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Symmetric Colorings of $\mathbb{Z}_p^n$

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Abstract: Symmetries on a group G are the mappings $G \ni x \mapsto gx^{-1}g \in G$, where $g \in G$. A coloring $\chi : G \to \{1,\dots,r\}$ of G is symmetric if it is invariant under some symmetry. We count the number $S_r(\mathbb{Z}_p^n)$ of symmetric r-colorings of $\mathbb{Z}_p^n$, the direct product of n copies of the cyclic group of prime order p. As a consequence we obtain that $S_r(\mathbb{Z}_p^n) = p^n r^{\frac{p^n+1}{2}} + S_r(\mathbb{Z}_p^{n-1})$.

Keywords: Symmetric coloring, equivalent colorings, elementary abelian p-group, Gaussian coefficient.

Let G be a finite group and let $r \in \mathbb{N}$. An *r-coloring* of G is any mapping $\chi : G \to \{1,\dots,r\}$. Let r^G denote the set of all r-colorings of G. The group G naturally acts on r^G. For any $\chi \in r^G$ and $g \in G$, $\chi g \in r^G$ is defined by $\chi g(x) = \chi(xg^{-1})$. Colorings χ and ψ are *equivalent* if there exists $g \in G$ such that $\chi(xg^{-1}) = \psi(x)$ for all $x \in G$ (that is, χ and ψ belong to the same orbit). Let $c_r(G)$ denote the number of equivalence classes of r-colorings of G (= the number of orbits of r^G). Applying Burnside's Lemma [1, 1.7] shows that

$$c_r(G) = \frac{1}{|G|}\sum_{g \in G} r^{|G:\langle g\rangle|},$$

where $\langle g\rangle$ is the subgroup generated by g. For $\mathbb{Z}_n$, the cyclic group of order n, this formula simplifies to

$$c_r(\mathbb{Z}_n) = \frac{1}{n}\sum_{d|n} \varphi(d) r^{\frac{n}{d}},$$

where φ is the Euler function [2].

For every $g \in G$, the *symmetry* on G with respect to g is the mapping

$$G \ni x \mapsto gx^{-1}g \in G.$$

This is an old notion, which can be found in the book [3]. For $\mathbb{Z}_n$, identifying it with the vertices of a regular n-gon, the symmetries are the reflections of the polygon in an axis through one of the vertices (if n is odd, the symmetries are all the reflections). A coloring $\chi \in r^G$ is *symmetric* if it is invariant under some symmetry (that is, there exists $g \in G$ such that $\chi(gx^{-1}g) = \chi(x)$ for all $x \in G$). A coloring equivalent to a symmetric one is also symmetric [4, Lemma 2.1]. Let $S_r(G)$ denote the number of symmetric r-colorings of G and $s_r(G)$ the number of equivalence classes of symmetric r-colorings of G (= the number of symmetric orbits of r^G). If G is abelian, then

$$S_r(G) = \sum_{X \le G}\sum_{Y \le X} \frac{\mu(Y,X)|G/Y|}{|(G/Y)[2]|} r^{\frac{|G/X|+|(G/X)[2]|}{2}},$$

$$s_r(G) = \sum_{X \le G}\sum_{Y \le X} \frac{\mu(Y,X)}{|(G/Y)[2]|} r^{\frac{|G/X|+|(G/X)[2]|}{2}},$$

where X runs over subgroups of G, Y over subgroups of X, $\mu(Y,X)$ is the Möbius function on the lattice of subgroups of G, and $H[2] = \{x \in H : x^2 = 1\}$ [5]. Similar but more complicated formulas hold also in the non-abelian case [4].

For $\mathbb{Z}_n$, the general formulas simplify to

$$S_r(\mathbb{Z}_n) = \begin{cases} \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}} & \text{if } n \text{ is odd} \\ \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1} & \text{if } n \text{ is even,} \end{cases}$$

$$s_r(\mathbb{Z}_n) = \begin{cases} r^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ \frac{1}{2}(r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}) & \text{if } n \text{ is even,} \end{cases}$$

where p is a variable of prime value and m is the greatest odd divisor of n [5]. For the dihedral group D_n, the semidirect product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$, the numbers $S_r(D_n)$ and $s_r(D_n)$ were counted in [6].

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In this note we consider elementary abelian p-group $\mathbb{Z}_p^n$, the direct product of n copies of $\mathbb{Z}_p$, where p is prime. If $p=2$, then every coloring is symmetric, so

$$S_r(\mathbb{Z}_2^n)=r^{2^n},$$

$$s_r(\mathbb{Z}_2^n)=c_r(\mathbb{Z}_2^n)=\frac{1}{2^n}(r^{2^n}+(2^n-1)r^{2^{n-1}}).$$

And if $p>2$, then

$$s_r(\mathbb{Z}_p^n)=r^{\frac{p^n+1}{2}},$$

which is a partial case of a more general fact (we prove it in the end of the note). In [7], $S_r(\mathbb{Z}_p^n)$ was counted for $n=2,3$. Notice that a symmetry of $\prod_{i=1}^n G_i$ is a mapping $\prod_{i=1}^n \sigma_i$, where σ_i is a symmetry of G_i, so the symmetries of $\mathbb{Z}_p^n$ $(p>2)$ are the coordinate-wise reflections.

The aim of this note is to count the number $S_r(\mathbb{Z}_p^n)$ for all n. We show that

Theorem 1. *For all $r,n\in\mathbb{N}$ and prime $p>2$,*

$$\begin{aligned}S_r(\mathbb{Z}_p^n)&=p^nr^{\frac{p^n+1}{2}}+p^{n-1}(1-p^n)r^{\frac{p^{n-1}+1}{2}}+\\&+p^{n-2}(1-p^{n-1})(1-p^n)r^{\frac{p^{n-2}+1}{2}}+\dots\\&+p(1-p^2)(1-p^3)\dots(1-p^n)r^{\frac{p+1}{2}}+\\&+(1-p)(1-p^2)\dots(1-p^n)r.\end{aligned}$$

Proof. The number of subgroups of $\mathbb{Z}_p^n$ of order p^k is

$$\binom{n}{k}_p=\frac{(p^n-1)(p^{n-1}-1)\dots(p^{n-k+1}-1)}{(p^k-1)(p^{k-1}-1)\dots(p-1)},$$

the Gaussian coefficient [1, 3.11], and if $Y\le X\le\mathbb{Z}_p^n$ and $|Y|=p^k$ and $|X|=p^m$, then

$$\mu(Y,X)=(-1)^{m-k}p^{\binom{m-k}{2}}$$

[1, 4.20]. Here,

$$\binom{n}{k}=\frac{n(n-1)\dots(n-k+1)}{1\cdot2\cdot\dots\cdot k},$$

and if $k>n$, then $\binom{n}{k}=0$ and $\binom{n}{k}_p=0$. Thus, the general formula for counting $S_r(G)$ gives us that

$$S_r(\mathbb{Z}_p^n)=\sum_{m=0}^{n}\binom{n}{m}_p\sum_{k=0}^{m}\binom{m}{k}_p(-1)^{m-k}p^{\binom{m-k}{2}+n-k}r^{\frac{p^{n-m}+1}{2}}.$$

Comparing, we conclude that in order to prove the theorem, it suffices to show that

$$\binom{n}{m}_p\sum_{k=0}^{m}\binom{m}{k}_p(-1)^{m-k}p^{\binom{m-k}{2}+n-k}=$$
$$=p^{n-m}(1-p^{n-m+1})\dots(1-p^n).$$

If $m=0$, both sides are equal to p^n, so let $m\ge1$.

The left-hand side of the equality is equal to

$$\begin{aligned}&\frac{(p^n-1)\dots(p^{n-m+1}-1)}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}\frac{(p^m-1)\dots(p^{m-k+1}-1)}{(p^k-1)\dots(p-1)}(-1)^{m-k}p^{\binom{m-k}{2}+n-k}\\&=(p^n-1)\dots(p^{n-m+1}-1)\sum_{k=0}^{m}\frac{(-1)^{m-k}p^{\binom{m-k}{2}+n-k}}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)}\\&=\frac{(p^n-1)\dots(p^{n-m+1}-1)}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}\frac{(p^m-1)\dots(p-1)\cdot(-1)^{m-k}p^{\binom{m-k}{2}+n-k}}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)}\\&=\frac{(1-p^n)\dots(1-p^{n-m+1})}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}\frac{(p^m-1)\dots(p-1)\cdot(-1)^kp^{\binom{m-k}{2}+n-k}}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)},\end{aligned}$$

and since

$$\frac{(p^m-1)\dots(p-1)}{(p^{m-k}-1)\dots(p-1)\cdot(p^k-1)\dots(p-1)}=\binom{m}{k}_p$$

and $n-k=(n-m)+(m-k)$ and

$$\binom{m-k}{2}+m-k=1+2+\dots+(m-k-1)+(m-k)=$$
$$=\binom{m-k+1}{2},$$

it is equal to

$$\frac{p^{n-m}(1-p^{n-m+1})\dots(1-p^n)}{(p^m-1)\dots(p-1)}\sum_{k=0}^{m}(-1)^k\binom{m}{k}_pp^{\binom{m-k+1}{2}}.$$

Then the next lemma finishes the proof.

Lemma 1.

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}_pp^{\binom{m-k+1}{2}}=(p^m-1)(p^{m-1}-1)\dots(p-1).$$

Proof. If $m=1$, the left-hand side is

$$(-1)^0\binom{1}{0}_pp^{\binom{2}{2}}+(-1)^1\binom{1}{1}_pp^{\binom{1}{2}}=p-1,$$

so the equality holds.

Now fix $m>1$ and suppose that the equality holds for $m-1$. Since

$$\binom{m}{k}_p=\binom{m-1}{k-1}+p^k\binom{m-1}{k}$$

[1, 3.34], we obtain that

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}_pp^{\binom{m-k+1}{2}}=\sum_{k=0}^{m}(-1)^k\binom{m-1}{k-1}_pp^{\binom{m-k+1}{2}}+$$
$$+\sum_{k=0}^{m}(-1)^k\binom{m-1}{k}_pp^{\binom{m-k+1}{2}+k}.$$

The first sum is equal to

$$-\sum_{k=1}^{m}(-1)^{k-1}\binom{m-1}{k-1}_p p^{\binom{(m-1)-(k-1)+1}{2}} = -\sum_{k=0}^{m-1}(-1)^{k}\binom{m-1}{k}_p p^{\binom{(m-1)-k+1}{2}},$$

and the second one to

$$\sum_{k=0}^{m-1}(-1)^{k}\binom{m-1}{k}_p p^{\binom{m-k+1}{2}+k} = p^m\sum_{k=0}^{m-1}(-1)^{k}\binom{m-1}{k}_p p^{\binom{(m-1)-k+1}{2}}$$

because

$$\binom{m-k+1}{2}+k = 1+2+\ldots+(m-k-1)+(m-k)+k =$$
$$=\binom{m-k}{2}+m.$$

Consequently, by the inductive hypothesis,

$$\sum_{k=0}^{m}(-1)^{k}\binom{m}{k}_p p^{\binom{m-k+1}{2}} = -(p^{m-1}-1)\ldots(p-1)+$$
$$+p^m(p^{m-1}-1)\ldots(p-1)$$
$$=(p^m-1)(p^{m-1}-1)\ldots(p-1).$$

As a consequence we obtain from Theorem 1 that

Corollary 1. *For all $r \geq 1$, $n \geq 2$, and prime $p > 2$,*

$$S_r(\mathbb{Z}_p^n) = p^n r^{\frac{p^n+1}{2}} + S_r(\mathbb{Z}_p^{n-1}).$$

We conclude this note by counting the number $s_r(G)$ for every finite abelian group G of odd order.

Let G be a finite group and let $r \in \mathbb{N}$. For every $\chi \in r^G$, let

$$[\chi] = \{\chi g : g \in G\} \text{ and } St(\chi) = \{g \in G : \chi g = \chi\},$$

and let

$$Z(\chi) = \{g \in G : \chi \text{ is symmetric with respect to } g\}.$$

For every symmetric $\chi \in r^G$ and for every $h \in G$, $Z(\chi h) = Z(\chi)h$ [4, Lemma 2.1], so there are colorings in $[\chi]$ symmetric with respect to 1, and their number is $\frac{|Z(\chi)|}{|St(\chi)|}$ [4, Lemma 2.5]. Furthermore, if χ is symmetric with respect to 1 and G is abelian, then $Z(\chi) = \{g \in G : g^2 \in St(\chi)\}$ [4, Corollary 2.9], and consequently, if the order of G is odd, then $Z(\chi) = St(\chi)$.

Lemma 2. *If G is a finite abelian group of odd order, then*

$$s_r(G) = r^{\frac{|G|+1}{2}}.$$

Proof. Every symmetric orbit of r^G has a coloring symmetric with respect to 1, and since G is abelian of odd order, there is only one such coloring. Consequently, $s_r(G)$ is equal to the number of r-colorings of G symmetric with respect to 1, which is equal to the number of r-colorings of the set $\{1\} \cup \{\{x, x^{-1}\} : x \in G \setminus \{1\}\}$ whose cardinality is $1 + \frac{|G|-1}{2} = \frac{|G|+1}{2}$.

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References

[1] M. Aigner, *Combinatorial Theory*. Springer-Verlag, Berlin-Heidelberg-New York, (1979).

[2] E. Bender and J. Goldman, On the applications of Möbius inversion in combinatorial analysis, *Amer. Math. Monthly*, **82**, 789-803, (1975).

[3] O. Loos, *Symmetric Spaces*. Benjamin: New York, NY, USA, (1969).

[4] Y. Zelenyuk, *Symmetric colorings of finite groups*, in Proceedings of Groups St Andrews 2009, Bath, UK, LMS Lecture Note Series, **388**, 580-590, (2011).

[5] Y. Gryshko (Zelenyuk), Symmetric colorings of regular polygons, *Ars Combinatoria*, **78**, 277-281, (2006).

[6] J. Phakathi, W. Toko, Y. Zelenyuk, and Yu. Zelenyuk, Symmetric colorings of the dihedral group, *Communications in Algebra*, **46**, 1554-1559, (2018).

[7] Yu. Zelenyuk, Computing the number of symmetric colorings of elementary Abelian groups, *Alexandria Engineering Journal*, **60**, 2075-2081, (2021).

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