfrak{M}^* \\
&\quad + \mathfrak{M}^{**}\sup_{\zeta \in I}(\|z(\zeta)\|)).s
\end{aligned}$$

Proof: Put $\zeta = b$, in (1.8) we have:

$$
\begin{aligned}
z(b) &= \omega(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s) \\
&+ \Psi_A(b,\zeta_0)\Big[z_0 + \varepsilon(z) - \omega\Big(\zeta_0, z_a(\zeta_0), \\
&\int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big] \\
&+ \int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\Big(r, z_a(r), \\
&\int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\Big)\Delta r \\
&+ \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))B(r)u(r)\Delta r \\
&+ \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))f\Big(r, z(r), \\
&\int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\Big)\Delta r \\
&+ \sum_{\zeta_0<\zeta_k<\zeta} \Psi_A(b,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k)) \\
&= \omega\left(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s\right) \\
&+ \Psi_A(b,\zeta_0)\Big[z_0 + \varepsilon(z) - \omega\Big(\zeta_0, z_a(\zeta_0), \\
&\int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big] \\
&+ \int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\Big(r, z_a(r), \\
&\int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\Big)\Delta r \\
&+ \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))B(r)B^T(r) \\
&\Psi_A^T(b,\rho(r))(W_{\zeta_0}^{b})^{-1}P(z)\Delta r \\
&+ \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))f\Big(r, z(r), \\
&\int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\Big)\Delta r \\
&+ \sum_{k=1}^{m} \Psi_A(b,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k^-)) \\
&= \omega\left(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s\right) \\
&+ \sum_{k=1}^{m} \Psi_A(b,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k^-)) \\
&+ \Psi_A(b,\zeta_0)\Big[z_0 + \varepsilon(z) - \omega\Big(\zeta_0, z_a(\zeta_0), \\
&\int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big] \\
&+ \int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\left(r, z_a(r), \int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\right)\Delta r \\
&+ (W_{\zeta_0}^{b})(W_{\zeta_0}^{b})^{-1}P(z) \\
&+ \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))f\left(r, z(r), \int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\right)\Delta r
\end{aligned}
$$

$$
\begin{aligned}
z(b) &= \omega\left(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s\right) \\
&+ \sum_{k=1}^{m} \Psi_A(b,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k^-)) \\
&+ \Psi_A(b,\zeta_0)\Big[z_0 + \varepsilon(z) - \omega\Big(\zeta_0, z_a(\zeta_0), \\
&\int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big] \\
&+ \int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\left(r, z_a(r), \int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\right)\Delta r \\
&+ z_b - \Psi_A(b,\zeta_0)(z_0 + \varepsilon(z) \\
&- \omega\left(\zeta_0, z_a(\zeta_0), \int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\right) \\
&- \omega\left(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s\right) \\
&- \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))f\left(r, z(r), \int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\right)\Delta r \\
&- \int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\left(r, z_a(r), \int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\right)\Delta r \\
&- \sum_{k=1}^{m} \Psi_A(b,\zeta_k)V_{1k}(\zeta_k, z(\zeta_k^-))\Big] \\
&+ \int_{\zeta_0}^{b} \Psi_A(b,\rho(r))f\left(r, z(r), \int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\right)\Delta r \\
&= z_b
\end{aligned}
$$

i.e., $z(b) = z_b$.
$\therefore$ control input (1.9) is suitable for the system

$$
\begin{aligned}
&\left[z(\zeta) - \omega\left(\zeta, z_a(\zeta), \int_{\zeta_0}^{\zeta} l(\zeta,s)E_3(\zeta,z(s))\Delta s\right)\right]^{\Delta} = \\
&A(\zeta)z(\zeta) + B(\zeta)u(\zeta) \\
&+ f\left(\zeta, z(\zeta), \int_{\zeta_0}^{\zeta} v(\zeta,s)G(s,z(s))\Delta s\right), \\
&\zeta \in I, \zeta \neq \zeta_k \\
&z(\zeta_k^+) - z(\zeta_k^-) = V_{1k}(\zeta_k, z(\zeta_k^-)), k = 1,2,\ldots,m \\
&z(\zeta_0) - \varepsilon(z) = z_0.
\end{aligned}
$$

Also, we have,

$$
\begin{aligned}
\|u(\zeta)\| &\le \|B^T(\zeta)\|\|\Psi_A^T(b,\rho(\zeta))\|\|(W_{\zeta_0}^{b})^{-1}\|\|P(z)\| \\
&\le \mathfrak{M}_{\mathfrak{B}} M_{\Psi} \mathfrak{M}_{\mathfrak{W}} \|P(z)\| \qquad (1.10)
\end{aligned}
$$

$$
\begin{aligned}
\|P(z)\| = &\|z_b - \Psi_A(b,\zeta_0)\Big[z_0+\varepsilon(z)-\omega\Big(\zeta_0, z_a(\zeta_0),\\
&\int_{\zeta_0}^{b} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big]\\
&-\omega\Big(b, z_a(b), \int_{\zeta_0}^{b} l(b,s)E_3(b,z(s))\Delta s\Big)\\
&-\int_{\zeta_0}^{b}\Psi_A(b,\rho(r))f\Big(r,z(r),\\
&\int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\Big)\Delta r\\
&-\int_{\zeta_0}^{b} A(r)\Psi_A(b,\rho(r))\omega\Big(r,z_a(r),\\
&\int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\Big)\Delta r\\
&-\sum_{k=1}^{m}\Psi_A(b,\zeta_k)V_{1k}(\zeta_k,z(\zeta_k^-))\|\\
\le &\|z_b\|+M_\Psi\|z_0+\varepsilon(z)\|\\
&+M_\Psi(\mathfrak{C}_\omega+\mathfrak{M}_\omega\sup_{\zeta\in I}\|z(\zeta)\|\\
&+\mathfrak{N}_\omega(\mathfrak{M}_\mathfrak{l}(\mathfrak{M}_{\mathfrak{E}_3}+\mathfrak{N}_{\mathfrak{E}_3}\sup_{\zeta\in I}\|z(\zeta)\|)))+\mathfrak{C}_\omega\\
&+\mathfrak{M}_\omega\sup_{\zeta\in I}\|z(\zeta)\|+\mathfrak{N}_\omega(\mathfrak{M}_\mathfrak{l}(\mathfrak{M}_{\mathfrak{E}_3}\\
&+\mathfrak{N}_{\mathfrak{E}_3}\sup_{\zeta\in I}\|z(\zeta)\|))\\
&+\mathfrak{M}_\mathfrak{A}M_\Psi b(\mathfrak{C}_\omega+\mathfrak{M}_\omega\sup_{\zeta\in I}\|z(\zeta)\|)\\
&+\mathfrak{N}_\omega(\mathfrak{M}_\mathfrak{l}(\mathfrak{M}_{\mathfrak{E}_3}+\mathfrak{N}_{\mathfrak{E}_3}\sup_{\zeta\in I}\|z(\zeta)\|)))\\
&+M_\Psi b(\mathfrak{C}_\mathfrak{f}+\mathfrak{M}_\mathfrak{f}\sup_{\zeta\in I}\|z(\zeta)\|\\
&+\mathfrak{N}_\mathfrak{f}(\mathfrak{M}_\mathfrak{v}(\mathfrak{M}_\omega+\mathfrak{N}_\omega\sup_{\zeta\in I}\|z(\zeta)\|)))\\
&+mM_\Psi(\mathfrak{M}_{\mathfrak{V}_1}+\mathfrak{N}_{\mathfrak{V}_1}\sup_{\zeta\in I}\|z(\zeta)\|)\\
\le &\|z_b\|+M_\Psi\|z_0+\varepsilon(z)\|+M_\Psi\mathfrak{C}_\omega+M_\Psi\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{M}_{\mathfrak{E}_3}\\
&+\mathfrak{C}_\omega+\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{M}_{\mathfrak{E}_3}+\mathfrak{M}_\mathfrak{A}M_\Psi b\mathfrak{C}_\omega\\
&+\mathfrak{M}_\mathfrak{A}M_\Psi b\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{M}_{\mathfrak{E}_3}\\
&+M_\Psi b\mathfrak{C}_\mathfrak{f}+M_\Psi b\mathfrak{N}_\mathfrak{f}\mathfrak{M}_\mathfrak{v}\mathfrak{M}_\mathfrak{G}+mM_\Psi\mathfrak{M}_{\mathfrak{V}_1}\\
&+\{M_\Psi(\mathfrak{M}_\omega+\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{N}_{\mathfrak{E}_3})+\mathfrak{M}_\omega+\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{N}_{\mathfrak{E}_3}\\
&+\mathfrak{M}_\mathfrak{A}M_\Psi b(\mathfrak{M}_\omega+\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{N}_{\mathfrak{E}_3})\\
&+M_\Psi b(\mathfrak{M}_\mathfrak{f}+\mathfrak{N}_\mathfrak{f}\mathfrak{M}_\mathfrak{v}\mathfrak{N}_\omega)+mM_\Psi\mathfrak{N}_{\mathfrak{V}_1}\}\sup_{\zeta\in I}\|z(\zeta)\|\\
\le &\|z_b\|+M_\Psi\|z_0+\varepsilon(z)\|+\mathfrak{M}^*+\mathfrak{M}^{**}\sup_{\zeta\in I}\|z(\zeta)\|.
\end{aligned}
$$

Then equation (1.10) becomes

$$
\begin{aligned}
\|u(\zeta)\| \le &\mathfrak{M}_\mathfrak{B}M_\Psi\mathfrak{M}_\mathfrak{W}(\|z_b\|+M_\Psi\|z_0+\varepsilon(z)\|+\mathfrak{M}^*\\
&+\mathfrak{M}^{**}\sup_{\zeta\in I}\|z(\zeta)\|)\\
=&M_u.
\end{aligned}
$$

Theorem 3. *If* $(\mathbf{H_1})-(\mathbf{H_8})$ *are hold with*

$$
\begin{aligned}
L_\alpha = &\Big[\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}+M_\Psi\Big(\mathfrak{L}_\varepsilon+\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\\
&+m\mathfrak{L}_{\mathfrak{V}_1}+\mathfrak{M}_\mathfrak{A}b(\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3})\\
&+b(\mathfrak{L}_{\mathfrak{f}_1}+\mathfrak{L}_{\mathfrak{f}_2}\mathfrak{M}_\mathfrak{v}\mathfrak{L}_\mathfrak{G})\Big)\Big](1+M_\Psi^2\mathfrak{M}_\mathfrak{B}{}^2\mathfrak{M}_\mathfrak{W}b)<1,
\end{aligned}
$$

then on I (1.1) *is controllable.*

Proof: For $\beta=\frac{\mathfrak{K}_1}{1-\mathfrak{K}_2}$, assume a subset $B\subseteq PC$ s.t.,

$$B=\{z\in PC:\|z\|_{PC}\le\beta\}.$$

Define $\Pi: B\to B$:

$$
\begin{aligned}
(\Pi z)\zeta = &\Psi_A(\zeta,\zeta_0)\Big[z_0+\varepsilon(z)\\
&-\omega\Big(\zeta_0,z_a(\zeta_0),\int_{\zeta_0}^{\zeta} l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\Big]\\
&+\omega\Big(\zeta,z_a(\zeta),\int_{\zeta_0}^{\zeta} l(\zeta,s)E_3(\zeta,z(s))\Delta s\Big)\\
&+\int_{\zeta_0}^{\zeta}\Psi_A(\zeta,\rho(r))\Big[B(r)u(r)\\
&+f\Big(r,z(r),\int_{r_0}^{r} v(r,s)G(r,z(s))\Delta s\Big)\Delta r\Big]\\
&+\int_{\zeta_0}^{b} A(r)\Psi_A(\zeta,\rho(r))\omega\Big(r,z_a(r),\\
&\qquad\int_{r_0}^{r} l(r,s)E_3(r,z(s))\Delta s\Big)\Delta r\\
&+\sum_{\zeta_0<\zeta_k<\zeta}\Psi_A(\zeta,\zeta_k)V_{1k}(\zeta_k,z(\zeta_k^-)).
\end{aligned}
$$

We need to show that $\Pi: B\to B$ to use the Banach contraction theorem. Now, for $\zeta\in I$ & $z\in B$, then

$$
\begin{aligned}
\|(\Pi z)\zeta\| \le &M_\Psi(\|z_0+\varepsilon(z)\|+\mathfrak{C}_\omega+\mathfrak{M}_\omega\sup_{\zeta\in I}\|z(\zeta)\|\\
&+\mathfrak{N}_\omega(\mathfrak{M}_\mathfrak{l}(\mathfrak{M}_{\mathfrak{E}_3}+\mathfrak{N}_{\mathfrak{E}_3}\sup_{\zeta\in I}\|z(\zeta)\|))\\
&+\mathfrak{C}_\omega+\mathfrak{M}_\omega\sup_{\zeta\in I}\|z(\zeta)\|\\
&+\mathfrak{N}_\omega(\mathfrak{M}_\mathfrak{l}(\mathfrak{M}_{\mathfrak{E}_3}+\mathfrak{N}_{\mathfrak{E}_3}\sup_{\zeta\in I}\|z(\zeta)\|))\\
&+\mathfrak{M}_\mathfrak{A}M_\Psi b\Big(\mathfrak{C}_\omega+\mathfrak{M}_\omega\sup_{\zeta\in I}\|z(\zeta)\|
\end{aligned}
$$

$$
\begin{aligned}
&+\mathfrak{N}_\omega(\mathfrak{M}_\mathfrak{l}(\mathfrak{M}_{\mathfrak{E}_3}+\mathfrak{N}_{\mathfrak{E}_3}\sup_{\zeta\in I}\|z(\zeta)\|))\Big)\\
&+M_\Psi^2 b M_B^2 \mathfrak{M}_\mathfrak{W}\|P(z)\|\\
&+M_\Psi b(\mathfrak{C}_\mathfrak{f}+\mathfrak{M}_\mathfrak{f}\sup_{\zeta\in I}\|z(\zeta)\|+\mathfrak{N}_\mathfrak{f}(\mathfrak{M}_\mathfrak{v}(\mathfrak{M}_\omega\\
&+\mathfrak{N}_\omega\sup_{\zeta\in I}\|z(\zeta)\|))+mM_\Psi(\mathfrak{M}_{\mathfrak{V}_1}\\
&+\mathfrak{N}_{\mathfrak{V}_1}\sup_{\zeta\in I}\|z(\zeta)\|)\\
\le{}& M_\Psi(\|z_0+\varepsilon(z)\|)+(1+M_\Psi+\mathfrak{M}_\mathfrak{A}M_\Psi b)\\
&(\mathfrak{C}_\omega+\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{M}_{\mathfrak{E}_3})+M_\Psi b(\mathfrak{C}_\mathfrak{f}+\mathfrak{N}_\mathfrak{f}\mathfrak{M}_\mathfrak{v}\mathfrak{M}_\omega)\\
&+mM_\Psi\mathfrak{M}_{\mathfrak{V}_1}+\Big[(1+M_\Psi+\mathfrak{M}_\mathfrak{A}M_\Psi b)\\
&(\mathfrak{M}_\omega+\mathfrak{N}_\omega\mathfrak{M}_\mathfrak{l}\mathfrak{N}_{\mathfrak{E}_3})+M_\Psi b(\mathfrak{M}_\mathfrak{f}+\mathfrak{N}_\mathfrak{f}\mathfrak{M}_\mathfrak{v}\mathfrak{N}_\omega)\\
&+mM_\Psi\mathfrak{N}_{\mathfrak{V}_1}\Big]b\sup_{\zeta\in I}\|z(\zeta)\|+M_\Psi^2M_B^2\mathfrak{M}_\mathfrak{W}b\Big[\\
&\|z_b+M_\Psi\|z_0+\varepsilon(z)\|+\mathfrak{M}^*+\mathfrak{M}^{**}\sup_{\zeta\in I}\|z(\zeta)\|\Big]\\
&\quad M_\Psi\|z_0+\varepsilon(z)\|+\mathfrak{M}^*+\mathfrak{M}^{**}\sup_{\zeta\in I}\|z(\zeta)\|\\
&+M_\Psi^2M_B^2\mathfrak{M}_\mathfrak{W}\|z_b\|+M_\Psi^2M_B^2\mathfrak{M}_\mathfrak{W}b[M_\Psi\|z_0+\varepsilon(z)\|\\
&+\mathfrak{M}^*+\mathfrak{M}^{**}\sup_{\zeta\in I}\|z(\zeta)\|]\\
\le{}&(1+M_\Psi^2M_B^2\mathfrak{M}_\mathfrak{W}b)(M_\Psi\|z_0+\varepsilon(z)\|+\mathfrak{M}^*)\\
&+M_\Psi^2M_B^2\mathfrak{M}_\mathfrak{W}b\|z_b\|+(1+M_\Psi^2M_B^2\mathfrak{M}_\mathfrak{W}b)\mathfrak{M}^{**}\beta\\
\le{}&\mathfrak{K}_1+\mathfrak{K}_2\beta\\
\le{}&\mathfrak{K}_1+\mathfrak{K}_2\frac{\mathfrak{K}_1}{1-\mathfrak{K}_2}\\
\le{}&\frac{\mathfrak{K}_1-\mathfrak{K}_1\mathfrak{K}_2+\mathfrak{K}_2\mathfrak{K}_1}{1-\mathfrak{K}_2}\\
\le{}&\frac{\mathfrak{K}_1}{1-\mathfrak{K}_2}\\
\le{}&\beta
\end{aligned}
$$

$\therefore \|\Pi z\|_{PC}\le\beta.\ \therefore \Pi : B\to B$
For $z,y\in B$ & $\zeta\in I$, then:

$$
\begin{aligned}
&\|(\Pi z)\zeta-(\Pi y)\zeta\|\le\\
&\Big\|\omega\Big(\zeta,z_a(\zeta),\int_{\zeta_0}^{\zeta}l(\zeta,s)E_3(\zeta,z(s))\Delta s\Big)\\
&-\omega\Big(\zeta,y_a(\zeta),\int_{\zeta_0}^{\zeta}l(\zeta,s)E_3(\zeta,y(s))\Delta s\Big)\Big\|\\
&+\Big\|\Psi_A(\zeta,\zeta_0)\Big[z_0+\varepsilon(z)-\omega\Big(\zeta_0,z_a(\zeta_0),\\
&\int_{\zeta_0}^{\zeta}l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)-y_0-\varepsilon(y)\\
&+\omega\Big(\zeta_0,y_a(\zeta_0),\int_{\zeta_0}^{\zeta}l(\zeta_0,s)E_3(\zeta_0,y(s))\Delta s\Big)\Big]\Big\|\\
&+\int_{\zeta_0}^{b}\Big\|A(r)\Psi_A(\zeta,\rho(r))\Big[\omega\Big(r,z_a(r),\int_{r_0}^{r}l(r,s)E_3(r,z(s))\Delta s\Big)\\
&-\omega\Big(r,y_a(r),\int_{r_0}^{r}l(r,s)E_3(r,y(s))\Delta s\Big)\Big]\Delta r\Big\|\\
&+\Big\|\int_{\zeta_0}^{\zeta}\Psi_A(\zeta,\rho(r))\Big[f\Big(r,z(r),\int_{r_0}^{r}v(r,s)G(r,z(s))\Delta s\Big)\\
&-f\Big(r,y(r),\int_{r_0}^{r}v(r,s)G(r,y(s))\Delta s\Big)\Big]\Delta r\Big\|\\
&+\Big\|\sum_{k=1}^{m}\Psi_A(\zeta,\zeta_k)[V_{1k}(\zeta_k,z(\zeta_k^-))-V_{1k}(\zeta_k,y(\zeta_k^-))]\Big\|\\
&+\Big\|\int_{\zeta_0}^{\zeta}\Psi_A(\zeta,\rho(r))B(r)B^T(r)\Psi_A^T(\zeta,\rho(r))(W_{\zeta_0}^{b})^{-1}\Big[z_b-y_b\\
&+\Psi_A(b,\zeta_0)\Big[z_0+\varepsilon(z)-y_0-\varepsilon(y)\\
&-\omega\Big(\zeta_0,z_a(\zeta_0),\int_{\zeta_0}^{b}l(\zeta_0,s)E_3(\zeta_0,z(s))\Delta s\Big)\\
&+\omega\Big(\zeta_0,y_a(\zeta_0),\int_{\zeta_0}^{b}l(\zeta_0,s)E_3(\zeta_0,y(s))\Delta s\Big)\Big]\\
&+\omega\Big(b,y_a(b),\int_{\zeta_0}^{b}l(b,s)E_3(b,y(s))\Delta s\Big)\\
&-\omega\Big(b,z_a(b),\int_{\zeta_0}^{b}l(b,s)E_3(b,z(s))\Delta s\Big)\\
&+\int_{\zeta_0}^{b}A(r)\Psi_A(b,\rho(r))\Big[\omega\Big(r,z_a(r),\int_{r_0}^{r}l(r,s)E_3(r,z(s))\Delta s\Big)\\
&-\omega\Big(r,y_a(r),\int_{r_0}^{r}l(r,s)E_3(r,y(s))\Delta s\Big)\Delta r\Big]\\
&+\int_{\zeta_0}^{b}\Psi_A(b,\rho(r))\Big[f\Big(r,z_a(r),\int_{r_0}^{r}v(r,s)G(r,z(s))\Delta s\Big)\\
&-f\Big(r,y_a(r),\int_{r_0}^{r}v(r,s)G(r,y(s))\Delta s\Big)\Big]\Delta r\\
&+\sum_{k=1}^{m}\Psi_A(b,\zeta_k)(V_{1k}(\zeta_k,z(\zeta_k^-))-V_{1k}(\zeta_k,y_(\zeta_k^-)))\Big]\Big\|
\end{aligned}
$$

$$
\begin{aligned}
&\|(\Pi z)\zeta-(\Pi y)\zeta\|\le\mathfrak{L}_{\omega_1}\|z-y\|+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\|z-y\|\\
&+M_\Psi(\mathfrak{L}_\varepsilon\|z-y\|+\mathfrak{L}_{\omega_1}\|z-y\|+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\|z-y\|)\\
&+\mathfrak{M}_\mathfrak{A}M_\Psi b(\mathfrak{L}_{\omega_1}\|z-y\|+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\|z-y\|)\\
&+M_\Psi b(\mathfrak{L}_{\mathfrak{f}_1}\|z-y\|+\mathfrak{L}_{\mathfrak{f}_2}\mathfrak{M}_\mathfrak{v}\mathfrak{L}_\mathfrak{G}\|z-y\|)+mM_\Psi\mathfrak{L}_{\mathfrak{V}_1}\|z-y\|\\
&bM_\Psi^2\mathfrak{M}_\mathfrak{B}{}^2\mathfrak{M}_\mathfrak{W}(M_\Psi(\mathfrak{L}_\varepsilon\|z-y\|+\mathfrak{L}_{\omega_1}\|z-y\|+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\|z-y\|)\\
&+\mathfrak{L}_{\omega_1}\|z-y\|+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\|z-y\|+\mathfrak{M}_\mathfrak{A}M_\Psi b(\mathfrak{L}_{\omega_1}\|z-y\|\\
&+\mathfrak{L}_{\omega_2}\mathfrak{M}_\mathfrak{l}\mathfrak{L}_{\mathfrak{E}_3}\|z-y\|)+M_\Psi b(\mathfrak{L}_{\mathfrak{f}_1}\|z-y\|+\mathfrak{L}_{\mathfrak{f}_2}\mathfrak{M}_\mathfrak{v}\mathfrak{L}_\mathfrak{G}\|z-y\|)\\
&+mM_\Psi\mathfrak{L}_{\mathfrak{V}_1}\|z-y\|)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1+M_{\Psi}^{2}\mathfrak{M}_{\mathfrak{B}}{}^{2}\mathfrak{M}_{\mathfrak{W}})(M_{\Psi}(\mathfrak{L}_{\varepsilon}+\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_{\mathfrak{l}}\mathfrak{L}_{\mathfrak{E}_3})\\
&+\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_{\mathfrak{l}}\mathfrak{L}_{\mathfrak{E}_3}+\mathfrak{M}_{\mathfrak{A}}M_{\Psi}b(\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_{\mathfrak{l}}\mathfrak{L}_{\mathfrak{E}_3})\\
&+M_{\Psi}b(\mathfrak{L}_{\mathfrak{f}_1}+\mathfrak{L}_{\mathfrak{f}_2}\mathfrak{M}_{\mathfrak{v}}\mathfrak{L}_{\mathfrak{G}})+mM_{\Psi}\mathfrak{L}_{\mathfrak{V}_1})\|z-y\|\\
&\leq (\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_{\mathfrak{l}}\mathfrak{L}_{\mathfrak{E}_3}+M_{\Psi}(\mathfrak{L}_{\varepsilon}+\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_{\mathfrak{l}}\mathfrak{L}_{\mathfrak{E}_3}+m\mathfrak{L}_{\mathfrak{V}_1}\\
&+\mathfrak{M}_{\mathfrak{A}}b(\mathfrak{L}_{\omega_1}+\mathfrak{L}_{\omega_2}\mathfrak{M}_{\mathfrak{l}}\mathfrak{L}_{\mathfrak{E}_3})+b(\mathfrak{L}_{\mathfrak{f}_1}+\mathfrak{L}_{\mathfrak{f}_2}\mathfrak{M}_{\mathfrak{v}}\mathfrak{L}_{\mathfrak{G}})))\\
&(1+M_{\Psi}^{2}\mathfrak{M}_{\mathfrak{B}}{}^{2}\mathfrak{M}_{\mathfrak{W}})\|z-y\|.
\end{aligned}
$$

Hence,

$$\|\Pi z-\Pi y\|_{PC}\leq L_{\alpha}\|z-y\|_{PC},$$

which gives a contradiction to our assumption $\Pi z=\Pi y \Rightarrow z=y$.
$\therefore$ Π holds strict contraction. Unique fixed point of Π, as result of (1.1) is given by Banach contraction theorem.

4 Conclusion

In this study, we investigated the existence & the controllability properties for a impulsive neutral functional integrodifferential time-varying equation on time-scales. These investigations were conducted employing the BFT. In the coming years, we can strive to determine the controllability results for fractional differential equations with impulses occurring at various time scales.

Conflict of Interest

The author declare that there is no conflict of interest regarding the publication of this article.

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